

ON SPHERICALLY SYMMETRIC LANDSBERG METRICS

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ABSTRACT. In this paper, we have studied spherically symmetric Landsberg metrics with isotropic E -curvature and isotropic S -curvature. In the first case we have shown that the metric reduces to a Berwald metric and therefore, it is of vanishing E -curvature. In the second case we have completely classified the spherically symmetric Landsberg metric with isotropic S -curvature.

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1. INTRODUCTION

Finsler metric is a smoothly varying family of Minkowski norm one on each tangent space. Riemannian metric is a special case of Finsler metric. Berwald metric is one of the important class of non-Riemannian Finsler metric in which all the norms are linearly isometric to each other. The Randers metric is of the

form $F = \alpha + \beta$, where α is a Riemannian metric and β is a one form, which is one of the most simplest non-Riemannian Finsler metric. Various generalizations of Berwald metrics have been introduced and studied by several geometers. Landsberg metric, Douglas metric, weakly Berwald metric are few of them. Berwald metric is one of the most important class of non-Riemannian Finsler metric. L. Zhou have studied spherically symmetric Douglas metric with isotropic S -curvature [14]. In this paper our focus is to study spherically symmetric Landsberg metric. In local coordinates, the Landsberg curvature is defined as $L := L_{jkl} dx^j \otimes dx^k \otimes dx^l$, where $L_{jkl} = -\frac{1}{2} F F_{y^i} \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$. Obviously, every Berwald metric is a Landsberg metric but the converse is still unknown for long time and widely called as “unicorn problem” following D. Bao [1].

In the study of Finsler geometry, we often encounter long and complicated calculations. However when we consider Finsler metrics with a certain symmetry, it would make things much easier. A Finsler metric F is called spherically symmetric which was introduced by Rutz [9] in 1996, if $F(Ax, Ay) = F(x, y)$, for all $A \in O(n)$, where $O(n)$ is the set of all $n \times n$ orthogonal matrices. It has been proved that a Finsler metric F on $\mathbb{B}^n(\mu)$ is a spherically symmetric if and only if there exist a function $\phi : [0, \mu) \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(1) \quad F(x, y) = \|y\| \phi \left(\|x\|, \frac{\langle x, y \rangle}{\|y\|} \right),$$

where $(x, y) \in T\mathbb{R}^n(\mu) \setminus \{0\}$, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

Funk metric is one of the most basic and important example of spherically symmetric metric which is defined by $F := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)} + \langle x, y \rangle}{1 - |x|^2}$ on $B^n(1)$. This Funk metric has some special geometric properties. Funk metric is an example of Randers metric. Although it is of constant S -curvature and of constant E -curvature it is neither Berwald nor Landsberg metric. Since a spherically symmetric Finsler metric does not need to be (α, β) -metric we can ask a more general question that, what are the properties of spherically symmetric Landsberg metrics when they possesses isotropic S -curvature or isotropic E -curvature. In this paper, we have studied spherically symmetric Landsberg metrics with isotropic E -curvature and isotropic S -curvature. In the first case we have shown that the metric

reduces to a Berwald metric and therefore, it is of vanishing E -curvature. In the second case we have completely classified the spherically symmetric Landsberg metric with isotropic S -curvature. Precisely we have proved the following results:

Theorem 1.1. *Let F be a spherically symmetric Finsler metric on an open set Ω in $\mathbb{R}^n$. Then F is a Landsberg metric with isotropic E -curvature if and only if F is exactly one of the following forms:*

- (i) F is a Riemannian metric.
- (ii) F is a Berwald metric of the form

$$F = u\psi \left(\frac{s^2}{g(r) + s^2 \int 4rc_2(r)g(r)dr} \right) e^{-\int (\frac{2}{r} - 2r^3 c_2(r))dr} s,$$

where $c_2(r)$ is a smooth function and $g(r) = e^{\int \frac{2}{r} - 4r^3 c_2(r)dr}$.

