

C^0 –SYMPLECTIC GEOMETRY UNDER DISPLACEMENTS

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ABSTRACT. This paper continues the study of the group $Hameo(M, \omega)$, of all Hamiltonian homeomorphisms of a closed symplectic manifold (M, ω) . After given a direct proof of the positivity result of the symplectic displacement energy, we show that the uniqueness theorem of generators of strong symplectic isotopies extends to any closed symplectic manifold: An explicit formula for the mass flow of any strong symplectic isotopy with respect to its generator is given. We show that $Hameo(M, \omega)$ inherits under the C^0 –Hamiltonian topology, the fragmentation property, the algebraic perfectness, and coincides with the commutator sub-group of the group of all strong symplectic homeomorphisms. This solves a Banyaga’s conjecture, and some other conjectures are also formulated.

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1. INTRODUCTION

The group $Hameo(M, \omega)$, of all Hamiltonian homeomorphisms of a closed symplectic manifold (M, ω) is known as a topological counterpart of the well-known

group $Ham(M, \omega)$, of all smooth Hamiltonian diffeomorphisms of (M, ω) [13]. In almost the same spirit, the group of strong symplectic homeomorphisms was defined and studied in [3]. For short, arguing with Theorem 3.3–[15], a strong symplectic (resp. a Hamiltonian) homeomorphism is nothing than an element of a certain non-trivial topological completion of the usual identity component $G_\omega(M)$ of the group of all symplectomorphisms of (M, ω) via some "kind" of twisted C^0 -metric, called "symplectic topology" (resp. "Hamiltonian topology"). Such homeomorphisms can be smooth or not, and since they belong to a group $G_\omega^0(M)$ that contains $G_\omega(M)$ (see [2, 3, 4]), it seems natural to think of the study of the spreading of the geometric, algebraic and topological properties of the sub-group $G_\omega(M) \subseteq G_\omega^0(M)$ over the ambient group $G_\omega^0(M)$. Many contributions in this sense have been done [6, 7, 8, 13, 14, 18]. But, so far, is still an open question to know whether the sub-group $Hameo(M, \omega) \subseteq G_\omega^0(M)$ is algebraically simple or not (a property that has the group $Ham(M, \omega)$). Idem, the algebraic perfectness and the fragmentation property for $Hameo(M, \omega)$ are still open questions.

Therefore, the main goal of the present note is to capture some possible affirmative answers of some of the above open questions. To that end, we prove the following results for any closed symplectic manifold (M, ω) .

Theorem 1.1. *Any strong symplectic isotopy determines a unique generator.*

Theorem 1.1 was proved in [7] for Lefschetz type closed symplectic manifolds. This restriction was caused by the technique of the proof of Lemma 4.5-[7] which has main ingredient the Fathi's Poincaré duality. When the manifold is simply connected, Theorem 1.1 is a result that was proved by Viterbo [18], and later by Buhovsky-Seyfaddini [8].

In order to introduce the second result of this note, let us recall that in the theory of smooth symplectic geometry, it is known that the Fathi's Poincaré duality result states that the flux of a volume-preserving isotopy determines its mass flow, while a factorization result found in [2] shows how the flux induced the symplectic form is linked with that induced by the corresponding symplectic volume form. These facts combined with the definition of strong symplectic isotopies, motivated the following result.

Theorem 1.2. *Any strong symplectic isotopy λ generated by $(U, \mathcal{H})$, has mass flow:*

$$(1.1) \quad \tilde{\theta}(\lambda)(f) = \frac{1}{(n-1)!} \int_M \left(\left(\int_0^1 \mathcal{H}_t dt \right) \wedge \omega^{(n-1)} \wedge f^*(\sigma) \right),$$

for all continuous mapping f from M to the circle $\mathbb{S}^1$, where σ is the volume form on $\mathbb{S}^1$, and $\tilde{\theta}$ is the Fathi's mass flow homomorphism.

In particular, Theorem 1.2 shows that the harmonic component of the generator of a strong symplectic isotopy completely determines its Fathi's mass flow. In the theory of smooth symplectic geometry, Theorem 1.2 is equivalent to the Fathi's Poincaré duality result. As a corollary, we have that any continuous Hamiltonian flow has a trivial Fathi's mass flow, and therefore the group of all Hamiltonian homeomorphisms of (M, ω) is contained in the kernel of the Fathi's mass flow (a well-known fact).

Here is one of the main results of this note.

Theorem 1.3. *The group $Hameo(M, \omega)$ coincides with the commutator sub-group of the group $G_\omega^0(M)$, of all strong symplectic homeomorphisms of (M, ω) .*

Theorem 1.3 gives an affirmative answer of a conjecture formulated in [3]. This can be seen as a topological counterpart of a well-known result from the usual smooth symplectic geometry which states that $[G_\omega(M), G_\omega(M)] = Ham(M, \omega)$. The proof of the inclusion $[G_\omega^0(M), G_\omega^0(M)] \subseteq Hameo(M, \omega)$, can be found in [3]. This follows from straightforward calculations. For the completeness of the proof of Theorem 1.3, we will only prove the inclusion $[G_\omega^0(M), G_\omega^0(M)] \supseteq Hameo(M, \omega)$. The strategy of the proof of this result is the following: Assuming that $Hameo(M, \omega)$ is not contained in the commutator sub-group $[G_\omega^0(M), G_\omega^0(M)]$, we shall exhibit a countable family $\{\mathcal{B}_j\}_{j \in \mathbb{N}}$, in which each element $\mathcal{B}_j$ is a non-empty open subsets of M , and then find out a positive integer δ (possibly large enough, but still an integer) such that

$$(1.2) \quad 0 < e_S(\mathcal{B}_\delta) \leq 0,$$

without appealing to any limit: Relation (1.2) is a contradiction. Here, e_S is the usual symplectic displacement energy, and this will still hold if we apply the capacity inequality theorem.

When M is simply connected, then $G_\omega^0(M) = \text{Hameo}(M, \omega)$. So, Theorem 1.3 reduces to the perfectness of $\text{Hameo}(M, \omega)$, i.e.,

$$[\text{Hameo}(M, \omega), \text{Hameo}(M, \omega)] = \text{Hameo}(M, \omega).$$

This motivated the following fact.

Theorem 1.4. *The group $\text{Hameo}(M, \omega)$ is algebraically perfect.*

On the other hand, according to [2], given any open cover $\mathcal{U} = \{U_i\}_{1 \leq i \leq k}$ of M and any Hamiltonian diffeomorphism ψ , one can always find a family of Hamiltonian diffeomorphisms $\{\phi_i\}_{1 \leq i \leq k}$, with each ϕ_i supported in U_i such that $\psi = \phi_1 \circ \cdots \circ \phi_k$.

Definition 1.5. The support of a homeomorphism h , denoted $\text{supp}(h)$, is the closure of the set $\{x \in M : h(x) \neq x\}$.

As we shall see later, the above smooth fragmentation property together with the lines of the proof of Theorem 1.3 suggest that $\text{Hameo}(M, \omega)$ inherits a fragmentation property from $\text{Ham}(M, \omega)$ through the C^0 -Hamiltonian topology.

