

SPHERICALLY SYMMETRIC STATIC CHARGED ANISOTROPIC FLUID IN ROSEN'S BIMETRIC THEORY OF GRAVITATION

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ABSTRACT. In this paper we have obtained an exact solution for the static spherically symmetric space-time with charged anisotropic fluid distribution in context of Rosen's Bimetric General Relativity (BGR).

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1. INTRODUCTION

Gravitation is one of the four fundamental interactions. Gravity is useful in understanding the large scale structure of the universe. Newton had proposed a theory of gravitation which is valid in a limited sense. Einstein gave a more general understanding of gravity in the General Theory of Relativity. These theories are regarded as successful theories. Many attempts are going on for further modifications and alternative theories. One of them is modification of GR given by Rosen [9], known as the Bimetric General Relativity (BGR). In this theory gravity is attributed to a

curved space-time described by the metric,

$$(1) \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

a second metric tensor in the background space is described by

$$(2) \quad d\sigma^2 = \gamma_{\mu\nu} dx^\mu dx^\nu,$$

in this way the curved space-time of interest is regarded as embedded in a flat space-time. The field equations in BGR are written in the form of Einstein's field equations, but with an additional term on the right hand side,

$$(3) \quad G_\nu^\mu = -8\pi T_\nu^\mu + S_\nu^\mu,$$

where G_ν^μ is the Einstein tensor, T_ν^μ energy-momentum tensor of matter distribution and

$$(4) \quad S_\nu^\mu = \frac{3}{a^2} (\gamma_{\nu\alpha} g^{\alpha\mu} - \frac{1}{2} \delta_\nu^\mu g^{\alpha\beta} \gamma_{\alpha\beta}),$$

where a is a constant scale parameter. The order of this scale parameter is related to the size of the universe.

In this paper we present an exact solution of the field equations for charged anisotropic fluid in context of BGR proposed by Rosen [7],[8],[9]. We have followed the method developed by Hojman and Santamarina [5], Berger et al. [1] in general relativity and Khadekar and Kondawar [6] for the case of perfect fluid in bimetric relativity to find the general static solution. The General Relativity analogue for the charged perfect fluid was considered by Singh et al. [10] and results obtained here match with those of obtained there if we consider the fluid to be isotropic. Also in absence of charge and anisotropy, results obtained in this paper match with the one obtained by Khadekar and Kondawar [6] for Rosen's Bimetric Theory of Gravitation.

2. METRICS AND FIELD EQUATIONS

The general static spherically symmetric line element may be expressed as

$$(5) \quad ds^2 = -dr^2 - R(r)^2 (d\theta^2 + \sin^2 \theta d\phi^2) + g(r)^2 dt^2.$$

Consider the background flat metric $\gamma_{\mu\nu}$ in a static de-Sitter form as

$$(6) \quad d\sigma^2 = -\left(1 - \frac{r^2}{a^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + \left(1 - \frac{r^2}{a^2}\right) dt^2.$$

For a region very small compared to a , i.e. for $r \ll a$, this line element has Minkowski form

$$(7) \quad d\sigma^2 = -dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + dt^2.$$

The convention used here for coordinates is

$$x^1 = r, x^2 = \theta, x^3 = \phi, x^4 = t.$$

The energy momentum tensor for charged anisotropic fluid is of the form

$$(8) \quad T_{\mu\nu} = (\rho + p_\perp)U_\mu U_\nu - p_\perp g_{\mu\nu} + (p_r - p_\perp)\chi_\mu \chi_\nu + \frac{1}{4\pi} \left(g^{\lambda\alpha} F_{\mu\lambda} F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F_{\lambda\alpha} F^{\lambda\alpha} \right),$$

with matter density ρ , p_r being the radial pressure in the direction of χ_μ , $p_\perp$ being the tangential pressure on the two-space orthogonal to χ_μ , the four-velocity vector of the fluid U_μ and χ_μ being the unit space-like vector orthogonal to U_μ .

