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ON ATTRACTOR AND PSEUDO ORBIT TRACING OF UNIFORM LIMIT DYNAMICAL SYSTEMS

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ABSTRACT. Some dynamical properties of the uniform limit dynamical system have been investigated. It has been found that for a sequences (X, f_n) of dynamical systems having same attractor A and converging uniformly to a dynamical system (X, f), the dynamical system (X, f) has the same attractor A. It has been found that a sequence (X, f_n) of uniformly rigid dynamical systems converging uniformly to a dynamical system (X, f), the dynamical system (X, f) is also uniformly rigid. We have proved that the uniform limit (X, f) of a uniformly convergent sequence of dynamical systems having pseudo orbit tracing property has pseudo orbit tracing property. The expansiveness and the equicontinuous set $\mathcal{E}$ of the uniform limit dynamical system have been investigated.

AMS Classification: 37B20, 37B55, 54H20
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1. INTRODUCTION

By a dynamical system we mean a pair (X, f); where X is a compact metric space with metric d, and f is a continuous self map on X. We consider X as an infinite compact metric space without isolated point. The uniform limit f of a uniformly convergent sequence (f_n) of continuous functions is continuous. Thus, the metric space (X, d), together with the uniform limit f constitute a dynamical system. It is interesting to investigate the dynamical properties inherited by the uniform limit from the sequence. Indeed, if (X, f_n) is a sequence of dynamical systems such that each member of the sequence has dynamical property P, then it is interesting to know whether the uniform limit (X, f) has this property P or not. In this section we shall define some terminologies of dynamical systems. The set $O(f, x) = \{f^n(x) : n \in N\}$ is called orbit of the point $x \in X$. In dynamical system, the main interesting task is to investigate the asymptotic orbit behaviour of each point. A point $x \in X$ of a dynamical system (X, f) is said to be periodic point if there exists a positive integer n such that $f^n(x) = x$. The least such positive integer is called the period of x. A point $x \in X$ of a dynamical system (X, f) is called a fixed point if $f(x) = x$. Thus a fixed point is a periodic point of period one. Let (f_n) be a sequence of continuous self maps on a metric space (X, d). Then f_n is called orbitally convergent[6] to a map $f : X \longrightarrow X$ if, for every $\epsilon > 0$, there exists $k \in N$, such that $d(f_n^m(x), f^m(x)) < \epsilon$, for all $x \in X$, for all $m \in N$ and for all $n \geq k$. A dynamical system (X, f) is said to be transitive if, for each pair U, V of non-empty open subsets of X, there exists a positive integer n such that $f^n(U) \cap V \neq \phi$. It is also equivalent to say that there exists a point $x \in X$ such that $\overline{O(f, x)} = X$. In this case the point x is called a transitive point. The set of all transitive points is denoted by trf. Das and Das [14] studied the topological transitivity of a sequence of transitive maps under group action. The limit function of a uniformly convergent sequence of topological transitive functions need not be topological transitive. But the limit f of a sequence f_n of topological transitive functions converging orbitally to f, is topological transitive [6]. It has also been shown in [13] that a sequence f_n of continuous and transitive maps that converge uniformly to f, is not necessarily topologically transitive. Raghib et al., have also given sufficient condition under which uniform limit function is topological transitive. Kexong et al., discussed a sufficient condition for the transitivity of the uniform limit of a sequence of a transitive system [18]. In [12] Raghib and Kifah have discussed a sufficient condition in which the uniform limit is transitive. They also prove that if (f_n) is a sequence of continuous function converging uniformly to a function f, then the sequence

