

OPTIMIZATION APPROACH FOR EXTREMITIES OF SUBMANIFOLDS IN LOCALLY CONFORMAL ALMOST COSYMPLECTIC MANIFOLD ADMITTING SSMC

POOJA BANSAL AND MOHAMMAD HASAN SHAHID

DEPARTMENT OF MATHEMATICS, FACULTY OF NATURAL SCIENCES, JAMIA

MILLIA ISLAMIA, NEW DELHI-110025, INDIA.

E-MAILS:POOJABANSAL811@GMAIL.COM, HASAN_JMI@YAHOO.COM

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ABSTRACT. The article is concerned with the extremities for the generalized normalized δ -Casorati curvatures of submanifolds in locally conformal almost cosymplectic manifold endowed with semi-symmetric metric connection using T. Oprea's technique. Applications of these inequalities give rise to several inequalities for bi-slant, hemi-slant, semi-slant, slant, invariant, anti-invariant, CR-submanifolds. Moreover, we characterize submanifolds on which equalities hold. Also, we establish an inequality in terms of the warping function, the scalar curvature and the Casorati curvature for warped product submanifold of locally conformal almost cosymplectic manifold with semi symmetric metric connection together with some of its applications.

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1. INTRODUCTION

Friedmann and Schouten in 1924 proposed the concept of one linear connection on smooth manifold known as semi-symmetric linear connection. Further, Hayden in 1930 proposed the concept of metric connection with torsion on Riemannian manifold [11]. Subsequently, the curvature and derivational conditions for semi-symmetric metric connection in Riemannian manifolds was deliberated by Yano [18].

However, the Chen invariant's conjecture involving the association of extrinsic and intrinsic invariants of a submanifold was first proposed by Chen and that tool is referred as Chen's invariants [6]. In 1993, he was the first who introduced this notion of Chen's invariants (or δ -invariants) [7] and obtained some optimal inequalities consisting of these invariants for any Riemannian submanifolds ([3], [7], [8]).

Moreover, the extended version of the notion of the dominant curvatures of a hypersurface of a Riemannian manifold known as Casorati curvature introduced by F. Casorati [5]. Some extremities containing the congruence essence and influence of the Casorati curvature have been examined by some well-known authors for different submanifolds of different ambient manifolds ([1], [2], [4], [5], [9], [10], [12], [13], [17], [20]). So, it is both important and very interesting to obtain some extremities concerning such kind of algebraic curvatures of submanifolds in any Riemannian manifolds.

The paper is organised as: we first study locally conformal almost cosymplectic manifold admitting semi-symmetric metric connection. Then, we obtain extremities for the generalized normalized δ -Casorati curvatures of submanifolds in same ambient manifold admitting semi-symmetric metric connection. Eventually, applying on extremities, we obtain extremal inequalities for different type of submanifolds (like bi-slant, hemi-slant, semi-slant, slant, invariant, anti-invariant, CR) and lastly we develop an extremity in terms of the warping function and the Casorati curvature for warped product submanifold together with its applications.

As long as, by virtue of simpleness, throughout a paper we denote locally conformal almost cosymplectic manifold, semi-symmetric metric connection, Levi-Civita connection and warped product by l.c.a.c. manifold, SSMC, LC connection and WP respectively.

2. L.C.A.C. MANIFOLD ADMITTING SSMC

Consider an almost contact manifold $\mathcal{N}^{2m+1}$ with an almost contact structure (ϕ, ξ, η) , where $\phi \in \Gamma(T^{(1,1)}\mathcal{N})$, ξ is a structure vector field and η is 1-form which holds

$$\phi^2 U = -U + \eta(U)\xi \text{ and } \eta(\xi) = 1.$$

Then, $\phi\xi = 0$ and $\eta \circ \phi = 0$ for $U \in T_p\mathcal{N}$. For $t \in \mathbb{R}$ and $\lambda \in C^\infty(\mathcal{N} \times \mathbb{R})$, if the induced almost complex structure J on $\mathcal{N} \times \mathbb{R}$ given by $J(U, \lambda \frac{d}{dt}) = (\phi U - \lambda\xi, \eta(U)\frac{d}{dt})$ is integrable, then the almost contact structure is said to be *normal*.

Let us assume that g is a reconcilable Riemannian metric with (ϕ, ξ, η) , i.e., $g(\phi U, \phi V) + \eta(U)\eta(V) = g(U, V)$ or, $\mathfrak{U}(U, V) = g(U, \phi V) = -g(\phi U, V)$ together with $\eta(U) = g(U, \xi)$, for all $U, V \in T_p\mathcal{N}$, where $\mathfrak{U}$ is the fundamental 2-form on $\mathcal{N}$. Then $\mathcal{N}$ is said to be an *almost contact metric manifold* endowed with an almost contact metric structure (ϕ, ξ, η, g) . Also, the almost contact structure is a contact structure with $\mathfrak{U} = d\eta$. Advancelly, if $d\mathfrak{U} = 0$ and $d\eta = 0$, then $\mathcal{N}$ is called *almost cosymplectic* manifold. A normal almost cosymplectic manifold is called *cosymplectic* manifold and necessary and sufficient condition for a manifold to be cosymplectic is $\check{\nabla}_U \phi = 0$ and $\check{\nabla}_U \eta = 0$ which shows that $\check{\nabla}_U \xi = 0$ for LC connection $\check{\nabla}$ of $\mathcal{N}$. For the theory of almost contact metric structure's conformal change (see [14]).

Definition 2.1. [14] A differentiable manifold $\mathcal{N}$ is known as *l.c.a.c. manifold* if $\exists$ 1-form Ω fascinating $d\mathfrak{U} = 2\Omega \wedge \mathfrak{U}$, $d\eta = \eta \wedge \Omega$ and $d\Omega = 0$.

Definition 2.2. [14] A structure is called *normal l.c.a.c.* if

$$(2.1) \quad (\check{\nabla}_U \phi)V = v(g(U, \phi V)\xi - \eta(V)\phi U),$$

where $\Omega = v\eta$ for $v \in C^\infty(\mathcal{N})$.

Then, (2.1) gives $\check{\nabla}_U \xi = f(U - \eta(U)\xi)$.