Corollary 1.1. *Any spherically symmetric Landsberg metric with isotropic E -curvature has vanishing E -curvature.*

Theorem 1.2. *Let F be a spherically symmetric Finsler metric on an open set Ω in $\mathbb{R}^n$ with spherically symmetric volume form. Then F is a Landsberg metric with isotropic S -curvature, if and only if F is one of the following forms:*

- (i) F is a Riemannian metric.
- (ii) F is a Berwald metric of the form

$$F = u\psi \left(\frac{s^2}{g(r) + s^2 \int 4rc_2(r)g(r)dr} \right) e^{-\int (\frac{2}{r} - 2r^3 c_2(r))dr} s,$$

- (iii) F is a Randers metric of the form

$$(2) \quad F = \sqrt{\frac{c}{r^2}|y|^2 + g(r)\langle x, y \rangle^2 + h(r)\langle x, y \rangle},$$

where the positive constant c and smooth functions $g(r), h(r)$ satisfy

$$g(r) - h(r)^2 > -\frac{c}{r^4}, \quad 2h' \left(\frac{c}{r^2} + r^2 g \right) - h \left(r^2 g' - \frac{4c}{r^3} \right) = 0.$$

2. PRELIMINARIES

Let M be an n -dimensional smooth manifold. $T_x M$ denotes the tangent space of M at x . The tangent bundle of M is the union of tangent spaces $TM := \bigcup_{x \in M} T_x M$. We denote the elements of TM by (x, y) where $y \in T_x M$ and the slit tangent bundle

$TM_0 := TM \setminus \{0\}$.

Definition. A Finsler metric on M is a function $F : TM \rightarrow [0, \infty)$ with the following properties:

- (i) F is C^∞ on TM_0 ,
 - (ii) F is a positively 1-homogeneous on the fibers of tangent bundle TM ,
 - (iii) The Hessian of $\frac{F^2}{2}$ with element $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0 .
- The pair (M, F) is called a Finsler space. The metric F is called the fundamental function and g_{ij} is called the fundamental tensor.

The spray coefficients of the Finsler metric F is defined by

$$(3) \quad G^i = \frac{1}{4} g^{il} \{ [F^2]_{x^m y^l} y^m - [F^2]_{x^l} \},$$

where $g^{ij} = (g_{ij})^{-1}$. The geodesic coefficients of a spherically symmetric Finsler metric $F(x, y) = \|y\| \phi \left(\|x\|, \frac{\langle x, y \rangle}{\|y\|} \right)$ are given by

$$(4) \quad G^i = u P y^i + u^2 Q x^i,$$

where

$$(5) \quad Q := \frac{1}{2r} \frac{r\phi_{ss} - \phi_r + s\phi_{rs}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}}, \quad u := |y|, \quad r := |x|, \quad s := \frac{\langle x, y \rangle}{|y|}$$

and

$$(6) \quad P := \frac{r\phi_s + s\phi_r}{2r\phi} - \frac{Q}{\phi} [s\phi + (r^2 - s^2)\phi_s].$$

The Berwald curvature of a Finsler metric is a tensor defined in local coordinates as follows: $B := B_{jkl}^i dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i}$, where $B_{jkl}^i = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$ and G^i is the geodesic spray coefficients. Suppose $\gamma(t)$ be the geodesic with $\gamma(0) = x$ and $\gamma'(0) = y$ on M . Then the S -curvature $S(x, y)$ which was introduced by Z-Shen [13, 12] for a given volume form $dV = \sigma(x)dx$ is defined as

$$S(x, y) = \frac{d}{dt} [\tau(\gamma(t), \gamma'(t))]_{t=0},$$

where $\tau(x, y)$ is the distortion of F .

The Finsler metric F is said to have isotropic S -curvature with respect to the volume form $dV = \sigma(x)dx$ if there exists a function $c = c(x)$ on M so that $S = (n+1)cF$. F is said to have constant S -curvature if $c = \text{constant}$.

Another quantity is associated with S -curvature called E -curvature. The E -tensor $\xi := E_{ij} dx^i \otimes dx^j$ is defined by $E_{ij} := \frac{1}{2} S_{y^i y^j}(x, y) = \frac{1}{2} \frac{\partial^3 G^m}{\partial y^i \partial y^j \partial y^m}$. F is said to

have isotropic E -curvature if there is a function $c = c(x)$ on M such that

$$E = \frac{1}{2}(n+1)cF_{y^i y^j}.$$

3. SPHERICALLY SYMMETRIC LANDSBERG METRIC

For a spherically symmetric Finsler metric we have the following lemmas:

Lemma 3.1. [7] *A spherically symmetric Finsler metric $F = u\phi(r, s)$ in $\mathbb{R}^n$ ($n \geq 3$) is a Berwald metric if and only if either F is Riemannian or there exists smooth function $c_2(r)$ so that $F = u\psi\left(\frac{s^2}{g(r)+s^2 \int 4rc_2(r)g(r)dr}\right) e^{-\int(\frac{2}{r}-2r^3 c_2(r))dr} s$, where $g(r) = e^{\int \frac{2}{r}-4r^3 c_2(r)dr}$*