$f_n^p \xrightarrow{uniformly} f^p$ for each p. Also in [7] there is a sufficient condition for the transitivity of the limit function. In the sense of Devaney, a dynamical system (X, f) is said to be chaotic if it is topological transitive and has a dense set of periodic points [4]. If $f_n \xrightarrow{uniformly} f$, in general there is no relation between chaotic behavior of the non-autonomous system generated by f_n and the chaotic behavior of f [5]. In [16], authors investigated chaos for a sequence of maps in a metric space. Transitivity on interval maps is equivalent to chaotic in Devaney sense if the map has a dense set of periodic points ([1, 15, 17]). Many results of transitivity can be found in [8]. A dynamical system (X, f) is said to be topological weak mixing, if the product system $(X \times X, f \times f)$ is transitive. A transitive dynamical system may not be weak mixing; or product of transitive dynamical system may not be transitive [10]. A dynamical system (X, f) is said to be topological mixing if for each pair U, V of non-empty open subsets, there exists a positive integer m such that $f^n(U) \cap V \neq \phi$ for all $n \geq m$. Mangang[11] has shown that the uniform limit function of a sequence of topological mixing functions is not necessarily topological mixing. Further, he gave a sufficient condition so that the uniform limit is topological mixing. A subset $A \subset X$ is said to be an invariant subset if $f(A) \subset A$. The orbit closure $\overline{O(f, x)}$ of a point $x \in X$ is an invariant set of the system (X, f). A system (X, f) is said to be minimal if there is no proper closed invariant subset of (X, f). Every point of a minimal system is a transitive point. The ω-limit set $\omega(x)$ of x is the set of all limit points of $O(x, f)$. Equivalently, $\omega(x) = \bigcap_{n\in N} \overline{\bigcup_{i\geq n} f^i(x)}$. A point $x \in X$ is said to be recurrent point for f if $x \in \omega(x)$, i.e x belongs to its ω- limit set. Equivalently, the point x is a recurrent point, if for all $\epsilon > 0$, there exists a positive n, such that $d(f^n(x), x) < \epsilon$. $R(f)$ denotes the set of all recurrent point of f. Periodic point is also a recurrent point. A point $x \in X$ ϵ- shadows a finite sequence $x_1, x_2, ...x_n$ if $d(f^i(x), x_i) < \epsilon$ for all $i < n$. A finite or infinite sequence $(x_i)_{i\geq 0}$ is said to be a δ- chain if $d(f(x_i), x_{i+1}) < \delta$ for all i. A dynamical system (X, f) is said to have pseudo orbit tracing property(POTP) if, for $\epsilon > 0$, there exists $\delta > 0$, such that any finite δ-chain is ϵ-shadowed by some point. A dynamical system (X, f) is said to be uniformly rigid, if for each $\epsilon > 0$ and for each $x \in X$, there exists a positive integer n such that $d(x, f^n(x)) < \epsilon$. A dynamical system (X, f) is said to be sensitive if, for all $x \in X$ and for all $\delta > 0$, there exists $\epsilon > 0$ such that $d(f^j(x), f^j(y)) \geq \epsilon$ for some $y \in B_\delta(x)$ and $j \geq 0$. A bijective dynamical system (X, f) is expansive if there is $\epsilon > 0$ such that for each pair of distinct points $x, y \in X$, there exists a positive integer j with $d(f^j(x), f^j(y)) \geq \epsilon$. A dynamical system (X, f) is positively expansive, if for each $\epsilon > 0$ and for each pair x, y of distinct points, there exists a positive integer n such that $d(f^n(x), f^n(y)) \geq \epsilon$. Let (X, f) be a dynamical system. A subset $Y \subseteq X$ is called an attractor if it is non-empty closed, $f(Y) = Y$ and for each $\epsilon > 0$, there exists $\delta > 0$, such that for any $x \in X$, $d(x, y) < \delta \Rightarrow d(f^n(x), Y) < \epsilon$ for all $n \geq 0$ and $\lim_{n\to\infty} d(f^n(x), Y) = 0$. Let (X, f) be a dynamical system. A point $x \in X$ is said to be equicontinuous if for each $\epsilon > 0$, there exists $\delta > 0$ such that $y \in B_\delta(x) \Rightarrow d(f^n(x), f^n(y)) < \epsilon$, for all $n \geq 0$. The set of all equicontinuous points is denoted by $\mathcal{E}$. A dynamical system is said to be equicontinuous if $\mathcal{E} = X$. We say that (X, f) is almost equicontinuous if $\mathcal{E}$ is dense in X. Mangang(2014) found that the limit function of an orbitally convergent sequence of equicontinuous function is equicontinuous. Detail discussion of equicontinuity, sensitivity and expansiveness are found in [2],[3], [9] and [10].

