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BIANCHI TYPE-I BULK VISCOUS STRING MODEL IN $f(R)$ GRAVITY

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ABSTRACT. Field equations of $f(R)$ theory of gravity are obtained with the aid of locally rotationally symmetric (LRS) Bianchi type-I metric when the matter source is a bulk viscous fluid containing one dimensional cosmic strings. Applying some physically possible conditions, we have obtained a determinate solution of the field equations. The deceleration parameter of our model exhibits a smooth transition from early decelerated phase to late time accelerating phase. We also find that the realistic energy conditions $\rho \geq 0$ and $\rho_p \geq 0$ are satisfied in our model. Some other properties of the model are also discussed.

AMS Classification: 83D05, 81T30
Keywords: LRS Bianchi type-I metric; Bulk viscosity; $f(R)$ gravity; Modified gravity; Cosmic strings

1. INTRODUCTION

Nowadays we have a strong belief that our universe experiences an accelerated expansion (Riess et al. [30]; Perlmutter et al. [21]). This cosmic acceleration can be introduced via dark energy or via modification of gravity (Nojiri and Odintsov

[16]). Recently, a general review of dark energy was given in Bamba et al. [3]. Dark energy should have strong negative pressure and can be characterized by an equation of state (EoS) parameter $\omega = p/\rho$, where p is the dark energy pressure known to be negative and ρ is the dark energy density. On the other hand modification of Einstein's gravity into different theories. The basic idea behind this way is to present the gravitational description of dark energy (Brevik et al. [4]). Among various modified theories of gravitation, $f(R)$ (Capozziello et al. [6]; Nojiri and Odintsov [17]) theory of gravity is significant in which a general function of the Ricci scalar, $f(R)$, replaces R in the standard Einstein-Hilbert Lagrangian. This provides (Nojiri and Odintsov [18]) an easy unification of early time inflation and late time acceleration. Nojiri and Odintsov [16] and Capozziello and Laurentis [5] have studied many aspects on $f(R)$ gravity.

It is found that some large-angle anomalies (Eriksen et al. [9]) appear in cosmic microwave background radiations which violate the statistical isotropy of the universe. Bianchi type models, which are homogeneous but not necessarily isotropic seem to be the most promising explanation of these anomalies. Jaffe et al. [10] explored that removing a Bianchi component from the WMAP data can account for several large-angle anomalies leaving the universe to be isotropic. Thus the universe may have achieved a slight anisotropic geometry in cosmological models regardless of the inflation. Recently, many authors have studied dark energy cosmological models with anisotropic background. Sharif and Shamir [37], Shamir [38], [39], Amir and Sattar [1] explored various exact vacuum solutions of anisotropic space-times in the $f(R)$ theory of gravity. Sharif and Shamir [40], Sharif and Kausar [41] and Singh and Singh [42] discussed various non-vacuum anisotropic Bianchi type models in $f(R)$ gravity. Yilmaz et al. [48] studied quark and strange quark matter in $f(R)$ gravity for Bianchi type I and V space-times.

Early stages of the formation of the universe is still an open problem. After the big bang, the universe may have undergone a series of phase transitions as its temperature lowered down. During the phase transition, the symmetry of the universe is broken spontaneously. It can give rise to topological defects like, domain walls, strings and monopoles (Kibble [13]). Among these three topological defects, only strings can lead to very interesting cosmological consequences (Vilenkin [46]). They are believed to give rise to density perturbation leading to the formation of galaxies. As these strings possess stress-energy and are coupled to the gravitational field, it may be interesting to study the gravitational effects that arise from strings. Stachel [43], Letelier [14], Reddy [29], Rao et al. [22]-[24], Tripathy et al. [44] and Sahoo and Mishra [32] are some of the authors who have investigated several important aspects of string cosmological models in various theories of gravitation. Sahoo et al. [33] discussed Bianchi type string cosmological models in $f(R,T)$ gravity. Sheykhi [35] studied magnetic strings in $f(R)$ gravity. Katore [12] has discussed Bianchi type-II, $VIII$ and IX string cosmological models in $f(R)$ gravity.