Proposition 1.6. *Given any open cover $\mathcal{U} = \{U_i\}_{1 \leq i \leq k}$ of M , and any Hamiltonian homeomorphism ψ , one can always find out a family of Hamiltonian homeomorphisms $\{\gamma_i\}_{1 \leq i \leq k}$, with each γ_i supported in U_i such that $\psi = \gamma_1 \circ \cdots \circ \gamma_k$.*

We organize the paper as follows: In the second Section, we recall the definitions of symplectic paths and their generators, the definition of the Hofer-like norm and the definition of the symplectic displacement energy. Therefore, we use Theorem 3.3–[15] to give a short proof of the positivity of the symplectic displacement energy (Theorem 2.2). Next, we revisit the construction of the Poincaré’s pairing, the construction of the flux homomorphism, and the construction of the Fathi’s mass flow; including some well-known results from flux geometry that we shall need. The third Section contains the definition of strong symplectic isotopies and their generators; including some useful notions and results found in [7, 8] that we shall need. In this section, in order to give a complete proof of Theorem 1.1, we prove a lemma (Lemma 3.8) that generalizes both Lemma 4.5-[7] and Lemma 4.6-[7]. Finally, the last Section contains a study of the mass flow for strong symplectic homeomorphisms: We recover a well-known result starting that the group of all Hamiltonian homeomorphisms is contains in the kernel of the Fathi’s mass flow.

This section includes the complete proofs of Theorem1.2, Theorem1.3, Theorem1.4, and Proposition 1.6.

2. PRELIMINARIES

Let M be an n -dimensional smooth closed manifold (i.e., a smooth compact, connected manifold without boundary). A differential form Ω on M of degree n that vanishes nowhere is called a volume form. In this situation the manifold (M, Ω) is said to be oriented. A diffeomorphism $\phi : M \rightarrow M$ is said to preserve a smooth differential q -form α on M if $\phi^*(\alpha) = \alpha$. Given a q -form α on M , we shall denote by $Diff^\infty(M, \alpha)$ the group of all diffeomorphisms of M that preserve α , and by $Iso(M, \alpha)$ we denote the group consists of all smooth paths $\Phi : [0, 1] \rightarrow Diff^\infty(M, \alpha)$, with $\Phi(0) = Id_M$ (see [2, 11]).

More often, in this paper, we will assume that M is a closed symplectic manifold of dimension $2n$, with symplectic form ω , and denote by Ω_0 the corresponding symplectic volume form. In this case the group $Diff^\infty(M, \omega)$ is usually denoted $Symp(M, \omega)$, and called the group of all symplectomorphisms of (M, ω) . Let $G_\omega(M)$ denote the group consists of all time-one maps of isotopies in $Iso(M, \omega)$, and $\widetilde{G_\omega(M)}$ the universal cover of $G_\omega(M)$ [2]. The sub-group of $G_\omega(M)$ formed by all the Hamiltonian diffeomorphisms of (M, ω) is denoted $Ham(M, \omega)$, and its universal cover is denoted $\widetilde{Ham(M, \omega)}$ (see [2, 17] for further details). We shall always assume that (M, ω) is a closed symplectic manifold of dimension $2n$ that carries a fixed Riemannian metric g .

2.1. Symplectic paths. The following notions can be found in [6]. Let $\Phi = \{\phi_t\}$ be a symplectic isotopy. The Hodge's decomposition of the closed 1-form $\iota(\dot{\phi}_t)\omega$ gives rise to a unique exact 1-form dU_t^Φ , and a unique harmonic 1-form $\mathcal{H}_t^\Phi$, where

$$(2.1) \quad \dot{\phi}_t(\phi_t(x)) := \frac{d}{dt}(\phi_t(x)),$$

for all $x \in M$, and for each t [4]. Denote by U the normalized Hamiltonian of $U^\Phi = (U_t^\Phi)$, i.e., for each t , we have

$$(2.2) \quad U_t = U_t^\Phi - \frac{\int_M U_t^\Phi \omega^n}{\int_M \omega^n},$$

and let $\mathcal{H}$, denote the smooth family of harmonic 1-forms $\mathcal{H}^\Phi = (\mathcal{H}_t^\Phi)$. The group $\mathfrak{T}(M, \omega, g)$, of all the pairs $(U, \mathcal{H})$ constructed as above is isomorphic to the group

$Iso(M, \omega)$ [6]. Furthermore, any symplectic isotopy Φ is denoted by $\phi_{(U, \mathcal{H})}$ to mean that Φ corresponds to $(U, \mathcal{H})$, and the pair $(U, \mathcal{H})$ is called the “generator” of the symplectic isotopy Φ . A symplectic isotopy $\phi_{(0, \mathcal{H})}$, is a harmonic isotopy, and a symplectic isotopy $\phi_{(U, 0)}$, is a Hamiltonian isotopy (see [6, 7, 15]). The product rule in $\mathfrak{T}(M, \omega, g)$ is given by,

$$(2.3) \quad (U, \mathcal{H}) \bowtie (V, \mathcal{K}) = (U + V \circ \phi_{(U, \mathcal{H})}^{-1} + \tilde{\Delta}(\mathcal{K}, \phi_{(U, \mathcal{H})}^{-1}), \mathcal{H} + \mathcal{K}).$$

The inverse of $(U, \mathcal{H})$, say $\overline{(U, \mathcal{H})}$ is given by,

$$(2.4) \quad \overline{(U, \mathcal{H})} = (-U \circ \phi_{(U, \mathcal{H})} - \tilde{\Delta}(\mathcal{H}, \phi_{(U, \mathcal{H})}), -\mathcal{H}).$$

In (2.3) and (2.4) the quantity $\tilde{\Delta}$ is defined as follows: for any symplectic isotopy $\Psi = \{\psi^t\}$, and for any smooth family of closed 1-forms $\alpha = (\alpha_t)$, we have $\Delta_t(\alpha, \Psi) := \int_0^t \alpha_t(\dot{\psi}^s) \circ \psi^s ds$, and $\tilde{\Delta}_t(\alpha, \Psi)$ is the normalized function of $\Delta_t(\alpha, \Psi)$, for all t (see [6])

2.2. Hofer-like metrics. Given a symplectic isotopy Φ generated by $(U, \mathcal{H})$, its the L^∞ –Hofer-like length is defined by,

$$(2.5) \quad l_{HL}^\infty(\Phi) = \max_{t \in [0, 1]} (osc(U_t) + |\mathcal{H}_t|),$$

where for each t , $osc(U_t) = \max U_t - \min U_t$, and $|\cdot|$ is a norm defined on the vector space of all harmonic 1-forms [4]. The L^∞ –energy of $\phi \in G_\omega(M)$ is defined as

$$e_0^\infty(\phi) = \inf(l_{HL}^\infty(\Phi)),$$

where the infimum is taken over the set of all symplectic isotopies Φ with time-one map equal to ϕ . Therefore, the L^∞ –Hofer-like norm of $\phi \in G_\omega(M)$ is defined as,

$$(2.6) \quad \|\phi\|_{HL}^\infty = (e_0^\infty(\phi) + e_0^\infty(\phi^{-1}))/2.$$

Note that the norm $\|\cdot\|_{HL}^\infty$ induces a right-invariant metric on $G_\omega(M)$, called Hofer-like metric defined by

$$(2.7) \quad d_{HL}(\phi, \psi) := \|\phi \circ \psi^{-1}\|_{HL}^\infty,$$

for all $\phi, \psi \in G_\omega(M)$.