Definition 2.3. [16] A connection $\check{\nabla}$ on $\mathcal{N}$ is *semi-symmetric linear connection* if its torsion tensor $\mathcal{T}$ holds

$$(2.2) \quad \mathcal{T}(U, V) = \check{\nabla}_U V - \check{\nabla}_V U - [U, V] = \eta(V)U - \eta(U)V.$$

Additionally, a semi-symmetric linear connection $\check{\nabla}$ is known as *SSMC* if it holds

$$(2.3) \quad \check{\nabla} g = 0.$$

Definition 2.4. [16] A SSMC $\overset{\circ}{\nabla}$ in terms of the LC connection $\overset{\circ}{\nabla}$ on $\mathcal{N}$ has the expression

$$(2.4) \quad \overset{\circ}{\nabla}_U V = \overset{\circ}{\nabla}_U V + \eta(V)U - g(U, V)\xi, \text{ for } U, V \in T_p\mathcal{N}.$$

Let us take a Riemannian manifold $\mathcal{N}^{2m+1}$ admitting SSMC $\overset{\circ}{\nabla}$ and the LC connection $\overset{\circ}{\nabla}$. Let $\mathcal{M}^n$ be a submanifold of $\mathcal{N}^{2m+1}$ with an induced SSMC ∇ and the induced LC connection $\bar{\nabla}$. Let $\check{R}$ be the curvature tensor of $\overset{\circ}{\nabla}$ and $\check{\bar{R}}$ the curvature tensor of $\overset{\circ}{\nabla}$. We put R and $\bar{R}$ the curvature tensors of ∇ and $\bar{\nabla}$ on $\mathcal{M}$ respectively. The Gauss formulas for $\mathcal{M}$ are given as

$$\overset{\circ}{\nabla}_U V = \nabla_U V + h(U, V) \text{ and } \overset{\circ}{\nabla}_U V = \bar{\nabla}_U V + \bar{h}(U, V),$$

for $U, V \in T_p\mathcal{M}$, where $\bar{h}$ is the second fundamental form of $\mathcal{M}$ and h is a (0,2) symmetric tensor on $\mathcal{M}$.

A l.c.a.c. manifold $\mathcal{N}$ with LC connection $\overset{\circ}{\nabla}$ is of positive constant ϕ -sectional curvature s if and only if its Riemannian curvature tensor $\check{\bar{R}}$ has the expression [14]

$$(2.5) \quad \begin{aligned} g(\check{\bar{R}}(U, V)W, W') &= \left(\frac{s-3v^2}{4}\right)[g(U, W')g(V, W) - g(U, W)g(V, W')] \\ &\quad + \left(\frac{s+v^2}{4}\right)[g(U, \phi W')g(V, \phi W) - g(U, \phi W)g(V, \phi W')] \\ &\quad - 2g(U, \phi V)g(W, \phi W')] - \left(\frac{s+v^2}{4} + v'\right)[\eta(V)\eta(W)g(U, W') \\ &\quad - \eta(V)\eta(W')g(U, W) + \eta(U)\eta(W')g(V, W) - \eta(U)\eta(W)g(V, W')] \end{aligned}$$

where $v \in C^\infty(\mathcal{N})$, $\Omega = v\eta$ and $v' = \xi v$.

Now, the Gauss equation for $\mathcal{M}$ follows

$$\begin{aligned} g(\check{\bar{R}}(U, V)W, W') &= g(\bar{R}(U, V)W, W') + g(\bar{h}(U, W), \bar{h}(V, W')) \\ &\quad - g(\bar{h}(U, W'), \bar{h}(V, W)). \end{aligned}$$

The curvature tensor $\check{R}$ of SSMC $\overset{\circ}{\nabla}$ is given by [16]

$$(2.6) \quad \begin{aligned} g(\check{R}(U, V)W, W') &= g(\check{\bar{R}}(U, V)W, W') - \Lambda(V, W)g(U, W') + \Lambda(U, W)g(V, W') \\ &\quad - g(V, W)\Lambda(U, W') + g(U, W)\Lambda(V, W'), \end{aligned}$$

for all $U, V, W, W' \in T_p\mathcal{M}$, where Λ denotes a (0,2) tensor field given by $\Lambda(U, V) = (\overset{\circ}{\nabla}_U \eta)V - \eta(U)\eta(V) + \frac{1}{2}\eta(P)g(U, V)$.

For a submanifold $\mathcal{M}^n$ of a l.c.a.c manifold N^{2m+1} , denote $\phi U = PU + FU$, with $PU = (\phi U)^T$ and $FU = (\phi U)^\perp$ respectively, for any $U \in T_p\mathcal{M}$. Consider an orthonormal basis $\{e_i\}_1^n$ of $T_p\mathcal{M}$, then the squared norm of P is defined as

$$(2.7) \quad ||P||^2 = \sum_{i,j=1}^n g^2(Pe_i, e_j)$$

Definition 2.5. A submanifold $\mathcal{M}$ with $\xi \in T_p\mathcal{M}$ is called *slant* if for any $p \in \mathcal{M}$ and any $X \in T_p\mathcal{M}$ linearly independent of ξ , the angle between ϕU and $T_p\mathcal{M}$ is invariant $\theta \in [0, \frac{\pi}{2}]$, known as the *slant angle* of $\mathcal{M}$ in $\mathcal{N}$.

Definition 2.6. A submanifold $\mathcal{M}$ of $\mathcal{N}$ with $\xi \in T_p\mathcal{M}$ is called *bi-slant* submanifold, if $\exists$ pair of orthogonal differentiable distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\mathcal{M}$, which holds

- (i) $T\mathcal{M} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$;
- (ii) $\mathcal{D}_i$ is slant distribution with slant angle θ_i for $i = 1, 2$.

Remark 2.7. If we take the $\dim \mathcal{D}_i = 2d_i$ for $i = 1, 2$, then we see that slant submanifold is a particular case of bi-slant submanifold by taking $d_1 = 0$ or $d_2 = 0$. However, we have particular cases of slant version as slant submanifold is *invariant* if $\theta = 0$ and *anti-invariant* if $\theta = \frac{\pi}{2}$, respectively. If we have bi-slant submanifold, then we can have following relation

$$||P||^2 = \sum_{i,j=1}^n g^2(Pe_i, e_j) = 2 \sum_{i=1}^2 (d_i \cos^2 \theta_i), \quad \dim \mathcal{D}_i = 2d_i \text{ for } i = 1, 2.$$

All the above discussion on classifications of submanifold $\mathcal{M}$ can be seen here:

$$(2.8) \quad \left\{ \begin{array}{ll} bi - slant, & \text{if } \mathcal{D}_1 \text{ is slant with } \angle \theta_1, \mathcal{D}_2 \text{ is slant with } \angle \theta_2; \\ hemi - slant, & \text{if } \mathcal{D}_1 \text{ is slant with } \angle \theta_1, \mathcal{D}_2 \text{ is anti-invariant with } \theta_2 = \frac{\pi}{2}; \\ semi - slant, & \text{if } \mathcal{D}_1 \text{ is invariant with } \theta_1 = 0, \mathcal{D}_2 \text{ is slant with } \angle \theta_2; \\ slant, & \text{if } \mathcal{D}_1 = 0 \text{ or } \mathcal{D}_2 = 0 \text{ with } \theta_1 = \theta_2 = \theta \text{ or } \theta_1 = \theta_2 \neq \theta; \\ invariant, & \text{if } \mathcal{D}_1 \text{ is slant with } \angle \theta_1 = 0, \mathcal{D}_2 = 0; \\ anti - invariant, & \text{if } \mathcal{D}_1 = 0, \mathcal{D}_2 \text{ is slant with } \angle \theta_2 = \frac{\pi}{2}; \\ CR, & \text{if } \mathcal{D}_1 \text{ is slant with } \theta_1 = 0, \mathcal{D}_2 \text{ is slant with } \theta_2 = \frac{\pi}{2}. \end{array} \right.$$

3. CASORATI INEQUALITY FOR SUBMANIFOLD OF L.C.A.C. MANIFOLD ADMITTING SSMC

Here, we obtain inequalities for the Mean curvature, the scalar and the sectional curvatures associated with the induced SSMC for submanifold $\mathcal{M}$ of l.c.a.c. manifold $\mathcal{N}$.