Due to large-scale distribution of galaxies in our universe the matter distribution is satisfactorily described by a perfect fluid. However, a realistic treatment of the problem requires the consideration of material distribution other than the perfect fluid. It is well known that when neutrino decoupling occurs, the matter behaves as a viscous fluid in an early stage of the universe. Viscous fluid cosmological models in early universe have been widely discussed in literature (Wang [47]; Bali and Pradhan [2]; Shri Ram et al. [36]; Sagar et al. [31]; Rao et al. [25, 26]). Very recently, we have investigated Bianchi type-III bulk viscous string cosmological models in $f(R)$ theory gravity with Hubble's law.

Motivated by above investigations, in this paper, we consider LRS Bianchi type-I space-time filled with bulk viscous fluid with one dimensional cosmic strings in metric $f(R)$ theory of gravity. The paper is organized as followed: After introduction, in section 2 we present metric and field equations. Section 3 is devoted to the solution of the field equations and the bulk viscous string model in $f(R)$ gravity. In section 4, we discuss some properties of the model. Obtained results are discussed in section 5.

2. Metric and field equations

We consider the spatially homogeneous but anisotropic space-time described by LRS Bianchi type-I metric in the form

$$ds^2 = dt^2 - X^2 dx^2 - Y^2(dy^2 + dz^2), \tag{1}$$

where X and Y are functions of cosmic time t only.

The field equations of $f(R)$ gravity are obtained from the action

$$S = \int \sqrt{-g}\left(\frac{1}{16\pi G} f(R) + L_m\right) d^4x, \tag{2}$$

where $f(R)$ is a general function of the Ricci scalar and L_m is the matter Lagrangian. Variation of action (2) with respect to metric gives the following field equations

$$F(R)R_{ij} - \frac{1}{2}f(R)g_{ij} - \nabla_i\nabla_j F(R) + g_{ij}\Box F(R) = \kappa T_{ij} \tag{3}$$

where $F(R) = \frac{df}{dR}$ and $\Box = \nabla^i\nabla_i$, ∇_i is the covariant derivative. Contracting the field equations (3), we get

$$F(R)R - 2f(R) + 3\Box F(R) = \kappa T. \tag{4}$$

Using above equation in equation (3), the field equations take the form

$$F(R)R_{ij} - \nabla_i\nabla_j F(R) - \kappa T_{ij} = g_{ij}\left(\frac{F(R)R - \Box F(R) - \kappa T}{4}\right). \tag{5}$$

Equation (4) is an important relationship between $f(R)$ and $F(R)$ which will be used to simplify the field equations and to evaluate $f(R)$.

The energy momentum tensor for a bulk viscous fluid containing one-dimensional cosmic strings is given by

$$T_{ij} = (\rho + \overline{p})u_i u_j - \overline{p}g_{ij} - \lambda x_i x_j \tag{6}$$

$$\overline{p} = p - 3\xi H \ (= \omega\rho) \tag{7}$$

Here $\overline{p}$ is the total pressure which includes the isotropic pressure p, λ is the string tension density, ρ is the rest energy density of the system, $\xi(t)$ is the coefficient of bulk viscosity, $3\xi H$ is generally known as bulk viscous pressure, H is the Hubble parameter of the model.

Equation of state (EoS) provides a relation between isotropic pressure and energy density and is given by

$$p = \gamma\rho \tag{8}$$

where γ is the EoS parameter and $\omega = \gamma - \zeta$ (where ζ is a constant). The values of $\gamma = -1,\ 0,\ 1/3,\ 1$ represent vacuum dominated, matter dominated, radiation dominated and stiff fluid era respectively.

Also, u_i is the four velocity vector, x_i is a space-like vector which represents the anisotropic directions of the string and they satisfies

$$g^{ij}u_i u_j = -x^i x_j = 1,\ u^i x_i = 0 \tag{9}$$

We assume the string to be lying along the x-axis. The one dimensional strings are assumed to be loaded with particle and energy density is $\rho_p = \rho - \lambda$.