In this paper, we investigate the attractor set of the uniform limit dynamical system. If (X, f_n) is a sequence of dynamical system having A as attractor for each $n \geq 1$, then the uniform limit dynamical system (X, f) has A as its attractor. If (X, f_n) is a sequence of dynamical systems converging uniformly to the dynamical systems (X, f); and if each dynamical system (X, f_n) has same equicontinuous set $\mathcal{E}$, then we show that (X, f) has same equicontinuous set $\mathcal{E}$. We also prove that if (X, f_n) is a sequence of positively expansive dynamical system converging to (X, f), then (X, f) is also positively expansive. If a sequence (f_i) of dynamical system converging uniformly to a function f and each system (X, f_n) has pseudo orbit tracing property, then f also has pseudo orbit tracing property. In a sequence of dynamical system (X, f_n), if each system has the same recurrent point x; and $f_n \xrightarrow{uniformly} f$, then (X, f) has x as its recurrent point. If (X, f_n) is uniformly rigid converging uniformly to (X, f), then (X, f) is uniformly rigid dynamical system.

2. Theorems

Theorem 2.1. *Let (f_i) be a sequence of dynamical systems having pseudo orbit tracing property and converging uniformly to a function f. Then f has pseudo orbit tracing property (POTP).*

Proof. Let $\epsilon > 0$ be an arbitrary number. For each $j \geq 1$, there exists $\delta_j > 0$, such that every δ_j-chain (x_i) is $\epsilon-$shadowed by some z_j. i.e $d(f_j^i(z_j), x_i) < \frac{\epsilon}{2}$. We have a sequence (z_j). Since X is a compact metric space, it converges to a point $z \in X$. Let $\delta = inf\{\delta_j : j \geq 1\}$. Let (y_i) be a δ-chain. Then, $d(f_j(y_i), y_{i+1}) < \delta < \delta_j$, $\forall\ j \geq 1, i \geq 1$. This shows that every δ- chain is a δ_j- chain, for each $j \geq 1$. Therefore, $d(f_j^i(z_j), y_i) < \frac{\epsilon}{2}$, $\forall\ i \geq 1, j \geq 1$. ——(1)

By triangle inequality, we get,

$d(f^i(z), y_i) \leq d(f^i(z), f_j^i(z_j)) + d(f_j^i(z_j), y_i)$, $\forall\ i \geq 1, j \geq 1$. ——(2).

We know that, $f_j^i \xrightarrow{uniformly} f^i$, then there exists N, such that

$d(f_j^i(z_j), f^i(z_j)) < \frac{\epsilon}{4}$, $\forall\ j \geq N$. ——(3)

Since, z_j converges to z and f^i is continuous, there exists N_1, such that

$d(f^i(z_j), f^i(z)) < \frac{\epsilon}{4}$, $\forall\ j \geq N_1$. ——(4)

Using (3)and (4) we get

$d(f_j^i(z_j), f^i(z)) \leq d(f_j^i(z_j), f^i(z_j)) + d(f^i(z_j), f^i(z)) < \frac{\epsilon}{2}$. ——(5)

From (1), (2) and (5), we get

$d(f^i(z), y_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, $\forall\ i \geq 1$.

Hence, f has pseudo orbit tracing property. □

Theorem 2.2. *Let (X, f_i) be a sequence of dynamical systems such that each member of the sequence has a common recurrent point x, and converging uniformly to (X, f). Then x is also a recurrent point of (X, f).*

Proof. Let $B_\epsilon(x)$ be an open ball of radius $\epsilon > 0$. Then for $i \geq 1$, there exists $n \geq 1$, such that $f_i^n(x) \in B_{\frac{\epsilon}{2}}(x)$. Since $f_i^n \xrightarrow{uniformly} f^n$, there exists $N \geq 1$, such that $d(f_i^n(x), f^n(x)) < \frac{\epsilon}{2}$, $\forall\ i \geq N$.

