

A NOTE ON FURSTENBURG FAMILIES AND SHADOWING IN TOPOLOGICAL SPACES

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ABSTRACT. In this note we use the concepts of *lower density* and *upper density* for elements of Furstenburg families to measure the set of error times in the shadowing process in non-metrizable topological spaces. We show that the measure of errors is the same for all iterations of a dynamical system.

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1. INTRODUCTION

The notion of *shadowing*, introduced in the 1970s independently by Anosov [2] and Bowen [3], is a well known property in the qualitative theory of dynamical systems [4, 7].

In [6] the authors introduced the concept of *partial* shadowing property formulated in terms of density of subsets of $\mathbb{N}$. The main idea of the definition of ergodic pseudo-orbit is to use the concepts of density.

2. PRIMINILARIES

We work with uniform space $(X, \mathcal{U})$ where the family $\mathcal{U}$ satisfying the following properties:

- U1) for any $E_1, E_2 \in \mathcal{U}$, the intersection $E_1 \cap E_2$ is also contained in $\mathcal{U}$, and if $E_1 \subset E_2$ and $E_1 \in \mathcal{U}$, then $E_2 \in \mathcal{U}$;
- U2) every set $E \in \mathcal{U}$ contains the diagonal $\Delta_X = \{(x, x) \mid x \in X\}$;
- U3) if $E \in \mathcal{U}$, then $E^T = \{(y, x) \mid (x, y) \in E\} \in \mathcal{U}$;
- U4) for any $E \in \mathcal{U}$ there exists $\hat{E} \in \mathcal{U}$ such that $\hat{E} \circ \hat{E} \subset E$, where

$$\hat{E} \circ \hat{E} = \{(x, y) \mid \text{there is a } z \in X \text{ with } (x, z) \in \hat{E}, (z, y) \in \hat{E}\}.$$

Each elements of $\mathcal{U}$ is called an *entourages* and $E[x] = \{y \in X \mid (x, y) \in E\}$ is called the cross section of E at a point $x \in X$. To avoid confusion, we denote subsets of the product space $X \times X$ by D, E, F , etc., and the subsets of the base space X by $\dot{U}, \dot{V}, \dot{W}$, etc. and denote

$$E^n := E \circ E \circ \dots \circ E \text{ (n times)}$$

$$= \{(x, y) \mid \text{there exist } z_0 = x, z_1, \dots, z_n = y; (z_{i-1}, z_i) \in E \text{ for } i = 1, 2, \dots, n\}.$$

By a dynamical system we mean a pair (X, f) where X is a completely regular topological space and $f : X \rightarrow X$ is a uniformly continuous map. For a given dynamical system (X, f) , we use the product map $F = f \times f$ on $X \times X$.

Definition 2.1. Let (X, f) be a dynamical system and let D and E be entourages of X . A (D, f) -chain of length n is a sequence $\xi = \{x_i\}_{i=0}^n$ such that $(f(x_i), x_{i+1}) \in D$ for $i = 0, \dots, n-1$. An (f, D) -pseudo-orbit is a infinite (f, D) -chain.

If there is no danger of confusion, we write D -chain and D -pseudo-orbit instead of (f, D) -chain and (f, D) -pseudo-orbit, respectively.

Definition 2.2. We say that a D -pseudo-orbit $\xi = \{x_i\}$ is E -shadowed by a point $y \in X$ if $(f^i(y), x_i) \in E$ for all $i \in \mathbb{N}_0$.

Definition 2.3. [5] A dynamical system (X, f) has the topological shadowing property if for every entourage E of X , we can find an entourage D of X such that every D -pseudo-orbit is E -shadowed by some point y in X .

Denote by $\mathcal{TS}$ the set of all dynamical systems with the topological shadowing property.

Definition 2.4. *A dynamical system (X, f) is called topologically chain transitive if for any entourage D of X and any two points x and y there exists a D -chain from x to y . It is said to be chain recurrent if for any entourage D and any $x \in X$ there is D -chain from x to itself. It is called topologically chain mixing if for any two points $x, y \in X$ and any entourage D of X there is a positive integer N such that for any $n \geq N$ there is a D -chain from x to y of length n .*