By adopting co-moving coordinates, the field equations (5) for the metric (1) yield the following equations:

$$\frac{\ddot{X}}{X}+2\frac{\dot{X}\dot{Y}}{XY}-\frac{f(R)}{2F}+\frac{\ddot{F}}{F}+2\frac{\dot{F}\dot{Y}}{FY} = -\frac{\kappa(\overline{p}-\lambda)}{F} \tag{10}$$

$$\frac{\ddot{Y}}{Y}+\frac{\dot{X}\dot{Y}}{XY}+\frac{\dot{Y}^2}{Y^2}-\frac{f(R)}{2F}+\frac{\ddot{F}}{F}+\frac{\dot{F}}{F}\left(\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}\right) = -\frac{\kappa\overline{p}}{F} \tag{11}$$

$$\frac{\ddot{X}}{X}+2\frac{\ddot{Y}}{Y}-\frac{f(R)}{2F}+\frac{\dot{F}}{F}\left(\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right) = \frac{\kappa\rho}{F} \tag{12}$$

where overhead dot stands for ordinary differentiation with respect to cosmic time t. We define the following parameters for the LRS Bianchi type-I model:
Hubble's parameter of the model

$$H=\frac{\dot{a}}{a}=\frac{1}{3}\left(\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right) \tag{13}$$

where $a=(XY^2)^{1/3}$ is average scale factor.
Anisotropic parameter A_h is given by

$$A_h = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i-H}{H}\right)^2, \tag{14}$$

where $H_1=\frac{\dot{X}}{X}, H_2=H_3=\frac{\dot{Y}}{Y}$ are directional Hubble's parameters, which express the expansion rates of the universe in the directions of x, y and z respectively.
Expansion scalar and shear scalar are defined as

$$\theta = u^i_{;i}=\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y} \tag{15}$$

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij}=\frac{1}{3}\left(\frac{\dot{X}}{X}-\frac{\dot{Y}}{Y}\right)^2, \tag{16}$$

where σ_{ij} is shear tensor, A_h is the deviation from isotropic expansion and the universe expands isotropically if $A_h=0$. Deceleration parameter is given by

$$q = \frac{d}{dt}\left(\frac{1}{H}\right)-1. \tag{17}$$

The universe exhibits accelerating volumetric expansion if $-1\leq q<0$, decelerating volumetric expansion if $q>0$, and exhibits constant-rate volumetric expansion if

$q = 0$.

The scalar curvature for the metric (1) is given by

$$R = 2\left(\frac{\ddot{X}}{X} + 2\frac{\ddot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} + 2\frac{\dot{X}\dot{Y}}{XY}\right). \tag{18}$$

3. Solution of the field equations

The system of the equations (10)-(12) is not fully determined, because there are three independent equations, but six unknown variables X, Y, $F(R)$, $\overline{p}$, ρ and λ. Thus some extra constraints are necessary to close the above nonlinear system. We use the following physically viable assumptions:

(i) We consider that shear scalar (σ) is proportional to the expansion scalar (θ). This leads to a relation between the metric potentials (Collins et al. [8]), i.e.,

$$X = Y^k \tag{19}$$

where $k > 1$ is a constant and takes care of anisotropy of the space-time (we have taken the integration constant as a unity).

(ii) Chiba et al. [7] have shown that $f(R)$ theory of gravity equivalent to scalar-tensor theory of gravity. The power-law relation between scalar field and average scale factor has already been used by Johri and Sudharsan [11] in the context of FRW Brans-Dicke models with bulk viscosity. Uddin et al. [45] have established a result in the context of $f(R)$ theory of gravity which shows that

$$F(R) \propto a^m \tag{20}$$

where m is an arbitrary constant. From equation (20), we have

$$F(R) = F_0 a^m \tag{21}$$

where F_0 is proportionality constant.

Subtracting (10) from (11), we obtain

$$\frac{\ddot{X}}{X} - \frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} + \frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{F}}{F}\left(\frac{\dot{Y}}{Y} - \frac{\dot{X}}{X}\right) = \frac{\kappa\lambda}{F} \tag{22}$$

By integrating the above equation, we have

$$\frac{\dot{X}}{X}-\frac{\dot{Y}}{Y}=\frac{F}{XY^2}\,exp\left(\int\frac{\kappa\lambda}{F\left(\frac{\dot{X}}{X}-\frac{\dot{Y}}{Y}\right)}dt\right)+\frac{c_1F}{XY^2} \tag{23}$$

where c_1 is positive constant of integration. To solve the integration in (23), we can choose λ such that

$$\lambda=\frac{F}{\kappa}\left(\frac{\dot{X}}{X}-\frac{\dot{Y}}{Y}\right). \tag{24}$$

If we take $k=1$ (i.e., isotropic model), from (19) and (24) we can observe that $\lambda=0$. This follows the well-known fact that string doesn't survive in isotropic universe. Thus, our choice is physically valid.

