

OPTIMAL INEQUALITIES FOR GENERALIZED NORMALIZED δ -CASORATI CURVATURES FOR BI-SLANT SUBMANIFOLDS OF KENMOTSU SPACE FORMS

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ABSTRACT. The main objective of the present paper is to generalize the inequalities obtained by the same author M. A. Lone, *Filomat*, 31:15, pp 4925-4932 and M. A. Lone, *Balkan Journal of Geometry and Its Applications*, 22(1), pp 41-50 in the contact version. We consider bi-slant submanifolds of Kenmotsu space forms to obtain the inequalities and also discuss the equality case.

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1. INTRODUCTION

Chen established a sharp inequality for a submanifold in a real space form using intrinsic invariants (the scalar curvature and the sectional curvature) and the main extrinsic invariant (squared mean curvature), initiating the theory of δ -invariants or so-called Chen invariants. Distinguished geometers work in this direction and obtain some interesting results [5].

The Casorati curvature(extrinsic invariant) of a submanifold of a Riemannian manifold defined as the normalized square length of the second fundamental form

introduced by Casorati [3]. The concept of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold [9]. The geometrical meaning and importance of the Casorati curvature discussed by some distinguished geometers [6, 7, 11, 20, 21]. Therefore, attracts the attention of geometers to obtain the optimal inequalities for the Casorati curvatures of the submanifolds of different ambient spaces [14, 15, 19, 17, 18].

In this paper, we generalize the inequalities for the generalized normalized δ -Casorati curvature for bi-slant submanifolds of Kenmotsu space forms.

2. PRELIMINARIES

Let $\overline{\mathcal{M}}$ be a $(2l+1)$ -dimensional almost contact metric manifold with structure (ϕ, ξ, η, g) where ϕ is a tensor field of type $(1, 1)$, ξ a vector field, η is a one form and g is the Riemannian metric on $\overline{\mathcal{M}}$. Then they satisfy [1]

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

These conditions also imply that

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi),$$

and

$$g(\phi X, Y) + g(X, \phi Y) = 0,$$

for all vector fields X, Y in $T\overline{\mathcal{M}}$. Where $T\overline{\mathcal{M}}$ denotes the Lie algebra of vector fields on $\overline{\mathcal{M}}$.

An almost contact metric manifold $\overline{\mathcal{M}}$ is said to be a Kenmotsu manifold if [10]

$$(1) \quad (\overline{\nabla}_X)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \quad \text{and} \quad \overline{\nabla}_X\xi = X - \eta(X)\xi$$

A Kenmotsu manifold with constant ϕ -holomorphic sectional curvature c is called a Kenmotsu space form and it is denoted by $\overline{\mathcal{M}}(c)$. The curvature tensor $\overline{R}$ for Kenmotsu space form $\overline{\mathcal{M}}(c)$ is defined as [10]

$$\begin{aligned} \overline{R}(X, Y)Z &= \frac{c-3}{4}\{g(Y, Z)X - g(X, Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ &\quad - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi \\ (2) \quad &\quad - g(\phi X, Z)\phi Y + g(\phi Y, Z)\phi X + 2g(X, \phi Y)\phi Z\} \end{aligned}$$

for all $X, Y, Z \in T\overline{\mathcal{M}}(c)$.

Let $\mathcal{M}$ be a submanifold of an almost contact metric manifold $\overline{\mathcal{M}}$ with induced metric g . We write

$$\phi X = PX + FX$$

for any $X \in T\mathcal{M}$, PX and FX denote the tangential and normal parts of ϕX respectively.

The equation of Gauss is given by

$$(3) \quad \begin{aligned} \overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) \\ - g(h(X, W), h(Y, Z)) \end{aligned}$$

for $X, Y, Z, W \in T\mathcal{M}$, where $\overline{R}$ and R represent the curvature tensors of $\overline{\mathcal{M}}$ and $\mathcal{M}$ respectively.