3. MAIN RESULTS

A family $\mathcal{F}$ of subsets of $\mathbb{N}_0$ is called a *Furstenburg family* if for any $\mathbb{S} \in \mathcal{F}$, the inclusion $\mathbb{S} \subset \mathbb{S}'$ implies that $\mathbb{S}' \in \mathcal{F}$. For any set $\mathbb{S} \subset \mathbb{N}_0$, the upper density of $\mathbb{S}$ is defined by

$$\bar{d}(\mathbb{S}) := \limsup_{n \rightarrow \infty} \frac{1}{n} |\mathbb{S} \cap \{0, 1, \dots, n-1\}|$$

and the lower density of $\mathbb{S}$ is defined by

$$\underline{d}(\mathbb{S}) := \liminf_{n \rightarrow \infty} \frac{1}{n} |\mathbb{S} \cap \{0, 1, \dots, n-1\}|.$$

If there exists a number $d(\mathbb{S})$ such that $\bar{d}(\mathbb{S}) = \underline{d}(\mathbb{S}) = d(\mathbb{S})$, then we say that the set $\mathbb{S}$ has density $d(\mathbb{S})$. For any $\alpha \in [0, 1)$ we define the following Furstenburg families:

$$\underline{\mathcal{D}}(\alpha) = \{\mathbb{S} \subset \mathbb{N}_0 \mid \underline{d}(\mathbb{S}) > \alpha\}, \quad \overline{\mathcal{D}}(\alpha) = \{\mathbb{S} \subset \mathbb{N}_0 \mid \bar{d}(\mathbb{S}) > \alpha\}.$$

If $\xi = \{x_i\}_{i=0}^\infty$ and $\eta = \{y_i\}_{i=0}^\infty$ are sequences in X , then for any entourage E of X we define

$$\Lambda(\xi, \eta, E) = \{i \in \mathbb{N}_0 \mid (x_i, y_i) \in E\}, \quad \Lambda^c(\xi, \eta, E) = \{i \in \mathbb{N}_0 \mid (x_i, y_i) \notin E\}.$$

Here we use the following notations

$$\begin{aligned} \Lambda(\xi, f, E) &:= \{i \in \mathbb{N}_0 \mid (x_{i+1}, f(x_i)) \in E\} \\ \Lambda^c(\xi, f, E) &:= \{i \in \mathbb{N}_0 \mid (x_{i+1}, f(x_i)) \notin E\} \\ \Lambda(\xi, z, f, E) &:= \{i \in \mathbb{N}_0 \mid (x_i, f^i(z)) \in E\} \\ \Lambda^c(\xi, z, f, E) &:= \{i \in \mathbb{N}_0 \mid (x_i, f^i(z)) \notin E\}. \end{aligned}$$

If $\mathbb{S} \subset \mathbb{N}_0$, then we denote by $\mathbb{S}_n$ the set $\mathbb{S} \cap \{0, 1, \dots, n-1\}$. A sequence $\xi = \{x_i\}_{i=0}^\infty$ is called an ergodic D -pseudo-orbit provided that the set $\Lambda^c(\xi, f, D)$ has density

zero, that is $\lim_{n \rightarrow \infty} 1/n |\Lambda_n^c(\xi, f, D)| = 0$. For an entourage E of X , an ergodic D -pseudo-orbit ξ is said to be ergodically E -shadowed by a point $x \in X$ provided that the set $\Lambda^c(\xi, z, f, E)$ has density zero, that is $\lim_{n \rightarrow \infty} 1/n |\Lambda_n^c(\xi, z, f, E)| = 0$.

Definition 3.1. *A dynamical system (X, f) has the topological ergodic shadowing property provided that for any entourage E of X there exist an entourage D of X such that any D -pseudo-orbit can be ergodically E -shadowed by some point in X .*

In [1] the author prove the following results.

Proposition 3.2. [1] *Let (X, f) be a dynamical system with the topological ergodic shadowing property. Then it is topologically chain transitive.*

Proposition 3.3. [1] *Let (X, f) be a dynamical system with the topological ergodic shadowing property. Then it has the topological shadowing property.*

Next we study a stronger form of shadowing formulated in terms of bounds on density of discrete times i where proper tracing appears.

Definition 3.4. *A dynamical system (X, f) has the topological $\overline{\mathcal{D}}(\alpha)$ -shadowing property (res. topological $\underline{\mathcal{D}}(\alpha)$ -shadowing Property) provided that for any entourage E of X there exists an entourage D of X such that for any ergodic D -pseudo-orbit ξ there exists a point $z \in X$ such that $\Lambda(\xi, z, E, f) \in \overline{\mathcal{D}}(\alpha)$ (res. $\Lambda(\xi, z, E, f) \in \underline{\mathcal{D}}(\alpha)$).*

Theorem 3.5. *Let (X, f) be a dynamical system and $\alpha \in [0, 1)$. Then the following statements are equivalent.*

- (1) (X, f) has the $\overline{\mathcal{D}}(\alpha)$ -shadowing property;
- (2) (X, f^k) has the $\overline{\mathcal{D}}(\alpha)$ -shadowing property for any positive integer k ;
- (3) (X, f^k) has the $\overline{\mathcal{D}}(\alpha)$ -shadowing property for some positive integer k .