Solving equation (23) using (19), (21) and (24), we get the metric potentials as

$$\begin{aligned} X &= \left(\frac{3(F_0\ e^t+c_1\ t+c_2)}{(k+2)(3-m)}\right)^{\frac{3k}{(k+2)(3-m)}} \\ Y &= \left(\frac{3(F_0\ e^t+c_1\ t+c_2)}{(k+2)(3-m)}\right)^{\frac{3}{(k+2)(3-m)}} \end{aligned} \tag{25}$$

and the function $F(R)$ as

$$F=F_0\left(\frac{3(F_0\ e^t+c_1\ t+c_2)}{(k+2)(3-m)}\right)^{\frac{m}{3-m}} \tag{26}$$

where c_2 is an another integrating constant.

Now the metric (1) can be written as

$$\begin{aligned} ds^2 &= dt^2-\left(\frac{3(F_0\ e^t+c_1\ t+c_2)}{(k+2)(3-m)}\right)^{\frac{6k}{(k+2)(3-m)}}dx^2 \\ &\quad -\left(\frac{3(F_0\ e^t+c_1\ t+c_2)}{(k+2)(3-m)}\right)^{\frac{6}{(k+2)(3-m)}}(dy^2+dz^2) \end{aligned} \tag{27}$$

Thus the metric (27) together with the following properties constitutes a LRS Bianchi type-I bulk viscous string cosmological model in metric $f(R)$ theory of gravity.

From equations (18) and (25), the expression for the Ricci scalar becomes

$$\begin{aligned} R=\frac{3}{(k+2)(3-m)(F_0e^t+c_1t+c_2)^2}\Bigg\{&F_0(k+2)e^t(c_1t+c_2-c_1^2-2c-1) \\ &+\frac{(F_0e^t+c_1)^2(k^2+k+1)}{(k+2)(3-m)}\Bigg\} \end{aligned} \tag{28}$$

From equations (24)-(26), we obtain the string density λ

$$\lambda = \frac{(27)^{\frac{1}{3-m}} F_0(k-1)(F_0e^t + c_1)}{\kappa((k+2)(3-m))^{\frac{3}{3-m}} (F_0e^t + c_1t + c_2)^{\frac{2m-3}{m-3}}}. \tag{29}$$

The energy density ρ is

$$\rho = \frac{3^{\frac{m}{3-m}} F_0(F_0e^t + c_1t + c_2)^{\frac{3(2-m)}{m-3}}}{\kappa(1+\gamma-\zeta)((k+2)(3-m))^{\frac{3}{3-m}}} \Big\{ F_0e^t(c_1t + c_2 - c_1^2 - 2c_1)(3(k+1) - m(k+2)) + \frac{(F_0e^t + c_1)^2(9(k^2-2k-1) + 6m(k+2) - (k+2)^2m^2)}{(k+2)(3-m)} \Big\} \tag{30}$$

Proper pressure p is

$$p = \frac{3^{\frac{m}{3-m}} \gamma F_0(F_0e^t + c_1t + c_2)^{\frac{3(2-m)}{m-3}}}{\kappa(1+\gamma-\zeta)((k+2)(3-m))^{\frac{3}{3-m}}} \Big\{ F_0e^t(c_1t + c_2 - c_1^2 - 2c_1)(3(k+1) - m(k+2)) + \frac{(F_0e^t + c_1)^2(9(k^2-2k-1) + 6m(k+2) - (k+2)^2m^2)}{(k+2)(3-m)} \Big\} \tag{31}$$