For a given orthonormal frame $\{e_1, e_2, \dots, e_l\}$ of a differentiable distribution D . The squared norm of P at $p \in \mathcal{M}$ is defined as

$$(4) \quad \|P\|^2 = \sum_{i,j=1}^l g^2(\varphi e_i, e_j).$$

For any $i = 1, 2, \dots, l$ where $\{e_1, \dots, e_l, e_{l+1} = \xi\}$ is a local orthonormal frame, we have

$$(5) \quad \sum_{j=1}^l g^2(\varphi e_i, e_j) = \cos^2 \theta.$$

It is well known that for a Kenmotsu manifolds

$$h(X, \xi) = 0.$$

A submanifold $\mathcal{M}$ is said to be a slant submanifold if for any $p \in \mathcal{M}$ and a non zero vector $X \in T_p\mathcal{M}$, the angle between ϕX and $T_p\mathcal{M}$ is constant, i.e., the angle does not depend on the choice of $p \in \mathcal{M}$ and $X \in T_p\mathcal{M}$. The angle $\theta \in [0, \frac{\pi}{2}]$ is called the slant angle.

In general, if a submanifold $\mathcal{M}$ admits two orthogonal distributions D^{θ_1} and D^{θ_2} , such that

$$(i) \quad T\mathcal{M} = D^{\theta_1} \oplus D^{\theta_2}$$

(ii) For any $i = 1, 2$, D^{θ_i} is the slant distribution with slant angle θ_i is known as *bi-slant submanifold* [4].

Naturally, the bi-slant submanifolds is a generalization of semi-slant submanifolds and hemi-slant submanifolds. The invariant, anti-invariant, CR-submanifolds, slant submanifolds appears as particular cases.

Let $\mathcal{M}$ be a Riemannian manifold of dimension l and $K(\pi)$ denotes the sectional curvature of $\mathcal{M}$ of the plane section $\pi \subset T_p\mathcal{M}$ at a point $p \in \mathcal{M}$. If $\{e_1, \dots, e_l\}$ and $\{e_{l+1}, \dots, e_{2l+1}\}$ be the orthonormal basis of $T_p\mathcal{M}$ and $T_p^\perp\mathcal{M}$ at any $p \in \mathcal{M}$, then the scalar curvature τ at that point is given by

$$\tau(p) = \sum_{1 \leq i < j \leq l} K(e_i \wedge e_j)$$

and the normalized scalar curvature ρ is defined as

$$\rho = \frac{2\tau}{l(l-1)}.$$

The mean curvature vector denoted by H is defined as

$$H = \frac{1}{l} \sum_{i=1}^l h(e_i, e_i)$$

and also we put

$$h_{ij}^\gamma = g(h(e_i, e_j), e_\gamma), \quad i, j \in 1, 2, \dots, l, \quad \gamma \in \{l+1, l+2, \dots, 2l+1\}.$$

The squared norm of mean curvature of the submanifold is defined by

$$\|H\|^2 = \frac{1}{l^2} \sum_{\gamma=l+1}^{2l+1} \left(\sum_{i=1}^l h_{ii}^\gamma \right)^2$$

and the squared norm of second fundamental form h is denoted by $\mathcal{C}$ defined as

$$\mathcal{C} = \frac{1}{l} \sum_{\gamma=l+1}^{2l+1} \sum_{i,j=1}^l (h_{ij}^\gamma)^2$$

known as Casorati curvature of the submanifold.

If we suppose that L is an r -dimensional subspace of $T\mathcal{M}$, $r \geq 2$, and $\{e_1, e_2, \dots, e_r\}$ is an orthonormal basis of L , then the scalar curvature of the r -plane section L is given as

$$\tau(L) = \sum_{1 \leq \gamma < \beta \leq r} K(e_\gamma \wedge e_\beta)$$

and the Casorati curvature $\mathcal{C}$ of the subspace L is as given below

$$\mathcal{C}(L) = \frac{1}{r} \sum_{\gamma=l+1}^{2l+1} \sum_{i,j=1}^l (h_{ij}^\gamma)^2.$$

A point $p \in \mathcal{M}$ is said to be an invariantly quasi-umbilical point if there exist l mutually orthogonal unit normal vectors $\xi_{l+1}, \dots, \xi_{2l+1}$ such that the shape operators with respect to all directions ξ_γ have an eigenvalue of multiplicity $l-1$ and that for each ξ_γ the distinguished eigen direction is the same. The submanifold is said to be an invariantly quasi-umbilical submanifold if each of its points is an invariantly quasi-umbilical point.