2.2.1. *The L^∞ -Metric on $\mathfrak{T}(M, \omega, g)$.* The L^∞ -metric topology on $\mathfrak{T}(M, \omega, g)$, denoted D^2 , is defined as follows:

$$(2.8) \quad D^2((U, \mathcal{H}), (V, \mathcal{K})) = \left(D_0((U, \mathcal{H}), (V, \mathcal{K})) + D_0(\overline{(U, \mathcal{H})}, \overline{(V, \mathcal{K})}) \right) / 2,$$

where,

$$(2.9) \quad D_0((U, \mathcal{H}), (V, \mathcal{K})) = \max_{t \in [0, 1]} (\text{osc}(U_t - V_t) + |\mathcal{H}_t - \mathcal{K}_t|).$$

(see [4, 15] for more details, or see [16] for a study of various Hofer-like metrics).

2.3. Symplectic displacement energy.

Definition 2.1. (Banyaga-Hurtubise-Spaeth, [5]) The symplectic displacement energy $e_S(A)$ of a non-empty open subset $A \subset M$ is defined as:

$$e_S(A) = \inf \{ \|\phi\|_{HL} \mid \phi \in G_\omega(M), \phi(A) \cap A = \emptyset \}.$$

The quantity $\|\cdot\|_{HL}$ is any version of the Hofer-like norms (see [4, 15]).

Theorem 2.2. *For any non-empty open subset $A \subset M$, its symplectic displacement energy $e_S(A)$, is a strict positive number.*

Theorem 2.2 was proved by Banyaga-Hurtubise-Spaeth [5]: Their given proof is technical, and one of its ingredients is a Polterovich's trick (see [5]). It occupies the main part of their paper [5]. In the present paper, using the following theorem that was recently generalized to various type of Hofer-like metrics (see [16]), we provide a short and direct alternate proof of Theorem 2.2.

Theorem 2.3. (Tchuiaga, [15]) *Let $\{\phi_i^t\}$ be a sequence of symplectic isotopies, let $\{\psi^t\}$ be another symplectic isotopy, and let $\phi : M \rightarrow M$ be a map such that*

- (ϕ_i^1) converges uniformly to ϕ , and
- $l_{HL}^\infty(\{\psi^t\}^{-1} \circ \{\phi_i^t\}) \rightarrow 0, i \rightarrow \infty$.

Then we must have $\phi = \psi^1$.

As we said above, Theorem 2.3 was recently generalized to various type of Hofer-like metrics (see [16]) without appealing to the positivity of any symplectic displacement energy. In fact, the results of the paper [15] follow from a particular choice of a linear section in [16]. Therefore, the given proof of our Theorem 2.2 seems to

suggest that, if one defines the displacement energies with respect to various Hofer-like metrics studied in [16], then they could be positive: This still need to be proved.

Proof of Theorem 2.2. Let us assume that there exists a non-empty open subset $\mathcal{O} \subset M$ such that $e_S(\mathcal{O}) = 0$. That is, for each positive integer l , there exists a symplectic diffeomorphism ψ_l , isotopic to the identity such that

$$(2.10) \quad 2e_0^\infty(\psi_l) \leq \|\psi_l\|_{HL}^\infty < 1/l,$$

and

$$(2.11) \quad \psi_l(\overline{B}) \cap \overline{B} = \emptyset,$$

for any non-empty open ball B such that $B \subset \overline{B} \subset \mathcal{O}$. From the definition of the Hofer-like norm, we also derive that for each positive integer l (fixed), there exists a sequence of symplectic isotopies $\{\Psi_{i,l}\}$, each of which with time-one map ψ_l such that

$$(2.12) \quad l_{HL}^\infty(\Psi_{i,l}) < e_0^\infty(\psi_l) + 1/i < (1/2l) + (1/i),$$

for all i . Now, for each l , extract a subsequence $\{\Psi_{i,l}\}_{(i \geq l)}$ of the sequence $\{\Psi_{i,l}\}$. Assume this done. Since for all i , the time-one map of $\Psi_{i,l}$ coincides everywhere with ψ_l , then the convergence $\Psi_{i,l}(1) \rightarrow \psi_l, i \rightarrow \infty$, is uniform. So, as l tends to infinity ($i \geq l$), then

- $\Psi_{i,l}(1) \xrightarrow{C^0} \psi_l$, and
- $l_{HL}^\infty(\{Id\}^{-1} \circ \Psi_{i,l}) = l_{HL}^\infty(\Psi_{i,l}) \rightarrow 0$,

Thus, it follows from Theorem 2.3 that as l tends to infinity, then ψ_l tends to Id_M . Therefore, we have proved that as l tends to infinity, the sequence of compact subsets $\psi_l(\overline{B})$ tends to $\overline{B}$: This is in contradiction with (2.11). Q.E.D $\diamond$.

2.4. Poincaré duality. The bilinear map,

$$\mathbb{I} : H^k(M, \mathbb{R}) \times H^{(2n-k)}(M, \mathbb{R}) \rightarrow \mathbb{R}, ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta,$$

is well defined, that is to say $\mathbb{I}([\alpha], [\beta])$ doesn't depend on the choice of representatives in the cohomology classes $[\alpha]$ or $[\beta]$ (this is an easy application of the Stokes formula). Moreover this bilinear form provides an isomorphism between $H^k(M, \mathbb{R})$ and the dual space of $H^{(2n-k)}(M, \mathbb{R})$. In particular, when $\alpha \in C^\infty(\wedge^k T^*M)$ is

a closed form and satisfies $\mathbb{I}([\alpha], [\beta]) = 0$, for all $[\beta] \in H^{(2n-k)}(M, \mathbb{R})$, then there exists $\theta \in C^\infty(\wedge^{(k-1)}T^*M)$ such that $\alpha = d\theta$.

2.5. The flux. Given a q -form α of a closed manifold M , together with a smooth path (h_t) in $Diff^\infty(M)$ such that $h_0 = id_M$ and $h_1 = h$, we have

$$(2.13) \quad h^*(\alpha) - \alpha = d\beta, \text{ with } \beta = \int_0^1 h_t^*(i_{\dot{h}_t}\alpha)dt.$$

The $(q-1)$ -form β is not uniquely determined by (2.13) in the sense that any other smooth isotopy (H_t) in $Diff^\infty(M)$ with the same endpoints than (h_t) satisfies the equation

$$d\left(\int_0^1 h_t^*(i_{\dot{h}_t}\alpha)dt\right) = d\left(\int_0^1 H_t^*(i_{\dot{H}_t}\alpha)dt\right).$$

If $h \in Diff^\infty(M, \alpha)$, then (2.13) implies that the $(q-1)$ -form $\left(\int_0^1 h_t^*(i_{\dot{h}_t}\alpha)dt\right)$ is closed, and hence defines an element in $H^{(q-1)}(M, \mathbb{R})$ (see [2]). This gives rise to a surjective group homomorphism $\tilde{S}_\alpha$ (see [2]), defined as follows,

$$(2.14) \quad \tilde{S}_\alpha: Iso(M, \alpha) \rightarrow H^{(q-1)}(M, \mathbb{R}), (h_t) \mapsto \left[\int_0^1 h_t^*(i_{\dot{h}_t}\alpha)dt\right].$$

When (M, ω) is a closed symplectic manifold, and α is its corresponding symplectic volume form Ω_0 , the study of the invariant $\tilde{S}_{\Omega_0}$ can be captured in the following proposition.