Particle energy density is

$$\rho_p = \frac{3^{\frac{m}{3-m}} F_0(F_0e^t + c_1t + c_2)^{\frac{3(2-m)}{m-3}}}{\kappa(1+\gamma-\zeta)((k+2)(3-m))^{\frac{3}{3-m}}} \Big\{ F_0e^t(c_1t + c_2 - c_1^2 - 2c_1)(3(k+1) - m(k+2)) + \frac{(F_0e^t + c_1)^2(9(k^2-2k-1) + 6m(k+2) - (k+2)^2m^2)}{(k+2)(3-m)} - \frac{(27)^{\frac{1}{3-m}}(k-1)(F_0e^t + c_1)}{(F_0e^t + c_1t + c_2)^{\frac{2m-3}{m-3}}} \Big\} \tag{32}$$

Coefficient of bulk viscosity ξ is

$$\xi = \frac{3^{\frac{2m-3}{3-m}}(\omega-\gamma) F_0(F_0e^t + c_1t + c_2)^{\frac{2m-3}{3-m}}}{\kappa(1+\gamma-\zeta)(F_0e^t + c_1)((k+2)(3-m))^{\frac{3}{3-m}}} \Big\{ F_0e^t(c_1t + c_2 - c_1^2 - 2c_1)(3(k+1) - m(k+2)) + \frac{(F_0e^t + c_1)^2(9(k^2-2k-1) + 6m(k+2) - (k+2)^2m^2)}{(k+2)(3-m)} \Big\} \tag{33}$$

In Nojiri and Odintsov [19] discussed reconstruction technique and obtained the solution of the field equations for FRW model. Assuming the average scale factor proportional to exponential function they have discussed different cosmological epochs, matter dominated phase, transition phase and accelerated phase. Here we have considered a LRS Bianchi type-I bulk viscous string cosmological models in $f(R)$ gravity and obtained exact solution of $f(R)$ gravity field equations. However following Nojiri and Odintsov [19] we have represented graphically the energy density in matter dominated, radiation and stiff fluid and discussed their physical significance. This discussion is quite similar to the discussion presented by Nojiri and Odintsov [19].

The behavior of rest energy density ρ and particle energy density ρ_p against time t for different phases of universe such as matter dominated ($\gamma = 0$), radiation dominated ($\gamma = 1/3$) and stiff fluid ($\gamma = 1$) eras is shown in figures 1 and 2 respectively. It can be observed that the both ρ and ρ_p for three models are positive throughout the evolution and increases with time. The realistic energy conditions, $\rho \geq 0$ and $\rho_p \geq 0$ are satisfied in our model. From figure 3 we observed that the energy tension density λ is negative throughout the evolution of model and decreases with cosmic time.

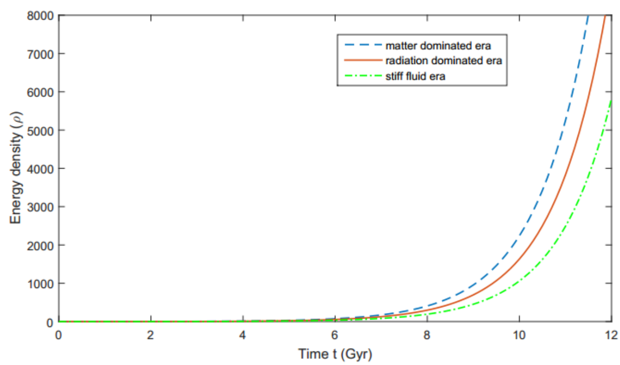

FIGURE 1. Plot of energy density versus time for $F_0 = 1.5$, $c_1 = 1.15$, $c_2 = 0.1$, $k = 0.95$, $\zeta = 0.1$ and $m = 1.38$.

4. Properties of the model

The following properties are important in discussion of cosmology of obtained anisotropic bulk viscous string model:
Spatial volume of the model

$$V = XY^2 = \left(\frac{3(F_0\ e^t + c_1\ t + c_2)}{(k+2)(3-m)} \right)^{\frac{3}{3-m}} \tag{34}$$

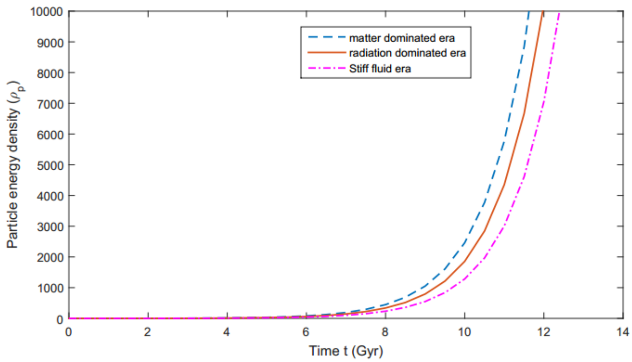

FIGURE 2. Plot of particle energy density versus time for $F_0 = 1.5$, $c_1 = 1.15$, $c_2 = 0.1$, $k = 0.95$, $\zeta = 0.1$ and $m = 1.38$.