The normalized δ -Casorati curvatures $\delta_c(l-1)$ and $\widehat{\delta}_c(l-1)$ are defined as

$$(6) \quad [\delta_c(l-1)]_p = \frac{1}{2}\mathcal{C}_p + \frac{l+1}{2l} \inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_p\mathcal{M}\}$$

and

$$(7) \quad [\widehat{\delta}_c(l-1)]_p = 2\mathcal{C}_p + \frac{2l-1}{2l} \sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_p\mathcal{M}\}.$$

Some authors use the coefficient $\frac{l+1}{2l(l-1)}$ instead of $\frac{2l-1}{2l}$ in the equation(7). It was pointed out by the authors in [13] that the coefficient $\frac{l+1}{2l(l-1)}$ is not suitable and therefore modified by the coefficient $\frac{2l-1}{2l}$. For a positive real number $t \neq l(l-1)$, the generalized normalized δ -Casorati curvatures $\delta_c(t; l-1)$ and $\widehat{\delta}_c(t; l-1)$ are given as

$$[\delta_c(t; l-1)]_p = t\mathcal{C}_p + \frac{1}{lt}(l-1)(l+t)(l^2-l-t) \inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_p\mathcal{M}\},$$

if $0 < t < l^2 - l$, and

$$[\widehat{\delta}_c(t; l-1)]_p = r\mathcal{C}_p + \frac{1}{lt}(l-1)(l+t)(l^2-l-t) \sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_p\mathcal{M}\},$$

if $t > l^2 - l$.

3. MAIN THEOREM

Theorem 3.1. *Let $\mathcal{M}$ be an $(l+1)$ -dimensional bi-slant submanifold of a Kenmotsu space form $\overline{\mathcal{M}}(c)$ of dimension $(2l+1)$. Then (i) The generalized normalized δ -Casorati curvature $\delta_c(t; l-1)$ satisfies*

$$(8) \quad \rho \leq \frac{\delta_c(t; l-1)}{l(l-1)} + \frac{c-3}{4} - \frac{c+1}{2l(l+1)} + \frac{3(c+1)}{4l(l+1)} \{l_1 \cos^2 \theta_1 + l_2 \cos^2 \theta_2\}$$

for any real number t such that $0 < t < l(l-1)$.

(ii) The generalized normalized δ -Casorati curvature $\widehat{\delta}_c(t; l-1)$ satisfies

$$(9) \quad \rho \leq \frac{\widehat{\delta}_c(t; l-1)}{l(l-1)} + \frac{c-3}{4} - \frac{c+1}{2l(l+1)} + \frac{3(c+1)}{4l(l+1)} \{l_1 \cos^2 \theta_1 + l_2 \cos^2 \theta_2\}$$

for any real number $t > l(l-1)$. Moreover, the equality holds in (8) and (9) iff $\mathcal{M}$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{\mathcal{M}}$, such that with respect to suitable tangent orthonormal frame $\{e_1, \dots, e_l\}$ and normal orthonormal frame $\{e_{l+1}, \dots, e_{2l+1}\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{l+1, \dots, 2l+1\}$, take the following form

$$(10) \quad S_{l+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 & 0 \\ 0 & a & 0 & \dots & 0 & 0 \\ 0 & 0 & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{l(l-1)}{t}a \end{pmatrix}, \quad S_{l+2} = \dots = S_{2l+1} = 0.$$

Proof. Let $\{e_1, \dots, e_l\}$ and $\{e_{l+1}, \dots, e_{2l+1}\}$ be the orthonormal basis of $T_p\mathcal{M}$ and $T_p^\perp\mathcal{M}$ respectively at any point $p \in \mathcal{M}$. Putting $X = W = e_i$, $Y = Z = e_j$, $i \neq j$

from (2), we can write

$$\begin{aligned}
 \overline{R}(e_i, e_j, e_j, e_i) &= \frac{c-3}{4} \{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\} \\
 &\quad + \frac{c+1}{4} \{\eta(e_i)\eta(e_j)g(e_j, e_i) \\
 &\quad - \eta(e_j)\eta(e_j)g(e_i, e_i) + \eta(e_j)\eta(e_i)g(e_i, e_j) - \eta(e_i)\eta(e_i)g(e_j, e_j) \\
 &\quad - g(\phi e_i, e_j)g(\phi e_j, e_i) + g(\phi e_j, e_i)g(\phi e_i, e_i) \\
 &\quad + 2g(e_i, \phi e_j)g(\phi e_j, e_i)\}
 \end{aligned}
 \tag{11}$$