For an orthonormal basis $\{e_i\}_1^n$ of $T_p\mathcal{M}$ and $\{e_\alpha\}_{n+1}^{2m+1}$ of $T_p^\perp\mathcal{M}$. The scalar curvature τ of $\mathcal{M}$ can be expressed as

$$(3.1) \quad \tau = \sum_{1 \leq i < j \leq n} \mathcal{K}(e_i \wedge e_j),$$

where $\mathcal{K}(\pi)$ is the sectional curvature of $\mathcal{M}$ associated with 2-plane section $\pi \subset T_p\mathcal{M}$ generated by $\{e_i, e_j\}$ given as $\mathcal{K}(e_i \wedge e_j) = g(R(e_i, e_j)e_j, e_i)$. The normalized scalar curvature ρ of $\mathcal{M}$ is given by

$$\rho = \frac{2\tau}{n(n-1)}.$$

We put $\mathcal{H}$ the Mean curvature vector field of $\mathcal{M}$ is defined as

$$\mathcal{H} = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i).$$

Conveniently, let $h_{ij}^\alpha = g(h(e_i, e_j), e_\alpha)$ for $i, j \in \{1, \dots, n\}$ and $\alpha \in \{n+1, \dots, 2m+1\}$. Then, we have the squared mean curvature $\|\mathcal{H}\|^2$ of $\mathcal{M}$ in $\mathcal{N}$ and the squared norm $\|h\|^2$ of h (second fundamental form) as follows

$$\|\mathcal{H}\|^2 = \frac{1}{n^2} \left(\sum_{\alpha=n+1}^{2m+1} \sum_{i,j=1}^n h_{ij}^\alpha \right)^2 \text{ and } \|h\|^2 = \sum_{\alpha=n+1}^{2m+1} \sum_{i,j=1}^n (h_{ij}^\alpha)^2$$

respectively, where $h_{ij}^\alpha = g(h(e_i, e_j), e_\alpha)$ for $\alpha \in \{n+1, \dots, 2m+1\}$.

Definition 3.1. [5] The *Casorati curvature* $\mathcal{C}$ of $\mathcal{M}$ in $\mathcal{N}$ is stated by

$$\mathcal{C} = \frac{\|h\|^2}{n} = \frac{1}{n} \sum_{\alpha=n+1}^{2m+1} \sum_{i,j=1}^n (h_{ij}^\alpha)^2.$$

Definition 3.2. [14] A submanifold $\mathcal{M}^n$ of $\mathcal{N}^{2m+1}$ is known as *invariantly quasi-umbilical* if $\exists$ an orthonormal basis $\{e_\alpha\}_{n+1}^{2m+1}$ of $T_p^\perp\mathcal{M}$ such that the caharacterstic value of the shape operators S_{e_α} is of multiplicity $n-1$ and the distinguished eigendirection of S_{e_α} is the same for $\alpha \in \{n+1, \dots, 2m+1\}$.

Now for $k \geq 2$, let us suppose that $\mathcal{L}$ be a k -dimensional subspace of $T_p\mathcal{M}$ such that $\{e_i\}_1^k$ is an orthonormal basis of $\mathcal{L}$. Then, $\mathcal{C}(\mathcal{L})$ is given by

$$\mathcal{C}(\mathcal{L}) = \frac{1}{k} \sum_{\alpha=n+1}^{2m+1} \sum_{i,j=1}^k (h_{ij}^\alpha)^2.$$

The normalized δ -Casorati curvatures $\delta_s(n-1)$ and $\delta_s^*(n-1)$ of $\mathcal{M}$ in $\mathcal{N}$ can be stated as [9]

$$(3.2) \quad [\delta_s(n-1)](p) = \frac{1}{2}\mathcal{C}(p) + \left(\frac{n+1}{2n}\right) \inf\{\mathcal{C}(\mathcal{L}) | \mathcal{L} \text{ is the hyperplane of } T_p\mathcal{M}\}$$

and

$$(3.3) \quad [\check{\delta}_s(n-1)](p) = 2\mathcal{C}(p) - \left(\frac{2n-1}{2n}\right) \sup\{\mathcal{C}(\mathcal{L}) | \mathcal{L} \text{ is the hyperplane of } T_p\mathcal{M}\}.$$

Then, the generalized normalized δ -Casorati curvatures $\delta_s(r; n-1)$ and $\delta_s^*(r; n-1)$ of $\mathcal{M}$ in $\mathcal{N}$ such that $A(r, s) = \frac{(n-1)(n+r)(n^2-n-r)}{rn}$ for $0 < r < n(n-1)$ and $r > n(n-1)$ respectively are formalised by [9]:

$$(3.4) \quad [\delta_s(r; n-1)](p) = r\mathcal{C}(p) + A(r, s) \inf\{\mathcal{C}(\mathcal{L}) | \mathcal{L} \text{ is the hyperplane of } T_p\mathcal{M}\},$$

$$(3.5) \quad [\delta_s^*(r; n-1)](p) = r\mathcal{C}(p) + A(r, s) \sup\{\mathcal{C}(\mathcal{L}) | \mathcal{L} \text{ is the hyperplane of } T_p\mathcal{M}\}.$$

From these two relations, one can note that the generalized normalized δ -Casorati curvatures $\delta_s(r; n-1)$ and $\delta_s^*(r; n-1)$ are the generalized versions of the normalized δ -Casorati curvatures $\delta_s(n-1)$ and $\delta_s^*(n-1)$ respectively, by substituting r by $\frac{n(n-1)}{2}$ as

$$(3.6) \quad \left[\delta_s\left(\frac{n(n-1)}{2}; n-1\right)\right](p) = n(n-1)[\delta_s(n-1)](p),$$

$$(3.7) \quad \left[\delta_s^*\left(\frac{n(n-1)}{2}; n-1\right)\right](p) = n(n-1)[\delta_s^*(n-1)](p)$$

for $p \in \mathcal{M}$.