The skew symmetric Maxwell Tensor $F_{\mu\nu}$ satisfies the Maxwell's equations in the form

$$(9) \quad F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} = 0,$$

$$(10) \quad F^{\mu\nu}{}_{;\nu} = -4\pi J^\mu,$$

where $J^\mu = \sigma U^\mu$ is the 4-current of the charge distribution with proper charge density σ within the sphere. It is known that due to spherical symmetry the only non-vanishing component of $F_{\mu\nu}$ is,

$$(11) \quad F_{14} = \frac{Q(r)g}{R^2},$$

where

$$(12) \quad Q(r) = 4\pi \int_0^r R^2 \sigma(r) dr,$$

represents the total charge contained within a sphere of coordinate radius r .

Choosing the comoving system we write

$$(13) \quad U^\mu = \left(0, 0, 0, \frac{1}{g}\right), \chi^\mu = (1, 0, 0, 0),$$

here $U_\mu U^\mu = -\chi_\mu \chi^\mu = 1$.

The non-vanishing components of the energy momentum tensor are

$$(14) \quad T_1^1 = -p_r + \frac{Q^2}{8\pi R^4},$$

$$(15) \quad T_2^2 = T_3^3 = -p_\perp - \frac{Q^2}{8\pi R^4},$$

$$(16) \quad T_4^4 = \rho + \frac{Q^2}{8\pi R^4}.$$

In the region $r \ll a$ and neglecting the terms which are small throughout this region, we can write the non-vanishing components of S_μ^ν in accordance with [2] as,

$$(17) \quad -S_1^1 = -S_2^2 = -S_3^3 = S_4^4 = \frac{3}{2a^2 g^2}.$$

Einstein tensor G_ν^μ for metric (5) is computed using Mathematica program described in [4] with relevant modifications. The calculations for the rest part of the field equations (3) are done using the energy-momentum tensor (8), background metric (7) and values from equation (17) following the procedure as given in [9]. Hence the field equations are written as,

$$(18) \quad \frac{R'^2}{R^2} + \frac{2R'g'}{Rg} - \frac{1}{R^2} = 8\pi p_r - \frac{Q^2}{R^4} - \frac{3}{2a^2 g^2},$$

$$(19) \quad \frac{R''}{R} + \frac{g''}{g} + \frac{R'g'}{Rg} = 8\pi p_\perp + \frac{Q^2}{R^4} - \frac{3}{2a^2 g^2},$$

$$(20) \quad \frac{2R''}{R} + \frac{R'^2}{R^2} - \frac{1}{R^2} = -8\pi\rho - \frac{Q^2}{R^4} + \frac{3}{2a^2 g^2}.$$

Here a prime for R and g denotes a differentiation with respect to r .

The energy-momentum conservation equation $T_{\nu;\mu}^\mu = 0$ gives,

$$(21) \quad (\rho + p_r) \frac{g'}{g} + p'_r + 2 \frac{R'}{R} (p_r - p_\perp) = \frac{1}{8\pi R^4} \frac{dQ^2}{dr}.$$

We now define the *effective* density ρ_e , *effective* radial pressure p_{r_e} and *effective* tangential pressure $p_{\perp_e}$ following [3] as,

$$(22) \quad \begin{aligned} \rho_e &= \rho - \frac{3}{16\pi a^2 g^2}, \\ p_{r_e} &= p_r - \frac{3}{16\pi a^2 g^2}, \\ p_{\perp_e} &= p_\perp - \frac{3}{16\pi a^2 g^2}. \end{aligned}$$

So the field equations (18)-(20) take the form,

$$(23) \quad \frac{R'^2}{R^2} + \frac{2R'g'}{Rg} - \frac{1}{R^2} = 8\pi p_{r_e} - \frac{Q^2}{r^4},$$

$$(24) \quad \frac{R''}{R} + \frac{g''}{g} + \frac{R'g'}{Rg} = 8\pi p_{\perp_e} + \frac{Q^2}{r^4},$$

$$(25) \quad \frac{2R''}{R} + \frac{R'^2}{R^2} - \frac{1}{R^2} = -8\pi \rho_e - \frac{Q^2}{r^4}.$$

Equation (21) can be rewritten as,

$$(26) \quad (\rho_e + p_{r_e}) \frac{g'}{g} + p'_{r_e} + 2 \frac{R'}{R} (p_{r_e} - p_{\perp_e}) = \frac{1}{8\pi R^4} \frac{dQ^2}{dr}.$$