Using Gauss equation and (11), we obtain

$$\begin{aligned}
 &\frac{c-3}{4} \{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\} + \frac{c+1}{4} \{\eta(e_i)\eta(e_j)g(e_j, e_i) \\
 &\quad - \eta(e_j)\eta(e_j)g(e_i, e_i) + \eta(e_j)\eta(e_i)g(e_i, e_j) - \eta(e_i)\eta(e_i)g(e_j, e_j) \\
 &\quad - g(\phi e_i, e_j)g(\phi e_j, e_i) + g(\phi e_j, e_i)g(\phi e_i, e_i) + 2g(e_i, \phi e_j)g(\phi e_j, e_i)\} \\
 &= R(e_i, e_j, e_j, e_i) - g(h(e_i, e_i), h(e_j, e_j)) + g(h(e_i, e_j), h(e_j, e_i))
 \end{aligned}
 \tag{12}$$

Taking summation $1 \leq i, j \leq l$, we reach at

$$\begin{aligned}
 (13) \quad 2\tau &= (l+1)^2 \|H\|^2 - (l+1)\mathcal{C} + \frac{c-3}{4}l(l+1) + \\
 &\quad \frac{c+1}{4} \{-2l + 3 \sum_{i,j=1}^{m+1} g^2(\phi e_i, e_j)\}
 \end{aligned}$$

As $\mathcal{M}$ is a bi-slant submanifold of a Kenmotsu space form $\overline{\mathcal{M}}(c)$. Assume that $(l+1) = 2l_1 + 2l_2 + 1$ and choose an adapted orthonormal frame as:

$$\begin{aligned}
 e_1, e_2 &= \frac{1}{\cos\theta} P e_1, \dots, e_{2l_1-1}, e_{2l_1} = \frac{1}{\cos\theta} P e_{2l_1-1}, e_{2l_1+1}, e_{2l_1+2} = \frac{1}{\cos\theta} P e_{2l_1+1}, \\
 &\dots, e_{2l_1+2l_2-1}, e_{2l_1+2l_2} = \frac{1}{\cos\theta} P e_{2l_1+2l_2-1}, e_{2l_1+2l_2+1} = \xi,
 \end{aligned}$$

we have

$$g(\phi e_1, e_2) = g(\phi e_1, \frac{1}{\cos\theta} P e_1) = \frac{1}{\cos\theta} g(P e_1, P e_1) = \cos\theta.$$

Similarly, we obtain

$$g^2(\phi e_1, e_2) = \cos^2\theta,$$

thus, we have

$$(14) \quad g^2(\phi e_i, e_j) = \begin{cases} \cos^2\theta_1, & \text{for } i = 1, \dots, 2l_1 - 1 \\ \cos^2\theta_2, & \text{for } i = 2l_1 + 1, \dots, 2l_1 + 2l_2 - 1. \end{cases}$$

Thus, we reach at

$$(15) \quad 2\tau = (l+1)^2 \|H\|^2 - (l+1)\mathcal{C} + \frac{c-3}{4}l(l+1) \\ - l\frac{c+1}{2} + 3\frac{c+1}{4}\{l_1\cos^2\theta_1 + l_2\cos^2\theta_2\}.$$

Define the following function, denoted by $\mathcal{Q}$, a quadratic polynomial in the components of the second fundamental form

$$(16) \quad \mathcal{Q} = t\mathcal{C} + a(t)\mathcal{C}(L) - 2\tau + \frac{c-3}{4}\{l(l+1)\} \\ + -l\frac{c+1}{2} + 3\frac{c+1}{4}\{l_1\cos^2\theta_1 + l_2\cos^2\theta_2\},$$

where L is the hyperplane of $T_p\mathcal{M}$. Without loss of generality, we assume that L is spanned by $e_1, \dots, e_{l-1}$, from (16) we have

$$\mathcal{Q} = \frac{l+t}{l} \sum_{\gamma=l+1}^m \sum_{i,j=1}^l (h_{ij}^\gamma)^2 + \frac{a(t)}{l-1} \sum_{\gamma=l+1}^m \sum_{i,j=1}^{l-1} (h_{ij}^\gamma)^2 - \sum_{\gamma=l+1}^m \left(\sum_{i=1}^l h_{ii}^\gamma \right)^2.$$