Now, $d(f^n(x), x) \leq d(f^n(x), f_i^n(x)) + d(f_i^n(x), x) < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Therefore, $f^n(x) \in B_\epsilon(x)$. This shows that, x is a recurrent point of f. Therefore, $\bigcap_{i=1}^{\infty} R(f_i) \subset R(f)$. □

Theorem 2.3. *Let (X, f_i) be a sequence of uniformly rigid dynamical systems converging uniformly to f. Then f is uniformly rigid.*

Proof. $f_i \xrightarrow{uniformly} f$. For each $\epsilon > 0$, there exists N_1,

such that $d(f_i(x), f(x) < \epsilon$, for all $i \geq N_1$ and for all $x \in X$. Also, $f_i^n \xrightarrow{uniformly} f^n$. Therefore for each $\epsilon > 0$, there exists N_2, such that $d(f_i^n(x), f^n(x)) < \frac{\epsilon}{2}$, for all $i \geq N_2$ and for all $x \in X$. It is given that (X, f_i) is uniformly rigid for each $i \geq 1$.

Therefore, for each $\epsilon > 0$ there exists positive integer n, such that $d(x, f_i^n(x)) < \frac{\epsilon}{2}$ for all $x \in X$. Now, $d(x, f^n(x)) \leq d(x, f_i^n(x)) + d(f_i^n(x), f^n(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Therefore, $d(x, f^n(x)) < \epsilon$. This shows that, (X, f) is uniformly rigid. □

Theorem 2.4. *Let (X, f_i) be a sequence of positively expansive dynamical systems converging uniformly to (X, f). Then (X, f) is positively expansive.*

Proof. (X, f_i) is positively expansive. Therefore, for each $\epsilon > 0$ and for each pair x, y of distinct points, there exists a positive integer m such that $d(f_i^m(x), f_i^m(y)) \geq \frac{3\epsilon}{2}$. We know that $f_i^m \xrightarrow{uniformly} f^m$. Then, there exists a positive integer N, such that $d(f_i^m(x), f^m(x)) < \frac{\epsilon}{4}$, for all $i \geq N$ and for all $x \in X$. By triangle inequality, we have;
$d(f_i^m(x), f_i^m(y)) \leq d(f_i^m(x), f^m(x)) + d(f^m(x), f^m(y)) + d(f^m(y), f_i^m(y))$
It implies that $\frac{3\epsilon}{2} \leq d(f_i^m(x), f_i^m(y)) < \frac{\epsilon}{4} + d(f^m(x), f^m(y)) + \frac{\epsilon}{4}$. Therefore, $d(f^m(x), f^m(y)) > \epsilon$. Hence, f is positively expansive. □

Theorem 2.5. *Let (X, f_i) be a sequence of dynamical systems converging uniformly to f. Let each dynamical system (X, f_i) has a common attractor $Y \subseteq X$. Then Y is also an attractor for (X, f).*

Proof. For each $i \geq 1, (X, f_i)$ has the attractor Y. Therefore for any $\epsilon > 0$ there exists $\delta_i > 0$, such that for any $x \in X$, with $d(Y, x) < \delta_i$ it follows that $d(f_i^m(x), Y) < \frac{\epsilon}{2}$, $\forall\ m \geq 0$, and $\lim\limits_{m\to\infty} d(f_i^m(x), Y) = 0$.
For each $m \geq 1$, there exists $N_m > 0$, such that

$$d(f_i^m(x), f^m(x)) < \frac{\epsilon}{2}\ \forall\ i \geq N_m.$$

$d(Y, f^m(x)) \leq d(Y, f_i^m(x)) + d(f_i^m(x), f^m(x)) < \epsilon$, whenever $d(Y, x) < \delta_i$, and $i \geq N_m$. Therefore, $d(Y, f^m(x)) < \epsilon$, $\forall\ m \geq 0$, whenever $d(Y, x) < \delta$, where $\delta = inf\{\delta_i : i \geq 1\}$. Now, $\lim\limits_{m\to\infty} d(f_i^m(x), Y) = 0$. Therefore, for $\epsilon > 0$, there exists M_i, such that $d(f_i^m(x), Y) < \epsilon$, $\forall\ m \geq M_i$. Now $\{M_i : i \geq 1\}$ is a bounded set. Let $M = sup\{M_i : i \geq 1\}$. Then, $d(f_i^m(x), Y) < \epsilon$, $\forall\ m \geq M$ and $\forall\ i \geq 1$. Therefore, $d(f^m(x), Y) < \epsilon$, $\forall\ m \geq M$. Since $\epsilon > 0$ was chosen arbitrarily, we have $\lim\limits_{m\to\infty} d(f^m(x), Y) = 0$.
Hence, Y is an attractor for (X, f). □