Now, we recall that if $\mathcal{M}$ is a submanifold of Riemannian manifold $\mathcal{N}$ with a real-valued smooth function $\mathcal{F}$ on $\mathcal{M}$ and if we consider the constrained extremum problem

$$(3.8) \quad \min \mathcal{F}(x), \text{ for } x \in \mathcal{M}.$$

Then, we can have the following result

Lemma 3.3. [15] *If $x_0 \in \mathcal{M}$ is the solution of the problem (3.8), then*

- (i) $(\text{grad}\mathcal{F})(x_0) \in T_{x_0}^\perp \mathcal{M}$;
- (ii) *the bilinear form $\mathcal{A} : T_{x_0} \mathcal{M} \times T_{x_0} \mathcal{M} \rightarrow \mathcal{R}$ defined as*

$$\mathcal{A}(U, V) = \text{Hess}\mathcal{F}(U, V) + g(h(U, V), (\text{grad}\mathcal{F})(x_0))$$

is positive semi-definite, where $\text{Hess}\mathcal{F}$ denotes Hessian of $\mathcal{F}$ and $\text{grad}\mathcal{F}$, the gradient of $\mathcal{F}$.

Now, we are going to present our main result

Theorem 3.4. *Let $\mathcal{M}^n$ be a submanifold of l.c.a.c. manifold $\mathcal{N}^{2m+1}$ admitting SSMC such that $\xi \in T_p \mathcal{M}$. Then*

$$(3.9) \quad (i) \rho \leq \frac{\delta_s(r; n-1)}{n(n-1)} - \frac{2\text{tr}\Lambda}{n} + \left(\frac{s+v^2}{4}\right) \left(\frac{3\|P\|^2}{n(n-1)} + \frac{n-2}{n}\right) - v^2 - \frac{2}{n}v',$$

$$(3.10) \quad (ii) \rho \leq \frac{\delta_s^*(r; n-1)}{n(n-1)} - \frac{2\text{tr}\Lambda}{n} + \left(\frac{s+v^2}{4}\right) \left(\frac{3\|P\|^2}{n(n-1)} + \frac{n-2}{n}\right) - v^2 - \frac{2}{n}v'.$$

Moreover, both relations (3.9) and (3.10) become equalities if and only if the submanifold $\mathcal{M}^n$ is invariantly quasi-umbilical with the flat normal connection $\nabla^\perp$ in $\mathcal{N}^{2m+1}$ such that with respect to an orthonormal basis $\{e_i\}_1^n$ of $T_p \mathcal{M}$ and $\{e_\alpha\}_{n+1}^{2m+1}$ of $T_p^\perp \mathcal{M}$, the shape operator $S_\alpha = S_{e_\alpha}$ for $\alpha \in \{n+1, \dots, 2m+1\}$ takes the matrix form

$$(3.11) \quad S_{n+1} = \begin{pmatrix} M & 0 \\ 0 & \frac{n(n-1)}{r}a \end{pmatrix}, S_{n+2} = \dots = S_{2m+1} = 0,$$

where M is the diagonal matrix of order $n-1$ with diagonal entry ' a '.

Proof. By taking $U = W' = e_i$ and $V = W = e_j, i \neq j$ in (2.6) and using (2.5), the scalar curvature τ with respect to SSMC has the following expression

$$(3.12) \quad \begin{aligned} 2\tau &= (n^2 - n) \left(\frac{s-3v^2}{4} \right) + 3 \left(\frac{s+v^2}{4} \right) \|P\|^2 - 2(n-1) \left(\frac{s+v^2}{4} + v' \right) \\ &\quad - 2(n-1)(\text{tr}\Lambda) + n^2 \|H\|^2 - n\mathcal{C} \end{aligned}$$

Now, we consider the quadratic polynomial $\mathcal{P}$

$$\begin{aligned} \mathcal{P} &= r\mathcal{C} + A(r, s)\mathcal{C}(\mathcal{L}) - 2\tau(p) - 2(n-1)(\text{tr}\Lambda) + (n^2 - n) \left(\frac{s-3v^2}{4} \right) \\ &\quad + 3 \left(\frac{s+v^2}{4} \right) \|P\|^2 - 2(n-1) \left(\frac{s+v^2}{4} + v' \right), \end{aligned}$$

where $\mathcal{L}$ is a hyperplane of $T_p\mathcal{M}$.

Let us assume that $\mathcal{L}$ is spanned by $\{e_i\}_1^{n-1}$. Then, it gives

$$\begin{aligned} \mathcal{P} = & \sum_{\alpha=n+1}^{2m+1} \left[\frac{r}{n} \sum_{i,j=1}^n (h_{ij}^\alpha)^2 + A(r, s) \sum_{i,j=1}^{n-1} (h_{ij}^\alpha)^2 \right] - 2\tau(p) - 2(n-1)(\text{tr}\Lambda) \\ & + (n^2 - n) \left(\frac{s - 3v^2}{4} \right) + 3 \left(\frac{s + v^2}{4} \right) \|P\|^2 - 2 \left(\frac{s + v^2}{4} + v' \right) (n-1) \end{aligned}$$

Using (3.12) in above relation together with the value of $A(r, s)$, one can obtain

$$\begin{aligned} \mathcal{P} = & \sum_{\alpha=n+1}^{2m+1} \sum_{i=1}^{n-1} \left[(h_{ii}^\alpha)^2 \left(\frac{n^2 + n(r-1) - 2r}{r} \right) + 2 \left(\frac{n+r}{n} \right) (h_{in}^\alpha)^2 \right] \\ & + \sum_{\alpha=n+1}^{2m+1} \left[\frac{(n-1)(n+r)}{n} \sum_{1 \leq i \neq j \leq n-1} (h_{ij}^\alpha)^2 - 2 \sum_{1 \leq i \neq j \leq n} h_{ii}^\alpha h_{jj}^\alpha + \frac{r}{n} (h_{nn}^\alpha)^2 \right] \\ \geq & \sum_{\alpha=n+1}^{2m+1} \left[\sum_{i=1}^{n-1} \left(\frac{n^2 + n(r-1) - 2r}{r} \right) (h_{ii}^\alpha)^2 - 2 \sum_{1 \leq i \neq j \leq n} (h_{ii}^\alpha h_{jj}^\alpha) + \frac{r}{n} (h_{nn}^\alpha)^2 \right] \end{aligned}$$

Now, we contemplate the optimization dilemma

$$\begin{aligned} \min \quad & \mathcal{F}_\alpha \\ \text{subjected to} \quad & \mathcal{F}_\alpha : h_{11}^\alpha + h_{22}^\alpha + \dots + h_{nn}^\alpha = \mathcal{B}_\alpha \end{aligned}$$

where $\mathcal{B}_\alpha$ is the trace of matrix h_{ij}^α and $\mathcal{F}_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ is quadratic form defined by

$$\begin{aligned} \mathcal{F}_\alpha(h_{11}^\alpha, h_{22}^\alpha, \dots, h_{nn}^\alpha) = & \sum_{\alpha=n+1}^{2m+1} \left[\sum_{i=1}^{n-1} \left(\frac{n^2 + n(r-1) - 2r}{r} \right) (h_{ii}^\alpha)^2 - 2 \sum_{1 \leq i \neq j \leq n} (h_{ii}^\alpha h_{jj}^\alpha) \right. \\ & \left. + \frac{r}{n} (h_{nn}^\alpha)^2 \right] \end{aligned}$$

Now, the partial derivative of $\mathcal{P}$ are given by

$$(3.13) \quad \begin{cases} \frac{\partial \mathcal{P}}{\partial h_{ii}^\alpha} = \frac{2(n+r)(n-1)}{r} h_{ii}^\alpha - 2 \sum_{k=1}^n h_{kk}^\alpha, \\ \frac{\partial \mathcal{P}}{\partial h_{nn}^\alpha} = \frac{2r}{n} h_{nn}^\alpha - 2 \sum_{k=1}^{n-1} h_{kk}^\alpha, \end{cases}$$

for $i \in \{1, 2, \dots, n-1\}$ and $\alpha \in \{n+1, \dots, 2m+1\}$.