The above equation can be simplified as

$$(17) \quad \mathcal{Q} = \sum_{\gamma=l+1}^m \sum_{i=1}^{l-1} \left[\left(\frac{l+t}{l} + \frac{a(t)}{l-1} \right) (h_{ii}^\gamma)^2 + \frac{2(l+t)}{l} (h_{in}^\gamma)^2 \right] \\ + \sum_{l+1}^m \left[2 \left(\frac{l+t}{l} + \frac{a(t)}{l-1} \right) \sum_{(i<j)=1}^l (h_{ij}^\gamma)^2 - 2 \sum_{(i<j)=1}^l h_{ii}^\gamma h_{jj}^\gamma + \frac{t}{l} (h_{ll}^\gamma)^2 \right]$$

From(17), we can see that the critical points

$$h^c = (h_{11}^{l+1}, h_{12}^{l+1}, \dots, h_{ll}^{l+1}, \dots, h_{11}^m, \dots, h_{ll}^m)$$

of $\mathcal{Q}$ are the solutions of the following system of homogenous equations:

$$(18) \quad \begin{cases} \frac{\partial \mathcal{Q}}{\partial h_{ii}^\gamma} = 2 \left(\frac{l+t}{l} + \frac{a(t)}{l-1} \right) (h_{ii}^\gamma) - 2 \sum_{k=1}^l h_{kk}^\gamma = 0 \\ \frac{\partial \mathcal{Q}}{\partial h_{il}^\gamma} = \frac{2t}{l} h_{il}^\gamma - 2 \sum_{k=1}^{l-1} h_{kk}^\gamma = 0 \\ \frac{\partial \mathcal{Q}}{\partial h_{ij}^\gamma} = 4 \left(\frac{l+t}{l} + \frac{a(t)}{l-1} \right) (h_{ij}^\gamma) = 0 \\ \frac{\partial \mathcal{Q}}{\partial h_{il}^\gamma} = 4 \left(\frac{l+t}{l} \right) (h_{il}^\gamma) = 0, \end{cases}$$

where $i, j = \{1, 2, \dots, l-1\}$, $i \neq j$, and $\gamma \in \{l+1, \dots, m\}$.

Thus, every solution h^c has $h_{ij}^\gamma = 0$ for $i \neq j$ and the corresponding determinant to the first two equations of the above system is zero. Moreover, the Hessian matrix of $\mathcal{Q}$ is of the following form

$$\mathcal{H}(\mathcal{Q}) = \begin{pmatrix} H_1 & O & O \\ O & H_2 & O \\ O & O & H_3 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} 2\left(\frac{l+t}{l} + \frac{a(t)}{l-1}\right) - 2 & -2 & \dots & -2 & -2 \\ -2 & 2\left(\frac{l+t}{l} + \frac{a(t)}{l-1}\right) - 2 & \dots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \dots & 2\left(\frac{l+t}{l} + \frac{a(t)}{l-1}\right) - 2 & -2 \\ -2 & -2 & \dots & -2 & \frac{2t}{l} \end{pmatrix},$$

H_2 and H_3 are the diagonal matrices and O is the null matrix of the respective dimensions. H_2 and H_3 are respectively given as

$$H_2 = \text{diag}\left(4\left(\frac{l+t}{l} + \frac{a(t)}{l-1}\right), 4\left(\frac{l+t}{l} + \frac{a(t)}{l-1}\right), \dots, 4\left(\frac{l+t}{l} + \frac{a(t)}{l-1}\right)\right),$$

and

$$H_3 = \text{diag}\left(\frac{4(l+t)}{l}, \frac{4(l+t)}{l}, \dots, \frac{4(l+t)}{l}\right).$$

Hence, we find that $\mathcal{H}(\mathcal{Q})$ has the following eigenvalues

$$\lambda_{11} = 0, \lambda_{22} = 2\left(\frac{2t}{l} + \frac{a(t)}{l-1}\right), \lambda_{33} = \dots = \lambda_{ll} = 2\left(\frac{l+t}{l} + \frac{a(t)}{l-1}\right),$$

$$\lambda_{ij} = 4\left(\frac{l+t}{l} + \frac{a(t)}{l-1}\right), \lambda_{in} = \frac{4(l+t)}{l}, \forall i, j \in \{1, 2, \dots, l-1\}, i \neq j.$$