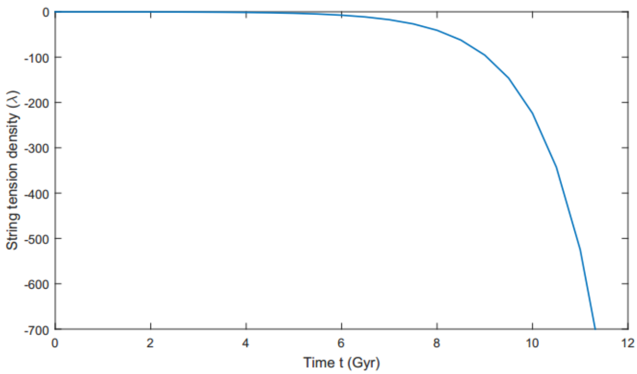

FIGURE 3. Plot of string tension density versus time for $F_0 = 1.5$, $c_1 = 1.15$, $c_2 = 0.1$, $k = 0.95$ and $m = 1.38$.

Directional Hubble's parameters are

$$H_1 = \frac{3(F_0e^t + c_1)}{(k+2)(3-m)(F_0e^t + c_1t + c_2)}, \tag{35}$$

$$H_2 = H_3 = \frac{3k(F_0e^t + c_1)}{(k+2)(3-m)(F_0e^t + c_1t + c_2)} \tag{36}$$

Hubble's parameter is

$$H = \frac{F_0e^t + c_1}{(3-m)(F_0e^t + c_1t + c_2)} \tag{37}$$

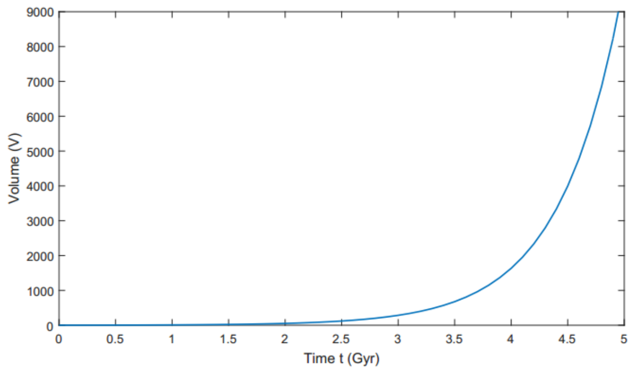

FIGURE 4. Plot of deceleration parameter versus time for $m = 1.38$, $F_0 = 1.5$, $c_1 = 1.15$, $c_2 = 0.1$.

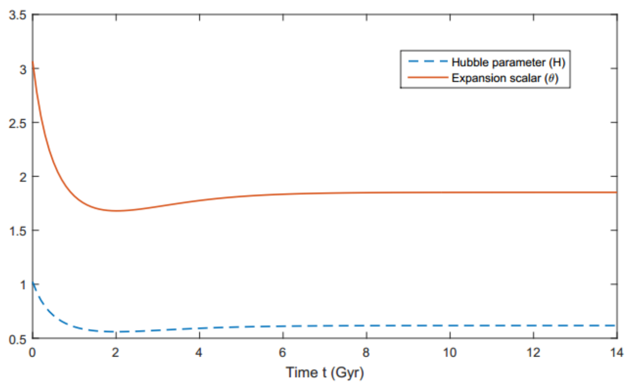

FIGURE 5. Plot of Hubble parameter and expansion scalar versus time for $F_0 = 1.5$, $c_1 = 1.15$, $c_2 = 0.1$, $m = 1.38$ and $k = 0.95$.