Now, we can see that the critical points of $\mathcal{P}$ are the optimal solutions of corresponding homogeneous system of (3.13) and are given by

$$(3.14) \quad \begin{aligned} h_{ii}^\alpha &= \frac{r\lambda}{n(n-1)}; \quad 1 \leq i \leq n-1 \\ h_{nn}^\alpha &= \lambda. \end{aligned}$$

Since, we have $\sum_{i=1}^n h_{ii}^\alpha = \mathcal{B}_\alpha$, together with (3.14) implies that $\frac{(r+n)\lambda}{n} = \mathcal{B}_\alpha$ or $\lambda = \frac{n\mathcal{B}_\alpha}{r+n}$. Hence, finally we have

$$(3.15) \quad h_{ii}^\alpha = \frac{r\mathcal{B}_\alpha}{(r+n)(n-1)}, h_{nn}^\alpha = \frac{n\mathcal{B}_\alpha}{n+r}, \quad 1 \leq i \leq n-1, n+1 \leq \alpha \leq 2m+1.$$

Now, we consider $\mathcal{A}(U, V) = \text{Hess}(\mathcal{F}_\alpha)(U, V) + \langle h(U, V), (\text{grad}\mathcal{F}_\alpha)(x_0) \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product defined on $\mathcal{R}^n$ and $\text{Hess}(\mathcal{F}_\alpha) = [B] = (b_{ij})$ is the matrix of order n , where

$$\begin{aligned} b_{ii} &= \frac{2(n+r)(n-1)}{r} - 2 \text{ for } i = 1, \dots, n-1 \\ b_{nn} &= \frac{2r}{n} \text{ and} \\ b_{ij} &= -2 \text{ for } 1 \leq i \neq j \leq n. \end{aligned}$$

Now, we take a vector $U = (U_1, \dots, U_n)$ tangent to $\mathcal{F}_\alpha$ at x on $\mathcal{F}_\alpha$ such that $\sum_{i=1}^n U_i = 0$ and also using the condition of $\mathcal{F}$ to be totally geodesic in $\mathcal{R}^n$, we derive

$$\mathcal{A}(U, U) = \frac{2(n^2 - n + nr - 2r)}{r} \sum_{i=1}^{n-1} U_i^2 + \frac{2r}{n} U_{m+1}^2 \geq 0.$$

Then, by lemma (3.3) and relation (3.14), we assert that the solution point $(h_{11}^\alpha, h_{22}^\alpha, \dots, h_{nn}^\alpha)$ is the global minimum point and $\mathcal{F}_\alpha(h_{11}^\alpha, h_{22}^\alpha, \dots, h_{nn}^\alpha) = 0$. Hence, $\mathcal{P} \geq 0$ which gives

$$(3.16) \quad \begin{aligned} \rho &\leq \frac{r}{n(n-1)} \mathcal{C} + \frac{(n+r)(n^2 - n - r)}{rn^2} \mathcal{C}(L) + \left(\frac{s+v^2}{4} \right) \left(1 + \frac{3\|P\|^2}{n(n-1)} - \frac{2}{n} \right) \\ &\quad - v^2 - \frac{2}{n} v' - \frac{2}{n} (\text{tr}\Lambda). \end{aligned}$$

Hence, the inequalities (3.9) and (3.10) follows from (3.16).

Furthermore, the equality sign holds if and only if

$$\begin{aligned} h_{ij}^\alpha &= 0 \text{ for } i \neq j \in \{1, 2, \dots, n\}, \\ h_{nn}^\alpha &= \frac{n(n-1)}{r} h_{11}^\alpha = \frac{n(n-1)}{r} h_{22}^\alpha = \dots = \frac{n(n-1)}{r} h_{n-1 \ n-1}^\alpha, \end{aligned}$$

for $\alpha = n+1, \dots, 2m+1$.

Hence, eventually we get equality cases of (3.9) and (3.10) if and only if the sub-manifold $\mathcal{M}^n$ in $\mathcal{N}^{2m+1}$ is invariantly quasi-umbilical with the normal connection in ambient space like the matrix of the shape operator is given by (3.11). $\square$

Now, Theorem (3.4) gives the following corollary

Corollary 3.5. *Let $\mathcal{M}^n$ be a submanifold of l.c.a.c. manifold $\mathcal{N}^{2m+1}$ with SSMC such that $\xi \in T_p\mathcal{M}$. Then*

(i) *the normalized δ -Casorati curvature $\delta_c(n-1)$ holds*

$$(3.17) \quad \rho \leq \delta_s(n-1) + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{3\|P\|^2}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda).$$

Moreover, relation (3.17) becomes an equality if and only if submanifold $\mathcal{M}^n$ is an invariantly quasi-umbilical with flat normal connection $\nabla^\perp$ in $\mathcal{N}^{2m+1}$ such that with respect to an orthonormal basis $\{e_i\}_1^n$ and $\{e_\alpha\}_{n+1}^{2m+1}$ of $T_p\mathcal{M}$ and $T_p^\perp\mathcal{M}$ respectively, the shape operator $S_\alpha = S_{e_\alpha}$ for $\alpha \in \{n+1, \dots, 2m+1\}$ takes the form

$$S_{n+1} = \begin{pmatrix} M & 0 \\ 0 & 2a \end{pmatrix}, S_{n+2} = \dots = S_{2m+1} = 0,$$

where M is the diagonal matrix of order $n-1$ with diagonal entry ' a '

(ii) *the normalized δ -Casorati curvature $\delta_c^*(n-1)$ holds*

$$(3.18) \quad \rho \leq \delta_s^*(n-1) + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{3\|P\|^2}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda).$$

Furthermore, the relation (3.18) becomes an equality if and only if submanifold $\mathcal{M}^n$ is an invariantly quasi-umbilical with flat normal connection $\nabla^\perp$ in $\mathcal{N}^{2m+1}$ such that with respect to an orthonormal basis $\{e_i\}_1^n$ and $\{e_\alpha\}_{n+1}^{2m+1}$ of $T_p\mathcal{M}$ and $T_p^\perp\mathcal{M}$ respectively, the shape operator $S_\alpha = S_{e_\alpha}$ for $\alpha \in \{n+1, \dots, 2m+1\}$ takes the matrix form

$$S_{n+1} = \begin{pmatrix} M & 0 \\ 0 & a \end{pmatrix}, S_{n+2} = \dots = S_{2m+1} = 0,$$

where M is the diagonal matrix of order $n-1$ with diagonal entry ' $2a$ '.