Proposition 2.4. (*Tchuiaga*, [15]) *Let $\mathcal{H} = (\mathcal{H}_t)$ be a smooth family of closed 1-forms, and let $\Phi = \{\phi_t\}$ be a symplectic isotopy. Then for each $t \in [0, 1]$, we have*

$$(2.15) \quad \int_M \Delta_t(\mathcal{H}, \Phi)\omega^n = n!\mathbb{I}(\tilde{S}_{\Omega_0}(\bar{\Phi}_t), [\mathcal{H}_t]),$$

where $\bar{\Phi}_t$ is the isotopy $s \mapsto \phi_{st}$.

Furthermore, we get a homomorphism $\tilde{S}_\omega: \widetilde{G_\omega(M)} \rightarrow H^1(M, \mathbb{R})$ (see [2]); and denoting by Γ_ω the image of $\pi_1(G_\omega(M))$ under $\tilde{S}_\omega$, the homomorphism $\tilde{S}_\omega$ induces an epimorphism S_ω from $G_\omega(M)$ onto $H^1(M, \mathbb{R})/\Gamma_\omega$. The study of the invariants $\tilde{S}_\omega$ and S_ω let to the following deep results.

Theorem 2.5. (*Banyaga*, [2])

- (1) $Ham(M, \omega) = \ker S_\omega$.
- (2) The group $Ham(M, \omega)$ is perfect.

- (3) $\widetilde{Ham(M, \omega)} = [\widetilde{G_\omega(M)}, \widetilde{G_\omega(M)}]$.
 (4) $[G_\omega(M), G_\omega(M)] = Ham(M, \omega)$.

Here is a consequence of Theorem 2.5.

Theorem 2.6. (*Banyaga, [2]*)

- The first homology group $H_1(\widetilde{G_\omega(M)}, \mathbb{Z})$, of $\widetilde{G_\omega(M)}$ with coefficients in $\mathbb{Z}$ is isomorphic to $H^1(M, \mathbb{R})$.
- The first homology group $H_1(G_\omega(M), \mathbb{Z})$, of $G_\omega(M)$ with coefficients in $\mathbb{Z}$ is isomorphic to $H^1(M, \mathbb{R})/\Gamma_\omega$.

2.6. The C^0 –metric. Let $Homeo(M)$ be the homeomorphisms' group of M equipped with the C^0 –compact-open topology. This is the metric topology induced by the distance $d_0(f, h) = \max(d_{C^0}(f, h), d_{C^0}(f^{-1}, h^{-1}))$, with $d_{C^0}(f, h) = \sup_{x \in M} d(h(x), f(x))$, where d is a distance on M induced by the Riemannian metric g . On the space of all continuous paths $\varrho : [0, 1] \rightarrow Homeo(M)$ (such that $\varrho(0) = Id_M$), we consider the C^0 –topology as the metric topology induced by the metric $\bar{d}(\lambda, \mu) = \max_{t \in [0, 1]} d_0(\lambda(t), \mu(t))$. Let $\mathcal{P}(Homeo(M), Id_M)$ denote the space of all continuous paths $\varrho : [0, 1] \rightarrow Homeo(M)$ such that $\varrho(0) = Id_M$. The topology on $Homeo(M)$ induced by the above metric is separable, and a countable sub-basis for the open sets consists of the sets $\{h \in Homeo(M) : h(\bar{U}) \subset V\}$, where U and V range over a countable basis for the topology of M (see [11]).

2.7. The mass flow. Let μ be a "good measure" on the manifold M (e.g. the measure induced by the symplectic volume form). Let $Homeo_0(M, \mu)$ denotes the identity component in the group of measure preserving homeomorphisms $Homeo(M, \mu)$, and $\widetilde{Homeo_0(M, \mu)}$ its universal covering. For $[h] = [(h_t)] \in \widetilde{Homeo_0(M, \mu)}$, and a continuous $f : M \rightarrow \mathbb{S}^1$ map, we lift the homotopy $fh_t - f : M \rightarrow \mathbb{S}^1$ to a map $\overline{fh_t - f}$ from M onto $\mathbb{R}$. Fathi [10], proved that the integral $\int_M \overline{fh_t - f} d\mu$ depends only on the homotopy class $[h]$ of (h_t) and the homotopy class $\{f\}$ of f in $[M, \mathbb{S}^1] \approx H^1(M, \mathbb{Z})$, and that the map $\tilde{\theta}((h_t))(f) := \int_M \overline{fh_t - f} d\mu$, defines a homomorphism,

$$\tilde{\theta} : \widetilde{Homeo_0(M, \mu)} \rightarrow Hom(H^1(M, \mathbb{Z}), \mathbb{R}) \approx H_1(M, \mathbb{R}).$$

This map induces a surjective group morphism $\mathcal{F}$ from $Homeo_0(M, \mu)$ onto a quotient of $H_1(M, \mathbb{R})$ by a discrete sub-group. This map is called the Fathi mass flow (see [10]).

3. TOPOLOGICAL SYMPLECTIC PATHS

Definition 3.1. ([13]) A homeomorphism h is called a Hamiltonian homeomorphism if there exists $\lambda \in \mathcal{P}(Homeo(M), Id_M)$ with $\lambda(1) = h$, and there exists a Cauchy sequence $\{(U^i, 0)\} \subset \mathfrak{T}(M, \omega, g)$ for the metric D^2 such that $\bar{d}(\phi_{(U^i, 0)}, \lambda) \rightarrow 0, i \rightarrow \infty$.

The set $Hameo(M, \omega)$, of all Hamiltonian homeomorphisms forms a group [13].

Definition 3.2. ([3]) A continuous map $\xi : [0, 1] \rightarrow Homeo(M)$ with $\xi(0) = Id_M$ is called a strong symplectic isotopy if there exists a D^2 -Cauchy sequence $\{(F_i, \lambda_i)\} \subset \mathfrak{T}(M, \omega, g)$ such that $\bar{d}(\phi_{(F_i, \lambda_i)}, \xi) \rightarrow 0, i \rightarrow \infty$.

Let $G_\omega^0(M)$ denote the set of all time-one maps of strong symplectic isotopies. It is shown that $G_\omega^0(M)$ is a group: the so-called group of strong symplectic homeomorphisms [3, 6].

3.1. Notations. The following notations can be found in [6, 7, 15].

- $\mathcal{H}^1(M, g)$: the vector space of all harmonic 1-forms on M with respect to a Riemannian metric g .
- $\mathcal{PH}^1(M, g)$: the space of all continuous maps from $[0, 1]$ into $\mathcal{H}^1(M, g)$.
- $\mathcal{PH}^1(M, g)_0$: the completion of the metric space $\mathcal{PH}^1(M, g)$ with respect to the uniform sup norm, $\|(\mathcal{H}_t)\|_0 = \sup_{[0, 1]} |\mathcal{H}_t|$.
- $\mathcal{N}([0, 1] \times M, \mathbb{R})$: the set of all smooth functions $F : [0, 1] \times M \rightarrow \mathbb{R}$ such that $\int_M F_t \omega^n = 0$, for every t (so-called normalized functions).
- $\mathcal{N}^0([0, 1] \times M, \mathbb{R})$: the completion of the metric space $\mathcal{N}([0, 1] \times M, \mathbb{R})$ with respect to the L^∞ -Hofer norm, $\|(F_t)\|_\infty = \max_t (osc(F_t))$.
- $J^0(M, \omega, g)$: the Cartesian product $\mathcal{N}^0([0, 1] \times M, \mathbb{R}) \times \mathcal{PH}^1(M, g)_0$.