Thus, $\mathcal{Q}$ is parabolic and reaches at minimum $\mathcal{P}(h^c) = 0$ for the solution h^c of the system (18). Hence $\mathcal{P} \geq 0$ and hence

$$2\tau \leq t\mathcal{C} + a(t)\mathcal{C}(L) + \frac{c-3}{4}l(l+1) - l\frac{c+1}{2} + 3\frac{c+1}{4}\{l_1\cos^2\theta_1 + l_2\cos^2\theta_2\},$$

whereby, we obtain

$$\rho \leq \frac{t}{l(l+1)}\mathcal{C} + \frac{a(t)}{l(l+1)}\mathcal{C}(L) + \frac{c-3}{4} - \frac{c+1}{2l(l+1)} + \frac{3(c+1)}{4l(l+1)}\{l_1\cos^2\theta_1 + l_2\cos^2\theta_2\}$$

for every tangent hyperplane L of $\mathcal{M}$. If we take the infimum over all tangent hyperplanes L , the result trivially follows. Moreover the equality sign holds iff

$$(19) \quad h_{ij}^\gamma = 0, \quad \forall i, j \in \{1, \dots, l\}, \quad i \neq j \text{ and } \gamma \in \{l+1, \dots, m\}$$

and

$$(20) \quad h_{il}^\gamma = \frac{l(l-1)}{t}h_{11}^\gamma = \dots = \frac{l(l-1)}{t}h_{l-1l-1}^\gamma, \quad \forall \gamma \in \{l+1, \dots, m\}.$$

From (19) and (20), we obtain that the equality holds if and only if the submanifold is invariantly quasi-umbilical with normal connections in $\overline{\mathcal{M}}$, such that the shape operator takes the form (10) with respect to the orthonormal tangent and orthonormal normal frames.

In the same way, we can prove (ii). □

Corollary 3.2. *Let $\mathcal{M}$ be an $(l+1)$ -dimensional bi-slant submanifold of a Kenmotsu space form $\overline{\mathcal{M}}(c)$ of dimension $(2l+1)$. Then*

(i) *The normalized δ -Casorati curvature $\delta_c(l-1)$ satisfies*

$$\rho \leq \delta_c(l-1) + \frac{c-3}{4} - \frac{c+1}{2l(l+1)} + \frac{3(c+1)}{4l(l+1)}\{l_1\cos^2\theta_1 + l_2\cos^2\theta_2\}$$

Moreover, the equality sign holds iff $\mathcal{M}$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{\mathcal{M}}$, such that with respect to suitable tangent

orthonormal frame $\{e_1, \dots, e_{l+1}\}$ and normal orthonormal frame $\{e_{l+2}, \dots, e_{2l+1}\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{l+1, \dots, 2l+1\}$, take the following form

$$(21) \quad S_{l+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 & 0 \\ 0 & a & 0 & \dots & 0 & 0 \\ 0 & 0 & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & 0 \\ 0 & 0 & 0 & \dots & 0 & 2a \end{pmatrix}, \quad S_{l+2} = \dots = S_{2l+1} = 0.$$

(ii) The normalized δ -Casorati curvature $\widehat{\delta}_c(l-1)$ satisfies

$$\begin{aligned} \rho \leq \widehat{\delta}_c(l-1) + \frac{c-3}{4} - \frac{c+1}{2l(l+1)} \\ + \frac{3(c+1)}{4l(l+1)} \{l_1 \cos^2 \theta_1 + l_2 \cos^2 \theta_2\} \end{aligned}$$

Moreover, the equality sign holds iff $\mathcal{M}$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{\mathcal{M}}$, such that with respect to suitable tangent orthonormal frame $\{e_1, \dots, e_{l+1}\}$ and normal orthonormal frame $\{e_{l+2}, \dots, e_{2l+1}\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{l+1, \dots, 2l+1\}$, take the following form

$$(22) \quad S_{l+1} = \begin{pmatrix} 2a & 0 & 0 & \dots & 0 & 0 \\ 0 & 2a & 0 & \dots & 0 & 0 \\ 0 & 0 & 2a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2a & 0 \\ 0 & 0 & 0 & \dots & 0 & a \end{pmatrix}, \quad S_{l+2} = \dots = S_{2l+1} = 0.$$

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