Proof. From (3.6) (resp. (3.7)) by taking $r = \frac{n(n-1)}{2}$ and using (3.2) (resp. (3.3)), we obtain our result for the normalized δ -Casorati curvature of submanifold $\mathcal{M}^n$ in l.c.a.c. manifold $\mathcal{N}^{2m+1}$ with SSMC. $\square$

4. WP SUBMANIFOLD OF L.C.A.C. MANIFOLD ADMITTING SSMC

In this section, we will study the WP submanifold of l.c.a.c. manifold admitting SSMC and then we develop inequalities consisting the warping function of a WP submanifold $\mathcal{M}$ of $\mathcal{N}$.

Consider two Riemannian manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ of dimensions n_1 and n_2 equipped with Riemannian metrics ς_1 and ς_2 respectively. Let ζ be a positive function on $\mathcal{M}_1$. The WP $\mathcal{M}_1 \otimes_{\zeta} \mathcal{M}_2$ is defined to be the product manifold $\mathcal{M}_1 \otimes \mathcal{M}_2$ with the warped metric $g = \varsigma_1 + \zeta^2 \varsigma_2$ [19].

Consider an isometric immersion $\Psi : \mathcal{M} = \mathcal{M}_1 \otimes_{\zeta} \mathcal{M}_2 \rightarrow \mathcal{N}(s)$ of a WP $\mathcal{M}_1 \otimes_{\zeta} \mathcal{M}_2$ into a Riemannian manifold $\mathcal{N}(c)$. Let h be the second fundamental form of Ψ and the Mean curvature vectors $H_i = \frac{1}{n_i} \text{tr} h_i$ where $\text{tr} h_i$ is the trace of h restricted to $\mathcal{M}_i$.

Theorem 4.1. *Let $\Psi : \mathcal{M}^n = \mathcal{M}_1 \otimes_{\zeta} \mathcal{M}_2 \rightarrow \mathcal{N}^{2m+1}(s)$ be an isometric immersion of a WP into l.c.a.c. manifold admitting SSMC with $\xi \in T_p \mathcal{M}_1$. Then*

$$(4.1) \quad n_2 \frac{\Delta \zeta}{\zeta} \leq \tau - \left(\frac{s - 3v^2}{8} \right) [n(n-1) - 2n_1 n_2] + \left(\frac{s + v^2}{4} + v' \right) (n_1 - 1) + n\mathcal{C}(\mathcal{L}) \\ - 3 \left(\frac{s + v^2}{4} \right) \left[\sum_{1 \leq i < j \leq n_1} g^2(Pe_i, e_j) + \sum_{n_1+1 \leq i < j \leq n} g^2(Pe_i, e_j) \right]$$

where $n_i = \dim \mathcal{M}_i$, for $i = 1, 2$ and Δ is the Laplacian operator of $\mathcal{M}_1$.

Proof. Let us consider an isometric immersion $\Psi : \mathcal{M} = \mathcal{M}_1 \otimes_{\zeta} \mathcal{M}_2 \rightarrow \mathcal{N}(s)$ of a WP $\mathcal{M}_1 \otimes_{\zeta} \mathcal{M}_2$ into l.c.a.c. manifold $\mathcal{N}(s)$ with SSMC whose structure vector field $\xi \in T_p \mathcal{M}_1$. Then, one can easily have [19]

$$\mathcal{K}(X \wedge Z) = \frac{1}{\zeta} \{ (\nabla_X X) \zeta - X^2 \zeta \}.$$

Now let us take an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_p \mathcal{M}$ such that $e_1, \dots, e_{n_1}$ are tangent to $\mathcal{M}_1$ and $e_{n_1+1}, \dots, e_n$ are tangent to $\mathcal{M}_2$. Then, with the virtue of above defined relation, we obtain

$$(4.2) \quad \frac{\Delta \zeta}{\zeta} = \sum_{1 \leq i \leq n_1} \sum_{n_1+1 \leq j \leq n} \mathcal{K}(e_i \wedge e_j).$$

Combining (3.1) and (4.2) yields

$$(4.3) \quad n_2 \frac{\Delta \zeta}{\zeta} = \tau - \sum_{1 \leq i \leq n_1} \mathcal{K}(e_i \wedge e_j) - \sum_{n_1+1 \leq j \leq n} \mathcal{K}(e_i \wedge e_j).$$

Using the Gauss equation and taking account of (4.3)

$$\begin{aligned}
 n_2 \frac{\Delta \zeta}{\zeta} = & \tau - \left(\frac{s-3v^2}{8} \right) (n(n-1) - 2n_1 n_2) - 3 \left(\frac{s+v^2}{4} \right) \left[\sum_{1 \leq i < j \leq n_1} g^2(Pe_i, e_j) \right. \\
 & + \sum_{n_1+1 \leq i < j \leq n} g^2(Pe_i, e_j) \Big] - \sum_{\alpha=n+1}^{2m+1} \left[\sum_{1 \leq i < j \leq n_1} h_{ii}^\alpha h_{jj}^\alpha - \sum_{1 \leq i < j \leq n_1} (h_{ij}^\alpha)^2 \right. \\
 (4.4) \quad & \left. - \sum_{n_1+1 \leq i < j \leq n} (h_{ij}^\alpha)^2 \right] + \left(\frac{s+v^2}{4} + v' \right) (n_1 - 1) - \left(\frac{n_2 - 1}{2} \right) \text{tr} \Lambda
 \end{aligned}$$

By applying the definition of Casorati curvature, (4.4) gives

$$\begin{aligned}
 n_2 \frac{\Delta \zeta}{\zeta} = & \tau - \left(\frac{s-3v^2}{8} \right) [n(n-1) - 2n_1 n_2] - 3 \left(\frac{s+v^2}{4} \right) \left[\sum_{1 \leq i < j \leq n_1} g^2(Pe_i, e_j) \right. \\
 & + \sum_{n_1+1 \leq i < j \leq n} g^2(Pe_i, e_j) \Big] - (n_1^2 \|\mathcal{H}_1\|^2 + n_2^2 \|\mathcal{H}_2\|^2) \\
 & + (n_1 \mathcal{C}(\mathcal{L}) + n_2 \mathcal{C}(\mathcal{L})) + \left(\frac{s+v^2}{4} + v' \right) (n_1 - 1) - \left(\frac{n_2 - 1}{2} \right) \text{tr} \Lambda \\
 \leq & \tau - \left(\frac{s-3v^2}{8} \right) [n^2 - n - 2n_1 n_2] - 3 \left(\frac{s+v^2}{4} \right) \left[\sum_{1 \leq i < j \leq n_1} g^2(Pe_i, e_j) \right. \\
 & + \sum_{n_1+1 \leq i < j \leq n} g^2(Pe_i, e_j) \Big] + n \mathcal{C}(\mathcal{L}) + \left(\frac{s+v^2}{4} + v' \right) (n_1 - 1) - \left(\frac{n_2 - 1}{2} \right) \text{tr} \Lambda
 \end{aligned}$$

from which we conclude our result. $\square$

5. APPLICATIONS: CASES FOR SLANT ANGLE ON THE DISTRIBUTION

This module is concerned with the applications of the conclusions which we have been obtained in the previous sections. We have,