3.2. The null symplectic set. Following [7], we consider the inclusion map

$$i_0 : \mathfrak{T}(M, \omega, g) \rightarrow J^0(M, \omega, g).$$

In fact, the map i_0 is uniformly continuous with respect of the topology induced by the metric D^2 on the space $\mathfrak{T}(M, \omega, g)$, and the natural topology of the complete

metric space $J^0(M, \omega, g)$. Let $L(M, \omega, g)$ denote the image of $\mathfrak{T}(M, \omega, g)$ under i_0 , and let $\mathfrak{T}(M, \omega, g)_0$ be the closure of the set $L(M, \omega, g)$ inside the complete metric space $J^0(M, \omega, g)$. That is, $\mathfrak{T}(M, \omega, g)_0$ consists of pairs $(U, \mathcal{H})$ where the maps $(t, x) \mapsto U_t(x)$ and $t \mapsto \mathcal{H}_t$ are continuous, and for each t , $\mathcal{H}_t$ lies in $\mathcal{H}^1(M, g)$ such that there exists a D^2 -Cauchy sequence $\{(U^i, \mathcal{H}^i)\} \subset \mathfrak{T}(M, \omega, g)$ that converges to $(U, \mathcal{H})$ inside $J^0(M, \omega, g)$.

Observe that the sequence (F_j, λ_j) which appears in the definition of a strong symplectic isotopy converges inside $\mathfrak{T}(M, \omega, g)_0$ to an element $(U, \mathcal{H})$ where the mappings $(t, x) \mapsto U_t(x)$ and $t \mapsto \mathcal{H}_t$ are continuous, and for each t , $\mathcal{H}_t$ is a smooth harmonic 1-form (see [7, 14]). The latter limit $(U, \mathcal{H})$ was called a "generator" of the corresponding strong symplectic isotopy. We shall call continuous harmonic flow any strong symplectic isotopy generated by an element of the form $(0, \mathcal{H})$, and continuous Hamiltonian flow any strong symplectic isotopy generated by an element of the form $(U, 0)$.

Definition 3.3. ([7]) We define the set $GSSympeo(M, \omega, g)$ as the space of all the pairs $(\xi, (U, \mathcal{H}))$ where ξ a strong symplectic isotopy admitting $(U, \mathcal{H})$ as a "generator", i.e., there exists a sequence $\{(F^i, \lambda^i)\} \subset \mathfrak{T}(M, \omega, g)$ which is Cauchy in D^2 such that $\bar{d}(\phi_{(F^i, \lambda^i)}, \xi) \rightarrow 0, i \rightarrow \infty$, and $\{(F^i, \lambda^i)\}$ converges to $(U, \mathcal{H})$ inside $\mathfrak{T}(M, \omega, g)_0$.

Definition 3.4. ([7]) We define the symplectic topology on the space $GSSympeo(M, \omega, g)$ as the sub-space topology induced by its inclusion inside the complete topological space $\mathcal{P}(\text{Homeo}(M), Id_M) \times \mathfrak{T}(M, \omega, g)_0$.

Theorem 3.5. (Banyaga-Tchuiaga, [7]). *$GSSympeo(M, \omega, g)$ is a topological group.*

Theorem 3.6. (Tchuiaga-Koivogui-Balibuno-Mbazumutima, [14]) *Any generator corresponds to a unique strong symplectic isotopy.*

In [7], following Buhovsky-Seyfaddini [8], the authors defined the null symplectic set as follows.

Definition 3.7. ([7]). We define the null symplectic set as the set of all the pairs $(U, \mathcal{H}) \in \mathfrak{T}(M, \omega, g)_0$ such that $(Id, (U, \mathcal{H})) \in GSSympeo(M, \omega, g)$, where Id is the constant path identity.

In the following, we denote by $\mathcal{G}^\infty$ the null symplectic set, and by $\mathcal{G}^0$ the set of time-independent elements of $\mathcal{G}^\infty$.

In order to give a proof of Theorem 1.1, we will prove the following general version of Lemma 4.5-[7] and Lemma 4.6-[7].

Lemma 3.8. *Let (M, ω) be a closed symplectic manifold. The set $\mathcal{G}^0$ is trivial, i.e., $\mathcal{G}^0 = \{(Id, (0, 0))\}$.*

The proof of Lemma 3.8 shall need the following result.

Lemma 3.9. *(Banyaga-Tchuiaga, [7]). The sets $\mathcal{G}^\infty$ and $\mathcal{G}^0$ have the following properties:*

- (1) $\mathcal{G}^\infty$ is closed under the sum operator and minus operator.
- (2) $\mathcal{G}^\infty$ is closed in L^∞ -topology while $\mathcal{G}^0$ is closed in C^0 -topology.
- (3) Let $(U, \mathcal{H}) \in \mathcal{G}^\infty$, and $\gamma : [0, 1] \rightarrow [0, 1]$ be a smooth increasing function. If U^γ is the continuous function $(t, x) \mapsto \gamma'(t)U_{\gamma(t)}(x)$, and $\mathcal{H}^\gamma$ is the continuous family of harmonic 1-forms $t \mapsto \gamma'(t)\mathcal{H}_{\gamma(t)}$, then the element $(U^\gamma, \mathcal{H}^\gamma)$ belongs to $\mathcal{G}^\infty$.
- (4) $\mathcal{G}^0$ is a vector space over $\mathbb{R}$.
- (5) If $(U, \mathcal{H}) \in \mathcal{G}^0$, then for any symplectic isotopy isotopic to the constant map identity ρ , we have $(Id, \rho^*((U, \mathcal{H}))) \in \mathcal{G}^0$, with

$$\rho^*((U, \mathcal{H})) := (U \circ \rho + \square^1\{\mathcal{H}, \Phi = (\phi_t)\}, \mathcal{H}).$$

Proof of Lemma 3.8. Let $(Id, (U, \mathcal{H})) \in \mathcal{G}^0$, we have to show that $(U, \mathcal{H}) = (0, 0)$. By definition of $\mathcal{G}^0$, we can find a D^2 -Cauchy sequence $\{(U^i, \mathcal{H}^i)\} \subset \mathfrak{T}(M, \omega, g)$ such that

$$(\phi_{(U_i, \mathcal{H}_i)}, (U_i, \mathcal{H}_i)) \xrightarrow{C^0 + L^\infty} (Id, (U, \mathcal{H})).$$

The Hodge decomposition theorem of symplectic isotopies [4] applied gives $\phi_{(U^i, \mathcal{H}^i)} = \rho_i \circ \Psi_i$, where ρ_i is the harmonic isotopy generated by $(0, \mathcal{H}^i)$, i.e., $\rho_i = \phi_{(0, \mathcal{H}^i)}$, and Ψ_i is a Hamiltonian isotopy. We have that $\phi_{(U^i, \mathcal{H}^i)} \xrightarrow{C^0} Id$, $\rho_i \xrightarrow{C^0} \phi_{(0, \mathcal{H})}$, and $\Psi_i := \rho_i^{-1} \circ \phi_{(U^i, \mathcal{H}^i)}$, for each i . Since the composition is continuous with respect to the C^0 -metric, then the sequence Ψ_i converges in the C^0 -metric to a continuous isotopy $\Psi = (\psi_t)$, and from the bi-invariance of the