Expansion scalar and shear scalar are given by

$$\theta = \frac{3(F_0e^t + c_1)}{(3-m)(F_0e^t + c_1t + c_2)}, \tag{38}$$

$$\sigma^2 = \frac{3(k-1)^2(F_0e^t + c_1)^2}{(k+2)^2(3-m)^2(F_0e^t + c_1t + c_2)^2} \tag{39}$$

Anisotropic parameter A_h and deceleration parameter q are given by

$$A_h = 3\left(\frac{k-1}{k+2}\right)^2 \tag{40}$$

$$q = 2 - m + \frac{F_0 e^t (m-3)(F_0 e^t + c_1 t + c_2)}{(F_0 e^t + c_1)^2}. \tag{41}$$

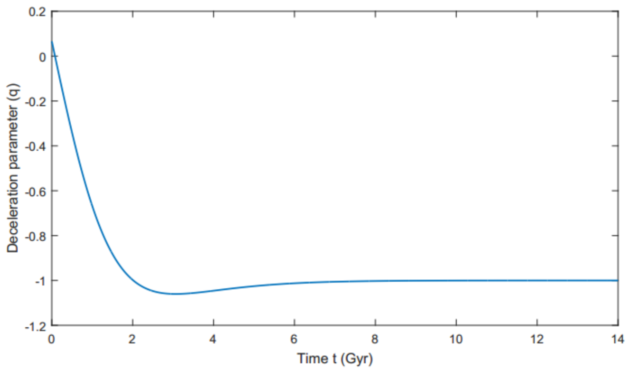

FIGURE 6. Plot of deceleration parameter versus time for $m = 1.38$, $F_0 = 1.5$, $c_1 = 1.15$, $c_2 = 0.1$.

From figure 4, we observed that the volume of the model increases exponentially with time. This shows that the model represents an expanding universe. It is observed from the figure 5 that the Hubble parameter and expansion scalar are decreasing functions with time and become constant at the present epoch, which is in good agreement with observations. We have plotted the variation of the deceleration parameter in terms of cosmic time t in figure 6. The sign of deceleration parameter q indicates whether the models accelerates or decelerates. A positive sign of q indicates decelerating model whereas negative sign of q shows acceleration of the model. It is observed that q decreases from positive region to negative region and finally approaches to -1. This shows that the model exhibits a smooth transition from early decelerated phase to present accelerated phase.

5. Conclusions

We have considered the $f(R)$ gravity field equations in the presence of bulk viscous fluid containing one dimensional cosmic strings in the anisotropic LRS Bianchi type-I spacetime. The field equations have been solved using some viable physical conditions. The solution exhibits some interesting features of the universe.

The model (27) is an expanding model of the universe since the volume of the model increases as time t increases for $0 < m < 3$. At initial epoch, the volume of our model is non-zero and other parameters are finite, thus the model free from initial singularity (i.e., at $t = 0$) which supports the analysis of Murphy [15] that the introduction of bulk viscosity avoids initial singularity. Moreover, we find that the H, θ and σ^2 decrease with cosmic time and attains a constant value in the future. Deceleration parameter indicates a smooth transition of the model from decelerated phase to accelerating phase (Fig. 6). It is also concluded that the nature of deceleration parameter agree with the recent cosmological observations. We observed that A_h is constant, which shows that the model does not approaches to isotropy throughout the evolution of the universe.

The rest energy density ρ and particle energy density ρ_p are positive functions of time and diverge as time increases for all three major eras of the universe (Figs. 1 and 2). Hence, the realistic energy conditions, $\rho \geq 0$ and $\rho_p \geq 0$ are satisfied in our model. We also observed that the energy tension density λ is negative throughout the evolution of universe. It is interesting to note that when $k = 1$ (i.e., isotropic model), we have $\lambda = 0$, $A_h = 0$ and $\sigma^2 = 0$, i.e., in this spacial case the strings in the universe do not survive and the model becomes isotropic and shear free.

Also, it is possible to generalize these results for other modified theories of gravity like $f(R,T)$ [27, 28]. It is interesting to consider bulk viscous string fluid matter distribution in $f(G)$ and $f(T)$ theories (a review of modified theories is given in Nojiri et al. [20]). We will investigate them in our future study.

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