Theorem 5.1. *Let $\mathcal{M}^n$ be a submanifold of l.c.a.c. manifold $\mathcal{N}^{2m+1}$ admitting SSMC with $\xi \in T_p \mathcal{M}$. Then, the inequalities for generalized normalized δ -Casorati curvatures $\delta_s(r; n-1)$ and $\delta_s^*(r; n-1)$ are given in the following table*

Submanifold $\mathcal{M}^n$	Inequalities
<i>bi-slant</i>	$\rho \leq \frac{\delta_s(r; n-1)}{n(n-1)} + \left(\frac{s+v^2}{4} \right) \left(1 + \frac{6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2)}{n(n-1)} - \frac{2}{n} \right) - v^2$ $- \frac{2}{n} v' - \frac{2}{n} (\text{tr} \Lambda)$
	$\rho \leq \frac{\delta_s^*(r; n-1)}{n(n-1)} + \left(\frac{s+v^2}{4} \right) \left(1 + \frac{6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2)}{n(n-1)} - \frac{2}{n} \right) - v^2$ $- \frac{2}{n} v' - \frac{2}{n} (\text{tr} \Lambda)$

<i>hemi-slant</i>	$\rho \leq \frac{\delta_s(r;n-1)}{n(n-1)} + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6d_1 \cos^2 \theta_1}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n} v'$ $- \frac{2}{n} (\text{tr} \Lambda)$ $\rho \leq \frac{\delta_s^*(r;n-1)}{n(n-1)} + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6d_1 \cos^2 \theta_1}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n} v'$ $- \frac{2}{n} (\text{tr} \Lambda)$
<i>semi-slant</i>	$\rho \leq \frac{\delta_s(r;n-1)}{n(n-1)} + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6(d_1+d_2 \cos^2 \theta_2)}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n} v'$ $- \frac{2}{n} (\text{tr} \Lambda)$ $\rho \leq \frac{\delta_s^*(r;n-1)}{n(n-1)} + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6(d_1+d_2 \cos^2 \theta_2)}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n} v'$ $- \frac{2}{n} (\text{tr} \Lambda)$
<i>slant</i>	$\rho \leq \frac{\delta_s(r;n-1)}{n(n-1)} + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{3 \cos^2 \theta}{n} - \frac{2}{n}\right) - v^2 - \frac{2}{n} v' - \frac{2}{n} (\text{tr} \Lambda)$ $\rho \leq \frac{\delta_s^*(r;n-1)}{n(n-1)} + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{3(n-1) \cos^2 \theta}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n} v'$ $- \frac{2}{n} (\text{tr} \Lambda)$
<i>anti-invariant</i>	$\rho \leq \frac{\delta_s(r;n-1)}{n(n-1)} + \left(\frac{s+v^2}{4}\right) \left(\frac{n-2}{n}\right) - v^2 - \frac{2}{n} v' - \frac{2}{n} (\text{tr} \Lambda)$ $\rho \leq \frac{\delta_s^*(r;n-1)}{n(n-1)} + \left(\frac{s+v^2}{4}\right) \left(\frac{n-2}{n}\right) - v^2 - \frac{2}{n} v' - \frac{2}{n} (\text{tr} \Lambda)$
<i>invariant</i>	$\rho \leq \frac{\delta_s(r;n-1)}{n(n-1)} + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{3}{n-1} - \frac{2}{n}\right) - v^2 - \frac{2}{n} v' - \frac{2}{n} (\text{tr} \Lambda)$ $\rho \leq \frac{\delta_s^*(r;n-1)}{n(n-1)} + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{3}{n-1} - \frac{2}{n}\right) - v^2 - \frac{2}{n} v' - \frac{2}{n} (\text{tr} \Lambda)$
<i>CR</i>	$\rho \leq \frac{\delta_s(r;n-1)}{n(n-1)} + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6d_1}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n} v' - \frac{2}{n} (\text{tr} \Lambda)$ $\rho \leq \frac{\delta_s^*(r;n-1)}{n(n-1)} + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6d_1}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n} v' - \frac{2}{n} (\text{tr} \Lambda)$

Moreover, all the above relations become equalities if and only if $\mathcal{M}^n$ is an invariantly quasi-umbilical with flat normal connection $\nabla^\perp$ in $\mathcal{N}^{2m+1}$ such that with respect to orthonormal basis $\{e_i\}_1^n$ and $\{e_\alpha\}_{n+1}^{2m+1}$ of $T_p\mathcal{M}$ and $T_p^\perp\mathcal{M}$ respectively, the shape operator $S_\alpha = S_{e_\alpha}$ for $\alpha \in \{n+1, \dots, 2m+1\}$ takes the following form

$$(5.1) \quad S_{n+1} = \begin{pmatrix} M & 0 \\ 0 & \frac{n(n-1)a}{r} \end{pmatrix}, S_{n+2} = \dots = S_{2m+1} = 0,$$

where M is the diagonal matrix of order $n-1$ with diagonal entry $'a'$.

Proof. Incorporating (2.7) together with (2.8) into the inequalities (3.9) and (3.10), one can derive the inequalities for bi-slant, hemi-slant, semi-slant and slant. Further, the inequalities for anti-invariant, invariant and CR submanifolds can be respectively obtained by substituting $\theta = \frac{\pi}{2}$ (in case of the inequalities for slant submanifold), $\|P\|^2 = n$ and $\theta_1 = 0, \theta_2 = \frac{\pi}{2}$ (in case of the inequalities for bi-slant submanifold) respectively. $\square$

Corollary 5.2. *Let $\mathcal{M}^n$ be a submanifold of l.c.a.c. manifold $\mathcal{N}^{2m+1}$ admitting SSMC with $\xi \in T_p\mathcal{M}$. Then, we have the following inequalities*

Submanifold $\mathcal{M}^n$	Inequalities
bi-slant	$\rho \leq \delta_s(n-1) + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2)}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$ $\rho \leq \delta_s^*(n-1) + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2)}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$
hemi-slant	$\rho \leq \delta_s(n-1) + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6d_1 \cos^2 \theta_1}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$ $\rho \leq \delta_s^* + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6d_1 \cos^2 \theta_1}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$
semi-slant	$\rho \leq \delta_s(n-1) + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6(d_1 + d_2 \cos^2 \theta_2)}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$ $\rho \leq \delta_s^*(n-1) + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6(d_1 + d_2 \cos^2 \theta_2)}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$
slant	$\rho \leq \delta_s(n-1) + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{3(n-1) \cos^2 \theta}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$ $\rho \leq \delta_s^*(n-1) + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{3(n-1) \cos^2 \theta}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$
anti-invariant	$\rho \leq \delta_s(n-1) + \left(\frac{s+v^2}{4}\right) \left(\frac{n-2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$ $\rho \leq \delta_s^*(n-1) + \left(\frac{s+v^2}{4}\right) \left(\frac{n-2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$
invariant	$\rho \leq \delta_s(n-1) + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{3}{n-1} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$ $\rho \leq \delta_s^*(n-1) + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{3}{n-1} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$
CR	$\rho \leq \delta_s(n-1) + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6d_1}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$ $\rho \leq \delta_s^*(n-1) + \left(\frac{s+v^2}{4}\right) \left(1 + \frac{6d_1}{n(n-1)} - \frac{2}{n}\right) - v^2 - \frac{2}{n}v' - \frac{2}{n}(\text{tr}\Lambda)$