Lemma 3.2. [7] *If a spherically symmetric Finsler metric $F = u\phi(r, s)$ in $\mathbb{R}^n$ ($n \geq 3$) is Landsberg type, where $u := |y|, r := |x|, s := \frac{\langle x, y \rangle}{|y|}$, then there exist the functions $c_0(r), c_1(r), c_2(r), c_3(r)$ so that its geodesic spray coefficients G^i satisfy*

$$G^i = uPy^i + u^2Qx^i,$$

where

$$(7) \quad P = c_1(r)s + c_2(r)\frac{\sqrt{r^2 - s^2}}{r^2} \quad \text{and} \quad Q = \frac{1}{2}c_0(r)s^2 - \frac{c_2(r)s\sqrt{r^2 - s^2}}{r^4} + c_3(r).$$

Lemma 3.3. [4] *On $\mathbb{B}^n(\mu)$, a spherically symmetric Finsler metric $F = u\phi(r, s)$ has isotropic E -curvature if and only if*

$$(8) \quad (P - sP_s)(n+1) + (r^2 - s^2)(Q_s - sQ_{ss}) = (n+1)(\phi - \phi_s)c(x),$$

where $c = c(x)$ is a scalar function on M .

Let $F = \alpha + \beta$ be a Randers metric on a n -dimensional manifold M , where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form. Then F will be a Finsler metric if and only if $\|\beta_x\|_\alpha < 1, \forall x \in M$. Let

$$(9) \quad \begin{aligned} r_{ij} &:= \frac{1}{2}(b_{i|j} + b_{j|i}), & s_{ij} &:= \frac{1}{2}(b_{i|j} - b_{j|i}), \\ s_j^i &:= a^{ik}s_{kj}, & s_j &:= b_i s_j^i = b^k s_{kj}, & e_{ij} &:= r_{ij} + b_i s_j + b_j s_i, \end{aligned}$$

where “ $|$ ” denotes the covariant derivative with respect to the Levi-Civita connection of α . The spray coefficients G^i of F and the spray coefficients G_α^i of α are related by

$$G^i = G_\alpha^i + Py^i + Q^i,$$

where

$$(10) \quad P := \frac{e_{00}}{F} - s_0, \quad Q^i = \alpha s_0^i, \quad e_{00} := e_{ij}y^i y^j, \quad s_0 := s_i y^i, \quad s_0^i := s_j^i y^j.$$

Lemma 3.4. [6] *A Randers metric $F = \alpha + \beta$ is a Landsberg metric if and only if β is parallel with respect to α , that is, $b_{i|j} = 0$.*

Lemma 3.5. [3] *Let $F = \alpha + \beta$ be a Randers metric on an n -dimensional manifold M . Then for a scalar function $c(x)$ following are equivalent:*

- (i) $e_{00} = 2c(\alpha^2 - \beta^2)$.
- (ii) $E = \frac{1}{2}(n+1)cF_{y^i y^j}$.

Proof of Theorem 1.1

From equation (7) we have,

$$(11) \quad \begin{aligned} P_s &= c_1(r) - \frac{c_2(r)s}{r^2\sqrt{r^2 - s^2}}, \\ Q_s &= c_0(r)s - \frac{c_2(r)(r^2 - 2s^2)}{r^4\sqrt{r^2 - s^2}}, \\ Q_{ss} &= c_0(r) + \frac{c_2(r)s(3r^2 - 2s^2)}{r^4(r^2 - s^2)^{\frac{3}{2}}}. \end{aligned}$$

At first assume $c(x) \neq 0$. Now using (11) in (8) we have the following differential equation,

$$(12) \quad \phi - s\phi_s = \frac{nc_2(r)}{(n+1)c(x)} \frac{1}{\sqrt{r^2 - s^2}}.$$

Equation (12) can be written as

$$(13) \quad \phi - s\phi_s = \frac{b(x)}{\sqrt{r^2 - s^2}}, \quad \text{where} \quad b(x) = \frac{nc_2(r)}{(n+1)c(x)}.$$

Solving the differential equation (13) we have

$$(14) \quad \phi = \frac{b(x)\sqrt{r^2 - s^2}}{r^2} + sk(x).$$

Therefore, F is a Randers metric.

Now by Lemma 3.4 a Randers metric will be a Landsberg metric if and only if $b_{i|j} = 0$. Therefore, using (10) we have, $e_{00} = 0$. Again from Lemma 3.5 we get $2c(\alpha^2 - \beta^2) = 0$. Which implies $(\alpha^2 - \beta^2) = 0$. This is a contradiction. Hence $c(x)$ has to be zero.