C^0 -metric, it follows that,

$$(3.1) \quad \bar{d}(\Psi, \phi_{(0, \mathcal{H})}^{-1}) \leq \bar{d}(\Psi, \rho_i^{-1} \circ \phi_{(U^i, \mathcal{H}^i)}) + \bar{d}(\rho_i^{-1} \circ \phi_{(U^i, \mathcal{H}^i)}, \rho_i^{-1}) + \bar{d}(\rho_i^{-1}, \phi_{(0, \mathcal{H})}^{-1}),$$

for each i . Each term in the right-hand side tends to zero when i goes to infinity, i.e., both paths Ψ and $\phi_{(0, \mathcal{H})}^{-1}$ coincide. Also, if for each i we set

$\Theta_i := \Psi_i^{-1} \circ \phi_{(0, \mathcal{H})}$, then the facts that $\Psi = \lim_{C^0}(\Psi_i)$ (i.e., $\Psi^{-1} = \lim_{\bar{d}}(\Psi_i^{-1})$), and both paths Ψ and $\phi_{(0, \mathcal{H})}^{-1}$ coincide, imply that,

$$(3.2) \quad \Theta_i \xrightarrow{C^0} Id.$$

By Corollary 2.1–[15] (see [15], page 73), for each i , the Hamiltonian path Ψ_i is generated by $(U_i \circ \phi_{(0, \mathcal{H}_i)}, 0)$. Thus, using (2.3) and (2.4) one derives that for each i , the path Θ_i is generated by

$$(3.3) \quad (\alpha_i, \beta_i) := \overline{(U_i \circ \phi_{(0, \mathcal{H}_i)}, 0)} \bowtie (0, \mathcal{H}) = (-U_i \circ \phi_{(0, \mathcal{H}_i)} \circ \Psi_i + \tilde{\Delta}(\mathcal{H}, \Psi_i), \mathcal{H}).$$

Since by assumption we have $\phi_{(0, \mathcal{H}_i)} \circ \Psi_i = \phi_{(U_i, \mathcal{H}_i)} \xrightarrow{C^0} Id$, and $(U_i, \mathcal{H}_i) \xrightarrow{L^\infty} (U, \mathcal{H})$, we derive that, $-U_i \circ (\phi_{(0, \mathcal{H}_i)} \circ \Psi_i) \xrightarrow{L^\infty} -U$. Therefore, to conclude that $(\alpha_i, \beta_i) \xrightarrow{L^\infty} (-U, \mathcal{H})$, we only need to show that $\tilde{\Delta}(\mathcal{H}, \Psi_i) \xrightarrow{L^\infty} 0$. To see this, apply the triangle inequality to get,

$$osc(\tilde{\Delta}(\mathcal{H}, \Psi_i)) \leq osc(\tilde{\Delta}(\mathcal{H}, \Psi_i) - \tilde{\Delta}(\mathcal{H}, \phi_{(0, \mathcal{H})}^{-1})) + osc(\tilde{\Delta}(\mathcal{H}, \phi_{(0, \mathcal{H})}^{-1})),$$

and derive from Proposition 2.5–[16] that

$$\tilde{\Delta}(\mathcal{H}, \phi_{(0, \mathcal{H})}^{-1}) = -\tilde{\Delta}(\mathcal{H}, \phi_{(0, \mathcal{H})}) \circ \phi_{(0, \mathcal{H})}^{-1},$$

and also, since $\mathcal{H}$ is time-independent, we have $\tilde{\Delta}(\mathcal{H}, \phi_{(0, \mathcal{H})}) = 0$ (see Example 3.1–[15]-page 81, or more genrally [16]). On the other hand, since $\bar{d}(\Psi_i, \phi_{(0, \mathcal{H})}^{-1}) \rightarrow 0, i \rightarrow \infty$, and $\phi_{(0, \mathcal{H})}$ is smooth, we derive from Lemma 3.10–[15] that there exists a rank j_0 such that

$$osc(\tilde{\Delta}(\mathcal{H}, \Psi_i) - \tilde{\Delta}(\mathcal{H}, \phi_{(0, \mathcal{H})}^{-1})) \leq 2|\mathcal{H}|\bar{d}(\Psi_i, \phi_{(0, \mathcal{H})}^{-1}),$$

for all $i \geq j_0$. That is, $osc(\tilde{\Delta}(\mathcal{H}, \Psi_i)) \rightarrow 0, i \rightarrow \infty$. Therefore, we have proved that

$$(\Theta_i, (\alpha_i, \beta_i)) \xrightarrow{L^\infty + C^0} (Id, (-U, \mathcal{H})) \in \mathcal{G}^0.$$

Finally, Lemma 3.9-item (4) applied gives

$$\left(\frac{1}{2}(Id, (U, \mathcal{H})) + \frac{1}{2}(Id, (-U, \mathcal{H})) \right) \in \mathcal{G}^0,$$

i.e., $(Id, (0, \mathcal{H})) \in \mathcal{G}^0$. Since $(0, \mathcal{H}) \in \mathfrak{T}(M, \omega, g)$, we apply Corollary 3.7–[15] to get $(0, \mathcal{H}) = (0, 0)$, i.e., $\mathcal{H} = 0$. From the above arguments, we have that $\mathcal{H} = 0$, and $(Id, (U, 0))$ identified with $(Id, U) \in Null$, where $Null$ is the Buhovsky-Seyfaddini time-independent null Hamiltonian set [8]. The fact that $U = 0$, follows from a verbatim repetition of the proof of similar fact in [8]. Q.E.D.□

Proof of Theorem 1.1. This proof follows from a verbatim repetition of the proof of Theorem 4.7–[7]. But, here we apply Lemma 3.8 instead of Lemma 4.5–[7] and Lemma 4.6–[7]. Q.E.D.□

As in [14], based on Theorem 1.1 and Theorem 3.6, we will denote any strong symplectic isotopy λ by $\lambda_{(U, \mathcal{H})}$ to mean that it is generated by $(U, \mathcal{H})$, or equivalently, to mean that $\lambda_{(U, \mathcal{H})}$ is the limit of a sequence of symplectic isotopies $\phi_{(U^i, \mathcal{H}^i)}$ with respect to the metric $(\bar{d} + D^2)$, i.e., $\lambda_{(U, \mathcal{H})} = \lim_{C^0+L^\infty} (\phi_{(U^i, \mathcal{H}^i)})$.