Now from equation (25),

$$(27) \quad R'^2 = 1 - \frac{2m_e(R)}{R} + \frac{Q^2}{R^2},$$

where $m_e(r)$ is the *effective* mass function defined as,

$$(28) \quad m_e(R) = \int \left(4\pi \rho_e R^2 + \frac{Q}{R} \frac{dQ}{dR} \right) dR,$$

i.e.,

$$(29) \quad \frac{dm_e(R)}{dR} = 4\pi \rho_e R^2 + \frac{Q}{R} \frac{dQ}{dR}.$$

Now from equation (26),

$$(30) \quad \frac{g'}{g} = -\frac{p'_{r_e}}{(\rho_e + p_{r_e})} - \frac{2R'(p_{r_e} - p_{\perp_e})}{R^2(\rho_e + p_{r_e})} + \frac{1}{8\pi R^4(\rho_e + p_{r_e})} \frac{dQ^2}{dr}.$$

Using equations (27) and (30) in (23), one can get

$$(31) \quad \begin{aligned} & \left(1 - \frac{2m_e}{R} + \frac{Q^2}{R^2} \right) \cdot \\ & \left(-\frac{2p'_{r_e}}{RR'(\rho_e + p_{r_e})} - \frac{4(p_{r_e} - p_{\perp_e})}{R^2(\rho_e + p_{r_e})} + \frac{1}{4\pi R^5 R'(\rho_e + p_{r_e})} \frac{dQ^2}{dr} + \frac{1}{R^2} \right) \\ & = 8\pi p_{r_e} + \frac{1}{R^2} - \frac{Q^2}{R^4}. \end{aligned}$$

Define a *generating function* $G(R)$ as,

$$(32) \quad G(R) = \frac{-R \left(1 - \frac{2m_e}{R} + \frac{Q^2}{R^2} \right)}{8\pi p_{r_e} + \frac{1}{R^2} - \frac{Q^2}{R^4}},$$

and introduce the *anisotropic function* $w(R)$ as,

$$(33) \quad w(R) = \frac{4(p_{r_e} - p_{\perp_e})}{(\rho_e + p_{r_e})} G(R).$$

Using equations (32) and (33) in equation (31), we get

$$(34) \quad 8\pi(\rho_e + p_{r_e}) = \frac{16\pi GR \frac{dp_{r_e}}{dR} - \frac{4GQ}{R^3} \frac{dQ}{dR}}{R^3 + G - w}.$$

Differentiation of equation (27) gives,

$$(35) \quad 2R'' = -\frac{2m'_e}{RR'} + \frac{2m_e}{R^2} + \frac{2QQ'}{R'R^2} - \frac{2Q^2}{R^3}.$$

Adding $8\pi p_{r_e}$ on both sides of equation (25) and then using equations (27), (32), (34) and (35), we get

$$(36) \quad -\frac{2m'_e}{R^2 R'} + \frac{2m_e}{R^3} + \frac{2QQ'}{R'R^3} - \frac{2Q^2}{R^4} - \left(8\pi p_{r_e} + \frac{1}{R^2} - \frac{Q^2}{R^4}\right) \left(1 + \frac{G}{R^3}\right) = -\left(\frac{16\pi GR \frac{dp_{r_e}}{dR} - \frac{4GQ}{R^3} \frac{dQ}{dR}}{R^3 + G - w}\right).$$

Differentiation of equation (32) gives,

$$(37) \quad -\frac{2m'_e}{R^2 R'} + \frac{2m_e}{R^3} + \frac{2QQ'}{R'R^3} - \frac{2Q^2}{R^4} = \left(\frac{G}{R^3} - \frac{G'}{R^2 R'}\right) \left(8\pi p_{r_e} + \frac{1}{R^2} - \frac{Q^2}{R^4}\right) - \frac{8\pi p'_{r_e} G}{R^2 R'} + \frac{2QQ'G}{R^6 R'} - \frac{4Q^2 G}{R^7} + \frac{2G}{R^5}.$$