Now suppose $c(x) = 0$. Then $c_2(r) = 0$ and F becomes a Berwald metric. Using Lemma 3.1 we have the result.

For any Berwald metric, the S -curvature vanishes, i.e., $S = 0$. Therefore, its E -curvature vanishes too. Hence, the converse part is also true.

Proof of Corollary 1.1

From Theorem 1.1 we have every spherically symmetric Landsberg metric with isotropic E -curvature is either Riemannian or Berwaldian. In both cases it has vanishing E -curvature.

Theorem 3.1. *Every spherically symmetric Landsberg metric F with isotropic S -curvature, is one of the following forms:*

- (i) F is a Riemannian metric.
- (ii) F is a Berwald metric of the form

$$F = u\psi \left(\frac{s^2}{g(r) + s^2 \int 4rc_2(r)g(r)dr} \right) e^{-\int (\frac{2}{r} - 2r^3 c_2(r))dr} s.$$

- (iii) F is a Randers metric.

Proof: The S -curvature with respect to the volume form $dV = \sigma(x)dx$ can be formulated by

$$(15) \quad S = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m} (\ln \sigma).$$

Since the spray coefficients of spherically symmetric metric is of the form

$G^m = uPy^m + u^2Qx^m$, we have

$$\begin{aligned} \frac{\partial G^m}{\partial y^m} &= \frac{\partial}{\partial y^m} (uPy^m + u^2Qx^m) \\ &= uP\delta_m^m + \left(\frac{P}{u} - \frac{s}{u}P_s \right) y^m y^m + P_s x^m y^m + uQ_s x^m x^m + (2Q - sQ_s)x^m y^m \\ &= uPn + \left(\frac{P}{u} - \frac{s}{u}P_s \right) u^2 + P_s us + uQ_s r^2 + (2Q - sQ_s)us \\ &= (n+1)uP + u(r^2 - s^2)Q_s + 2usQ. \end{aligned}$$

Moreover, we have

$$y^m \frac{\partial}{\partial x^m} (\ln \sigma(r)) = y^m \frac{\sigma'(r)x^m}{\sigma(r)r} = us \frac{\sigma'(r)}{r\sigma(r)}.$$

Therefore, for a spherically symmetric the S -curvature is given by

$$(16) \quad S = (n+1)uP + u(r^2 - s^2)Q_s + 2usQ + usp(r),$$

where $p(r) = -\frac{\sigma'(r)}{r\sigma(r)}$.

Therefore, for a spherically symmetric Finsler metric with isotropic S -curvature we have,

$$(17) \quad (n+1)P + (r^2 - s^2)Q_s + 2sQ + sp(r) = (n+1)c\phi.$$

Now for a spherically symmetric Landsberg metric with isotropic S -curvature using (7) and (11) we get,

$$(18) \quad \begin{aligned} (n+1) \left(c_1(r)s + c_2(r) \frac{\sqrt{r^2 - s^2}}{r^2} \right) + (r^2 - s^2) \left(c_0(r)s - \frac{c_2(r)(r^2 - 2s^2)}{r^4 \sqrt{r^2 - s^2}} \right) \\ + 2s \left(\frac{1}{2}c_0(r)s^2 - \frac{c_2(r)s\sqrt{r^2 - s^2}}{r^4} + c_3(r) \right) + sp(r) = (n+1)c\phi. \end{aligned}$$

Simplifying we have,

$$(19) \quad r^2sc_0(r) + (n+1)c_1(r)s + 2c_3(r)s + sp(r) + \frac{nc_2(r)\sqrt{r^2 - s^2}}{r^2} = (n+1)c\phi.$$

At first suppose $c(x) \neq 0$. Then we have,

$$(20) \quad \phi = k_1(x)s + k_2(x)\sqrt{r^2 - s^2},$$

where $k_1(x) = \frac{r^2c_0(r)}{(n+1)c(x)} + \frac{c_1(r)}{c(x)} + \frac{2c_3(r)}{(n+1)c(x)} + \frac{p(r)}{(n+1)c(x)}$ and $k_2(x) = \frac{nc_2(r)}{(n+1)r^2c(x)}$.

Therefore, F is a Randers metric.

If $c(x) = 0$, then from (19) we have $c_2(r) = 0$. Then we have from Lemma 3.1 that F is a Berwald metric of the given form.