4. THE MASS FLOW FOR STRONG SYMPLECTIC ISOTOPIES

Let (Φ_i) be a sequence of symplectic isotopies which converges in the C^0 –metric to λ . According to Fathi [10], the map $\lambda \mapsto \tilde{\theta}(\lambda)$ is continuous with the respect to the C^0 –metric, i.e., $\lim_{C^0} \left(\tilde{\theta}(\Phi_i)(f) \right) = \tilde{\theta}(\lambda)(f)$, for all continuous mapping $f : M \rightarrow \mathbb{S}^1$. This implies that

$$(4.1) \quad \lim_{C^0} \mathbb{I} \left(\tilde{S}_\omega(\Phi_i), \left[\frac{\omega^{(n-1)}}{(n-1)!} \wedge f^*(\sigma) \right] \right) = \tilde{\theta}(\lambda)(f),$$

for all continuous mapping $f : M \rightarrow \mathbb{S}^1$. But, in (4.1) one cannot permute the C^0 –limit symbol with the integral symbol since the convergence of the sequence of isotopies Φ_i in the C^0 –metric does not guarantee the convergence of the sequence of de Rham cohomology classes $\left(\tilde{S}_\omega(\Phi_i) \right)$ inside $H^1(M, \mathbb{R})$ with respect to the topology induced by the vector space structure on $H^1(M, \mathbb{R})$. For instance, consider the following continuous mappings,

$$T_1 : GSSympeo(M, \omega, g) \rightarrow \mathcal{PH}^1(M, g)_0,$$

$$(\xi, (F, \alpha)) \mapsto \alpha,$$

and

$$T_2 : GSSympeo(M, \omega, g) \rightarrow \mathcal{P}(\text{Homeo}(M), Id_M),$$

$$(\xi, (F, \alpha)) \mapsto \xi_{(F, \alpha)} := \xi.$$

Since any smooth symplectic isotopy $\phi_{(U, \mathcal{H})}$ belongs to $GSSympeo(M, \omega, g)$, we have,

$$(4.2) \quad \mathbb{I}([\int_0^1 T_1(\phi_{(U, \mathcal{H})})(t)dt], [\frac{\omega^{(n-1)}}{(n-1)!} \wedge f^*(\sigma)]) = \tilde{\theta}(T_2(\phi_{(U, \mathcal{H})}))(f),$$

where

$$[\int_0^1 T_1(\phi_{(U, \mathcal{H})})(t)dt] = \int_0^1 [\mathcal{H}_t]dt = \tilde{S}_\omega(\phi_{(U, \mathcal{H})}).$$

4.0.1. *Proof of Theorem 1.2.* Let $\lambda_{(U, \mathcal{H})}$ be a strong symplectic isotopy such that $\lambda_{(U, \mathcal{H})} = \lim_{(C^0 + L^\infty)}(\phi_{(U^i, \mathcal{H}^i)})$. According to equation (4.2), we have,

$$\mathbb{I}([\int_0^1 T_1(\phi_{(U^i, \mathcal{H}^i)})(t)dt], [\frac{\omega^{(n-1)}}{(n-1)!} \wedge f^*(\sigma)]) = \tilde{\theta}(\phi_{(U^i, \mathcal{H}^i)})(f),$$

for each i , and taking the limit with respect to the $(C^0 + L^\infty)$ -topology in both sides leads to,

$$\begin{aligned} \tilde{\theta}(\lambda_{(U, \mathcal{H})})(f) &= \lim_{(C^0 + L^\infty)} \mathbb{I}([\int_0^1 T_1(\phi_{(U^i, \mathcal{H}^i)})(t)dt], [\frac{\omega^{(n-1)}}{(n-1)!} \wedge f^*(\sigma)]) \\ &= \mathbb{I}(\lim_{L^\infty} \tilde{S}_\omega(\phi_{(U^i, \mathcal{H}^i)}), [\frac{\omega^{(n-1)}}{(n-1)!} \wedge f^*(\sigma)]) \\ &= \mathbb{I}([\int_0^1 \mathcal{H}_t dt], [\frac{\omega^{(n-1)}}{(n-1)!} \wedge f^*(\sigma)]). \end{aligned}$$

Therefore, it follows from the above equalities that the mass flow of any strong symplectic isotopy $\lambda_{(U, \mathcal{H})}$ is given by:

$$(4.3) \quad \tilde{\theta}(\lambda_{(U, \mathcal{H})})(f) = \frac{1}{(n-1)!} \int_M \left(\left(\int_0^1 \mathcal{H}_t dt \right) \wedge \omega^{(n-1)} \wedge f^*(\sigma) \right),$$

for all continuous mapping $f : M \rightarrow \mathbb{S}^1$. This completes the proof. ■

Proof of Theorem 1.3. Let $\phi \in Hameo(M, \omega)$, we have to show that $\phi \in [G_\omega^0(M), G_\omega^0(M)]$.

Case (1): Assume that $\phi \in Ham(M, \omega) \subseteq Hameo(M, \omega)$. Then, from the following inclusions of sub-groups, $[G_\omega(M), G_\omega(M)] \subseteq [G_\omega^0(M), G_\omega^0(M)]$, and $Ham(M, \omega) \subseteq [G_\omega(M), G_\omega(M)]$, we derive that $\phi \in [G_\omega^0(M), G_\omega^0(M)]$.

Case (2): Assume that ϕ is not a Hamiltonian diffeomorphism and does not belong to the commutator sub-group $[G_\omega^0(M), G_\omega^0(M)]$. So, ϕ cannot be written as a

product of commutators $[\alpha, \beta] := \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$, where α and β are arbitrary elements of $G_\omega^0(M)$. That is, for all countable families $\{a_j\}_{j \in J \subset \mathbb{N}} \subset G_\omega^0(M)$ and $\{b_j\}_{j \in J \subset \mathbb{N}} \subset G_\omega^0(M)$, we may assume that

$$(4.4) \quad \phi \neq \prod_{j \in J} [a_j, b_j] =: \prod (a, b),$$

where $\mathbb{N}$ stands for the set of all nonnegative integers. On the other hand, from the definition of a continuous Hamiltonian homeomorphism ϕ , there exists a sequence of smooth Hamiltonian isotopies $\Phi_i := \{\phi_i^t\}$, which is Cauchy in the $(L^\infty + C^0)$ -metric such that its sequence of time-one maps ϕ_i^1 converges in the C^0 -metric to ϕ . Since the isotopy Φ_i is Hamiltonian for each i , we derive from Theorem 2.5-item-2 that Φ_i can be written as the following product of commutators,

$$(4.5) \quad \Phi_i = \prod_{q \in Q \subset \mathbb{N}} [\alpha_i^q, \beta_i^q] =: \prod (\alpha_i, \beta_i),$$

where for each pair $(i, q) \in \mathbb{N} \times Q$, α_i^q and β_i^q are smooth paths in $G_\omega(M)$ with

$$(4.6) \quad \alpha_i^q(0) = Id_M = \beta_i^q(0).$$

From (4.4) and (4.5) we derive that for each integer i , we also have

$$(4.7) \quad \phi \neq \prod_{q \in Q} [\alpha_i^q(1), \beta_i^q(1)] =: \prod (\alpha_i(1), \beta_i(1)),$$

i.e.,

$$(4.8) \quad \left(\prod (\alpha_i(1), \beta_i(1)) \right)^{-1} \circ \phi \neq Id_M.$$

That is, to each positive integer k , attached is a non-empty open ball $\mathcal{B}_k \subset M$, whose closure $\overline{\mathcal{B}_k}$, is completely displaced by the mapping $(\prod (\alpha_k(1), \beta_k(1)))^{-1} \circ \phi$. In other words, for each positive integer k , there exists a non-empty open ball $\mathcal{B}_k \subset M$ such that,

$$(4.9) \quad \left\{ \left(\prod (\alpha_k(1), \beta_k(1)) \right)^{-1} \circ \phi \right\} (\overline{\mathcal{B}_k}) \cap \overline{\mathcal{B}_k} = \emptyset.$$

Now, fix a sufficiently large integer $\delta > 0$ (if necessary, take δ large enough as you want). Assume this done. Since the sequence of time-one maps