Using equations (37) into equation (36), we get

$$(38) \quad \frac{dp_{r_e}}{dR} + \frac{(R^2 + \frac{dG}{dR})(R^3 + G - w)}{G(-R^3 + G - w)} p_{r_e} = -\frac{(R^2 + \frac{dG}{dR})(1 - \frac{Q^2}{R^2})(R^3 + G - w)}{8\pi GR^2(-R^3 + G - w)} + \frac{(1 - \frac{2Q^2}{R^2})(R^3 + G - w)}{4\pi R^3(-R^3 + G - w)} + \frac{Q}{4\pi R^4} \frac{dQ}{dR},$$

which is a linear equation in p_{r_e} . We obtain its solution as,

$$(39) \quad p_{r_e} = e^{-\int B(R)dR} [p_0 + \int C(R)e^{\int B(R)dR} dR],$$

where p_0 is a constant of integration,

$$(40) \quad \left. \begin{aligned} B(R) &= \frac{(R^2 + \frac{dG}{dR})(R^3 + G - w)}{G(-R^3 + G - w)}, \\ C(R) &= -\frac{(R^2 + \frac{dG}{dR})(1 - \frac{Q^2}{R^2})(R^3 + G - w)}{8\pi GR^2(-R^3 + G - w)} + \\ &\quad \frac{(1 - \frac{2Q^2}{R^2})(R^3 + G - w)}{4\pi R^3(-R^3 + G - w)} + \frac{Q}{4\pi R^4} \frac{dQ}{dR}. \end{aligned} \right\}$$

From equation (28),

$$(41) \quad \frac{dm_e(R)}{dR} = 4\pi\rho_e R^2 + \frac{Q}{R} \frac{dQ}{dR}.$$

Using equation (41) in (37) and then using equation (32), we get

$$(42) \quad 8\pi\rho_e = -\frac{1}{R^2} \left[-1 + \frac{Q^2}{R^2} - \left(8\pi p_{r_e} + \frac{1}{R^2} - \frac{Q^2}{R^4} \right) \frac{dG}{dR} - \left(-\frac{2}{R^3} + 8\pi \frac{dp_{r_e}}{dR} - \frac{2Q}{R^4} \frac{dQ}{dR} + \frac{4Q^2}{R^5} \right) G \right],$$

which is the expression for effective density ρ_e . Equation (33) gives,

$$(43) \quad p_{r_e} - p_{\perp_e} = \frac{w(R)}{4G(R)}(\rho_e + p_{r_e}).$$

Equations (27) and (32) yields,

$$(44) \quad -\frac{RR'^2}{G} = 8\pi p_{r_e} + \frac{1}{R^2} - \frac{Q^2}{R^4}.$$

Using this above equation in equation (23) and then multiplying it on both sides by $\frac{R}{R'^2}$, we obtain

$$(45) \quad \frac{1}{R} + \frac{2}{g} \frac{dg}{dR} = -\frac{R^2}{G},$$

which after integration gives,

$$(46) \quad g^2 = \frac{g_0^2}{R} \exp \left[-\int \frac{R^2}{G} dR \right],$$

where g_0 is the constant of integration.

We can recover the relation between the metric coefficient R and the original variable r to complete the integration. From equation (27), we have

$$(47) \quad r = \int \frac{dR}{\left[1 - \frac{2m_e(R)}{R} + \frac{Q^2}{R^2} \right]^{\frac{1}{2}}}.$$

3. CONCLUSION

In the above procedure, we have presented an exact solution for static spherically symmetric anisotropic distribution of charged matter in context of Bimetric General Relativity by introducing the *generating function* and *anisotropic function* as defined in (32) and (33). From equation (28), we note that the electromagnetic anisotropy also contributes to the mass in addition to material density. It can also be noted that for $Q = 0$ and isotropic pressure, the solution obtained here match with the solution of Khadekar and Kondawar [6] for a neutral matter; thus the results obtained here are more general. Moreover for a physical system which is small compare to the size of the universe, the term $\frac{3}{2a^2g^2}$ in the field equations (18)-(20) is negligible. So the present result reduces to the Einstein's general relativity and matches with the one obtained by Singh et al. [10] for isotropic charged fluid for the General Relativity.

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