Theorem 2.6. *Let (X, f_i) be a sequence of dynamical systems converging uniformly to the dynamical systems (X, f). Suppose each dynamical system (X, f_i) has the common equicontinuous set $\mathcal{E}$, then $\mathcal{E}$ is contained in the equicontinuous set for (X, f).*

Proof. For a given $\epsilon > 0$, there exists $N_m > 0$, such that $d(f_i^m(x), f^m(x)) < \frac{\epsilon}{3}$, $\forall\ i \geq N_m$, $\forall\ x \in X$. Let $x \in \mathcal{E}$, then there exists $\delta > 0$, such that $d(f_i^m(x), f_i^m(y)) < \frac{\epsilon}{3}$ whenever $d(x, y) < \delta, \forall\ m \geq 0, i \geq 0$. By triangle inequality, we have $d(f^m(x), f^m(y)) \leq d(f^m(x), f_i^m(x)) + d(f_i^m(x), f_i^m(y)) + d(f_i^m(y), f^m(y))$,$\forall\ i \geq N_m, m \geq 0$. It follows that $d(f^m(x), f^m(y)) < \epsilon, \forall m \geq 0$. Therefore, $\mathcal{E}$ is contained in the equicontinuous set of (X, f). □

References

[1] L.S. Block and W.A. Coppel, *Dynamics in one dimension*, Springer Verlag, 1992.

[2] M. Brin and Garrett Stuck, *Introduction to dynamical systems*, Cambridge University Press, 2002.

[3] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic theory on compact spaces*, Lecture Notes in Math., **527**, Springer, (1977).

[4] R. Devaney, *An introduction to chaotic dynamical systems*, 2nd ed., Westview, 2003.

[5] J. Dvorakova, *Chaos in non-autonomous discrete dynamical systems*, Communications in Non-linear Svience and Numerical aimulation , **17**(2012), 4649-4652.

[6] A. Fedeli and A.L. Donne, *A note on the uniform limit of transitive dynamical systems*, Bull, Belg. Math. Soc.Simen Stevin, **16**(2009), 59-66.

[7] R.F. Heriberto, *Uniform convergence and transitivity*, Chaos, Solitons and Fractal, **38**(2006), 148-153.

[8] S. Kolyada and L. Snoha, *Some aspects of topololgical transitivity-a survey*, Grazer. Math. Ber., **334**,3(1997).

[9] P. Kurka, *Topological and symbolic dynamics*, Societe Mathematique de France, 2003.

[10] K.B. Mangang, *Equicontinuity of the limit function of a sequence of equicontinuous functions*, Journal of Indian Math. Soc., **81**(1-2)(2014), 115-121.

[11] K.B Mangang, *Topological mixing of the limit function of a sequence of dynamical systems*, Bull, Cal. Math. Soc., **109**(1)(2017), 45-54.

[12] A.S. Raghib, A.S and A.H. Kifah, *Uniform convergence and chaotic behaviour*, Nonlinear Analysis, **65**(2006), 993.

[13] A.S. Raghib, M.G. Felix and P. Alfredo, *Erratum to uniform convergence and chaotic behaviour*,[Nonlinear Analysis, **65**(4)(2006), 933-937], Non-linear Analysis, **68**(2008), 1406-1407.

[14] D. Ruchi and D. Tarun, *Topological transitivity of uniform limit functions on G-spaces*, Int. Journal Math. Analysis, **6**(2012), 1491-1499.

[15] A.N. Sharkovsky, *Nonwandering points and the centre of a continuous mapping of the line in itself*, Dopovidi Ukrain. Acad. Sci.,**7**(1964), 865.(in Ukrainian)

[16] C. Tian and G. Chen, *Chaos of sequence of maps in a metric space*, Chaos, Solitons and Fractals, **28**(2006), 1067-1075.

[17] M. Vellokoop and R. Berlund, *On interval, transitivity = Chaos*, Amer. Math. Monthly, **101**(1994), 353.

[18] K. Yang, F. Zeng and G. Zhang, *Devaney's chaos on uniform limit maps*, Chaos, Solitons, Fractals, **44**(2011), 522-525.