Moreover, equalities hold for δ_s for all type of given submanifolds if and only if submanifold $\mathcal{M}^n$ is an invariantly quasi-umbilical with flat normal connection in $\mathcal{N}^{2m+1}$ so that with respect to an orthonormal basis $\{e_i\}_1^n$ of $T_p\mathcal{M}$ and $\{e_\alpha\}_{n+1}^{2m+1}$ of $T_p^\perp\mathcal{M}$, the shape operator $S_\alpha = S_{e_\alpha}$ for $\alpha \in \{n+1, \dots, 2m+1\}$ takes the form

$$S_{n+1} = \begin{pmatrix} M & 0 \\ 0 & 2a \end{pmatrix}, S_{n+2} = \dots = S_{2m+1} = 0,$$

where M is the diagonal matrix of order $n - 1$ with diagonal entry ' a ' and the equalities hold for δ_s^* for all type of given submanifolds if and only if submanifold $\mathcal{M}^n$ is an invariantly quasi-umbilical with flat normal connection $\nabla^\perp$ in $\mathcal{N}^{2m+1}$ so that with respect to an orthonormal basis $\{e_i\}_1^n$ of $T_p\mathcal{M}$ and $\{e_\alpha\}_{n+1}^{2m+1}$ of $T_p^\perp\mathcal{M}$, the shape operator $S_\alpha = S_{e_\alpha}$ for $\alpha \in \{n+1, \dots, 2m+1\}$ takes the following form

$$S_{n+1} = \begin{pmatrix} M & 0 \\ 0 & a \end{pmatrix}, S_{n+2} = \dots = S_{2m+1} = 0,$$

where M is the diagonal matrix of order $n - 1$ with diagonal entry ' a '.

Proof. By taking $r = \frac{n(n-1)}{2}$ in all the inequalities of theorem (5.1), we obtain our result for normalized δ -Casorati curvature of all types of submanifolds $\mathcal{M}^n$ in l.c.a.c. manifold $\mathcal{N}^{2m+1}$ with SSMC. $\square$

Theorem 5.3. Let $\Psi : \mathcal{M}^n = \mathcal{M}_1 \otimes_\zeta \mathcal{M}_2 \rightarrow \mathcal{N}^{2m+1}(c)$ be an isometric immersion of a WP into l.c.a.c. manifold admitting SSMC with $\xi \in T_p\mathcal{M}_1$. Then where

Submanifold $\mathcal{M}^n$	Inequalities
bi-slant	$n_2 \frac{\Delta\zeta}{\zeta} \leq \tau - \left(\frac{s-3v^2}{8}\right)[n(n-1) - 2n_1n_2] + \left(\frac{s+v^2}{4} + v'\right)(n_1 - 1) - 3\left(\frac{s+v^2}{4}\right)[n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2] + n\mathcal{C}(\mathcal{L}) - \left(\frac{n_2-1}{2}\right)\text{tr}\Lambda$
hemi-slant	$n_2 \frac{\Delta\zeta}{\zeta} \leq \tau - \left(\frac{s-3v^2}{8}\right)[n(n-1) - 2n_1n_2] + \left(\frac{s+v^2}{4} + v'\right)(n_1 - 1) - 3\left(\frac{s+v^2}{4}\right)[n_1 \cos^2 \theta_1] + n\mathcal{C}(\mathcal{L}) - \left(\frac{n_2-1}{2}\right)\text{tr}\Lambda$
semi-slant	$n_2 \frac{\Delta\zeta}{\zeta} \leq \tau - \left(\frac{s-3v^2}{8}\right)[n(n-1) - 2n_1n_2] + \left(\frac{s+v^2}{4} + v'\right)(n_1 - 1) - 3\left(\frac{s+v^2}{4}\right)[n_1 + n_2 \cos^2 \theta_2] + n\mathcal{C}(\mathcal{L}) - \left(\frac{n_2-1}{2}\right)\text{tr}\Lambda$
slant	$n_2 \frac{\Delta\zeta}{\zeta} \leq \tau - \left(\frac{s-3v^2}{8}\right)[n(n-1) - 2n_1n_2] + \left(\frac{s+v^2}{4} + v'\right)(n_1 - 1) - 3\left(\frac{s+v^2}{4}\right)[n \cos^2 \theta] + n\mathcal{C}(\mathcal{L}) - \left(\frac{n_2-1}{2}\right)\text{tr}\Lambda$
anti-invariant	$n_2 \frac{\Delta\zeta}{\zeta} \leq \tau - \left(\frac{s-3v^2}{8}\right)[n(n-1) - 2n_1n_2] + \left(\frac{s+v^2}{4} + v'\right)(n_1 - 1) + n\mathcal{C}(\mathcal{L}) - \left(\frac{n_2-1}{2}\right)\text{tr}\Lambda$
invariant	$n_2 \frac{\Delta\zeta}{\zeta} \leq \tau - \left(\frac{s-3v^2}{8}\right)[n(n-1) - 2n_1n_2] + \left(\frac{s+v^2}{4} + v'\right)(n_1 - 1) - 3\left(\frac{s+v^2}{4}\right)n + n\mathcal{C}(\mathcal{L}) - \left(\frac{n_2-1}{2}\right)\text{tr}\Lambda$
CR	$n_2 \frac{\Delta\zeta}{\zeta} \leq \tau - \left(\frac{s-3v^2}{8}\right)[n(n-1) - 2n_1n_2] + \left(\frac{s+v^2}{4} + v'\right)(n_1 - 1) + n\mathcal{C}(\mathcal{L}) - 3\left(\frac{s+v^2}{4}\right)n_1 - \left(\frac{n_2-1}{2}\right)\text{tr}\Lambda$

$n_i = \dim \mathcal{M}_i$, for $i = 1, 2$ and Δ is the Laplacian operator of $\mathcal{M}_1$.

Proof. By substituting

$$\sum_{1 \leq i < j \leq n_1} g^2(Pe_i, e_j) = n_1 \cos^2 \theta_1 \text{ and } \sum_{n_1+1 \leq i < j \leq n} g^2(Pe_i, e_j) = n_2 \cos^2 \theta_2$$

in Theorem 4.1 and then applying (2.8), we conclude all the inequalities of Theorem 5.3. $\square$

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