4. PROOF OF THEOREM 1.2

The S -curvature is defined for an arbitrary given volume form $dV = \sigma(x)dx$ on a Finsler manifold. By Theorem 3.1 we conclude that spherically symmetric Landsberg metric with isotropic S -curvature must be either a Randers metric or a Berwald metric of the form given in Lemma 3.1, if the volume form is also spherically symmetric. Now we will classify the Randers metric with isotropic S -curvature. Therefore, it is enough to classify the Randers metric with isotropic S -curvature. Any Randers metric $F = \alpha + \beta$ can be written as

$$(21) \quad F = \sqrt{f(r)|y|^2 + g(r)\langle x, y \rangle^2} + h(r)\langle x, y \rangle,$$

where $f(r), g(r)$ and $h(r)$ are smooth functions.

In this case,

$$(22) \quad \alpha = \sqrt{f(r)|y|^2 + g(r)\langle x, y \rangle^2} \quad \text{and} \quad \beta = h(r)\langle x, y \rangle.$$

Therefore,

$$(23) \quad a_{ij} = f(r)\delta_j^i + g(r)x^i x^j, \quad b_i = h(r)x^i.$$

We also have,

$$\det(a_{ij}) = (f + r^2 g)f^{(n-1)}, \quad a^{ij} = \frac{1}{f}\delta^{ij} - \frac{g}{f(f + r^2 g)}x^i x^j$$

and

$$(24) \quad \|\beta\|_\alpha^2 = b_i b_j a^{ij} = \frac{r^2 h^2}{f + r^2 g}.$$

Since F is a Randers metric, α is a positive definite and $\|\beta\|_\alpha^2 < 1$. Therefore,

$$(25) \quad f(r) > 0, \quad f(r) + r^2 g(r) > 0, \quad f(r) + r^2(g(r) - h(r)^2) > 0.$$

The Christoffel symbols $\Gamma_{ij}^k(x)$ of α are given by

$$(26) \quad \begin{aligned} \Gamma_{ij}^k &= \frac{a^{kl}}{2} \left\{ \frac{\partial a_{li}}{\partial x^j} + \frac{\partial a_{lj}}{\partial x^i} - \frac{\partial a_{ij}}{\partial x^l} \right\} \\ &= \frac{1}{2} \left(\frac{f g' - 2 f' g}{r f (f + r^2 g)} x^i x^j x^k + \frac{2 r g - f'}{r (f + r^2 g)} x^k \delta_{ij} + \frac{f'}{r f} (x^i \delta_{kj} + x^j \delta_{ki}) \right). \end{aligned}$$

Therefore,

$$(27) \quad \begin{aligned} b_{i|j} &= \frac{\partial b_i}{\partial x^j} - b_k \Gamma_{ij}^k \\ &= \frac{1}{2} \frac{h(r f' + 2 f)}{f + r^2 g} \delta_{ij} + \left(\frac{h'}{r} - \frac{1}{2} \frac{h(r^2 g' + 2 f')}{r (f + r^2 g)} \right) x^i x^j. \end{aligned}$$

Now from Lemma 3.4 we have, $b_{i|j} = 0$. Therefore, from (27) we have

$$(28) \quad \frac{1}{2} \frac{h(r f' + 2 f)}{f + r^2 g} = 0, \quad \frac{h'}{r} - \frac{1}{2} \frac{h(r^2 g' + 2 f')}{r (f + r^2 g)} = 0.$$

From the first equation of (28) we have either

$$h = 0 \quad \text{or} \quad r f' + 2 f = 0.$$

If $h = 0$, then F is a Riemannian metric.

If $r f' + 2f = 0$, then solving this differential equation we have $f = \frac{c}{r^2}$.

Now putting the value of f in (28) we have

$$(29) \quad 2h' \left(\frac{c}{r^2} + r^2 g \right) - h \left(r^2 g' - \frac{4c}{r^3} \right) = 0.$$

Since, F is a Randers metric, from (25) we get c is a positive constant and

$$g(r) - h(r)^2 > -\frac{c}{r^4}.$$

Conversely, if F is a Riemannian or a Berwaldian, then F is a Landsberg metric and the S -curvature of F vanishes.

Otherwise, if $F = \sqrt{\frac{c}{r^2}|y|^2 + g(r)\langle x, y \rangle^2} + h(r)\langle x, y \rangle$ and $g(r), h(r)$ satisfy (29) then β is parallel to α . Thus F is again a Landsberg metric with isotropic S -curvature.

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