$\left(\left\{ \left(\prod (\alpha_\delta(1), \beta_\delta(1)) \right)^{-1} \circ \phi_j^1 \right\}_j \right)$ converges in the C^0 -metric to the mapping $(\prod (\alpha_\delta(1), \beta_\delta(1)))^{-1} \circ \phi$, and $\overline{\mathcal{B}_\delta}$ is compact, then we derive from (4.9) that,

$$(4.10) \quad \left\{ \left(\prod (\alpha_\delta(1), \beta_\delta(1)) \right)^{-1} \circ \phi_j^1 \right\} (\overline{\mathcal{B}_\delta}) \cap \overline{\mathcal{B}_\delta} = \emptyset,$$

for all integer j sufficiently large. Therefore, since by assumption the open ball $\mathcal{B}_\delta$ is non-empty, it follows from (4.10) together with Theorem 2.2 that,

$$(4.11) \quad 0 < e_S(\mathcal{B}_\delta) \leq l_{HL}^\infty \left\{ \left(\prod (\alpha_\delta, \beta_\delta) \right)^{-1} \circ \Phi_j \right\} = l_{HL}^\infty \{ \Phi_\delta^{-1} \circ \Phi_j \},$$

for all j sufficiently large. In particular, for $j = \delta$, the right-hand side in (4.11) is equal to zero: This contradicts the positivity of $e_S(\mathcal{B}_\delta)$. Q.E.D ♠

Proof of Theorem 1.4. It is clear that this proof can be obtained as a verbatim repetition of the proof of Theorem 1.3. Q.E.D ♣

Proof of Proposition 1.6. Assume there exists an open cover $\mathcal{U} = \{U_i\}_{1 \leq i \leq k}$ of M , and that there exists a Hamiltonian homeomorphism ψ such that for all k -fold of Hamiltonian homeomorphisms $\gamma_1, \dots, \gamma_k$, with each γ_i supported in U_i , one has

$$(4.12) \quad \psi \neq \gamma_1 \circ \dots \circ \gamma_k =: \prod(\gamma).$$

From the definition of a continuous Hamiltonian homeomorphism ψ , there exists a sequence of smooth Hamiltonian isotopies $\Psi_i := \{\psi_i^t\}$ which is Cauchy in the $(L^\infty + C^0)$ -metric such that its sequence of time-one maps $\{\psi_i^1\}$ converges in the C^0 -metric to ψ . Since for each i , we have $\psi_i \in \text{Ham}(M, \omega)$, then the fragmentation property of $\text{Ham}(M, \omega)$ applied with respect to the above open cover of M implies that for each i , one can always find out a family of Hamiltonian diffeomorphisms $\{\theta_j^i\}_{1 \leq j \leq k}$, with each θ_j^i supported in U_j such that

$$(4.13) \quad \psi_i^1 = \theta_1^i \circ \dots \circ \theta_k^i =: \prod(\theta_i).$$

In particular, since $\text{Ham}(M, \omega) \subset \text{Hameo}(M, \omega)$, then (4.12) and (4.13) imply that for each i , we have $\psi \neq \prod(\theta_i)$. Therefore, as in the proof of Theorem 1.3 we conclude. Q.E.D ■

Remark 4.1. A quite way to describe the ideas of the present paper is that, it studies the spreading of flux geometry under the C^0 -symplectic topology.

5. FINAL REMARKS

The uniqueness results of Theorem 1.1 and Theorem 3.6 seem to suggest that there is a well-defined surjective group homomorphism $\tilde{S}_0$, which assigns to each strong symplectic isotopy $\lambda_{(U, \mathcal{H})}$ the de Rham cohomology class $\int_0^1 [\mathcal{H}_t] dt \in H^1(M, \mathbb{R})$

so that any continuous Hamiltonian flow belongs to $\ker \tilde{S}_0$. By analogy with the usual flux geometry, the study the mapping $\tilde{S}_0$ may furnish some important properties of $Hameo(M, \omega)$ (e.g. it is not yet know whether any strong symplectic isotopy in $Hameo(M, \omega)$ is a continuous Hamiltonian flow or not).

If it happens that the above invariant $\tilde{S}_0$ is constructed, then Theorem 1.2 will reads: For any strong symplectic isotopy λ , we have

$$(5.1) \quad \tilde{\theta}(\lambda)(f) = \frac{1}{(n-1)!} \mathbb{I} \left(\tilde{S}_0(\lambda), [\omega^{(n-1)} \wedge f^*(\sigma)] \right),$$

so that by setting

$$(5.2) \quad \tilde{S}_{\Omega_0}^0(\lambda) := \tilde{S}_0(\lambda) \wedge \frac{[\omega^{(n-1)}]}{(n-1)!},$$

then in the C^0 -symplectic context, the usual Fathi mass flow will be nothing than the Poincaré's dual of the invariant $\tilde{S}_{\Omega_0}^0$: Theorem 1.2 seems to be a generalization of the usual Fathi's Poincaré duality result. Furthermore, Theorem 1.2 with the invariant $\tilde{S}_0$ could be key ingredients in solving the following open conjecture due to Oh-Müller.

Conjecture, [13]: $Hameo(M, \omega)$ is a proper sub-group of the kernel of the Fathi's mass flow.

On the other hand, we know that for an algebraic group G its first homology group $H_1(G, \mathbb{Z})$ with coefficients in $\mathbb{Z}$ is defined as the following quotient space: $H_1(G, \mathbb{Z}) := G/[G, G]$. The group $H_1(G, \mathbb{Z})$ is sometimes called the abelianization of G . For instance, Theorem 1.3 implies that

$$H_1(G_\omega^0(M), \mathbb{Z}) := G_\omega^0(M)/[G_\omega^0(M), G_\omega^0(M)] = G_\omega^0(M)/Hameo(M, \omega).$$

If the manifold M is simply connected we have $H^1(M, \mathbb{R}) = \{0\}$, and $G_\omega^0(M) = Hameo(M, \omega)$: $H_1(G_\omega^0(M), \mathbb{Z}) = \{0\}$, is isomorphic to a quotient of $H^1(M, \mathbb{R}) = \{0\}$. Otherwise, we still have that the first homology group $H_1(Hameo(M, \omega), \mathbb{Z})$ is trivial; in this case, one way to compare $H_1(G_\omega^0(M), \mathbb{Z})$ with a quotient space of $H^1(M, \mathbb{R})$ is algebraically equivalent to find out a surjective group homomorphism from $G_\omega^0(M)$ onto that quotient space, whose kernel coincides with $Hameo(M, \omega)$ (see [17, 2] for similar construction in the smooth case). The above arguments have

motivated the following conjecture.

Conjecture: $H_1(G_\omega^0(M), \mathbb{Z})$ is isomorphic to a quotient space of $H^1(M, \mathbb{R})$ (probably with some restriction on the dimension of M).

It is also well-known that there is a paradigm that the algebraic perfectness may imply the algebraic simplicity (see Anderson [1]). From the lines of the proof of similar fact in [1, 2], it seems that Theorem 1.4 may imply that $Hameo(M, \omega)$ is algebraically simple. To that end, we formulate the following conjecture.

Conjecture: $Hameo(M, \omega)$ is algebraically simple (probably with some restriction on the dimension of M).

Also, Proposition 1.6 shows that $Hameo(M, \omega)$ has a fragmentation property: This seems to be a useful property in some local constructions on M . However, for some suitable studies (e.g. the fragmented topological Hofer geometry), it could be interesting to give another proof of Proposition 1.6 by adapting the famous Kirby's "torus trick" into our context, [12, 9]: This is one goal of a future work.

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