Proof. (1 $\Rightarrow$ 2) Suppose that E is an entourage and choose an entourage $\hat{E}$ with $\hat{E}^4 \subset E$. By the uniform continuity of f there exists $\hat{D} \subset \hat{E}$ such that $\cup_{i=0}^k (f^i(x), f^i(y)) \subset \hat{E}$ for all $(x, y) \in \hat{D}$. Since f has the $\overline{\mathcal{D}}(\alpha)$ -shadowing property, there exists an entourage $D \subset \hat{D}$ such that for every ergodic D -pseudo-orbit ξ of f we can find some point z in X with $\Lambda(\xi, z, f, \hat{D}) \in \overline{\mathcal{D}}(\alpha)$. Suppose that $\xi = \{x_i\}_{i=0}^\infty$ is an ergodic D -pseudo-orbit of f^k . For $i \in \mathbb{N}_0$ and $0 \leq j \leq k$ put $y_{ik+j} = f^j(x_i)$. We see that $\eta = \{y_i\}_{i=0}^\infty$ is an ergodic D -pseudo-orbit of f . Hence there exists $z \in X$

such that $\Lambda(\eta, z, f, \hat{D}) \in \overline{\mathcal{D}}(\alpha)$. We are going to show that $\Lambda(\eta, z, f^k, E) \in \overline{\mathcal{D}}(\alpha)$. Indeed, if $\Lambda(\eta, z, f^k, E) \notin \overline{\mathcal{D}}(\alpha)$, then

$$\begin{aligned} \epsilon &= \liminf_{n \rightarrow \infty} \frac{1}{n} |\Lambda_n^c(\eta, z, f^k, E)| \\ &= 1 - \limsup_{n \rightarrow \infty} |\Lambda_n(\eta, z, f^k, E)| \geq 1 - \alpha. \end{aligned}$$

By the definition of limit inferior for any fixed positive integer m we can find a positive integer a_m such that $\frac{1}{n} |\Lambda_n(\eta, z, f^k, E)| \geq \epsilon - 1/(2m)$ for any $n \geq a_m$. On the other hand ξ is an ergodic D -pseudo-orbit of f^k , so

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\Lambda_n^c(\xi, f^k, D) + 1| = 1.$$

Hence if we put $\mathbb{E}_n = (\Lambda_n(\xi, f^k, D) + 1) \cap \Lambda_n^c(\xi, z, f^k, E)$, then

$$\liminf_{n \rightarrow \infty} \frac{1}{n} |\mathbb{E}_n| \geq \epsilon - \frac{1}{2m}.$$

Thus there exists $b_m > a_m$ such that $\frac{1}{n} |\mathbb{E}_n| \geq \epsilon - \frac{1}{m}$ for any $n \geq b_m$.

Note that $f^j(y_{ik-j}) = f^j(y_{(i-1)k+k-j}) = f^j(f^{k-j}(x_{i-1})) = f^k(x_{i-1})$, so if $i \in \mathbb{E}_n$ then $(f^j(y_{ik-j}), y_{ik}) = (f^j(y_{ik-j}), x_i) \in D$ and $(f^{ik}(z), y_{ik}) = (f^{ik}(z), x_i) \notin E$. We claim that $(f^{ik}(z), f^j(y_{ik-j}))$ does not belong to $\hat{E}^3$. Indeed if this is not true, since $(f^j(y_{ik-j}), y_{ik}) \in \hat{E}$ then we must have $(f^j(f^{ik}(z)), y_{ik}) \in \hat{E}^3 \circ \hat{E} \subset E$ which is a contradiction. Therefore $(f^{ik}(z), f^j(y_{ik-j})) \notin \hat{E}^3$, so $(f^{ik-j}(z), y_{ik-j}) \notin \hat{D}$ for $n \geq b_m$, $i \in \mathbb{E}_n$ and $1 \leq j < k$. Hence for any $n \geq b_m$,

$$\bigcup_{j=1}^{k-1} (k\mathbb{E}_n - j) \subset \Lambda_{nk}^c(z, \eta, f, \hat{D}).$$

Since $(k\mathbb{E}_n + p) \cap (k\mathbb{E}_n + q) = \emptyset$ for $0 \leq p < q < k$, we conclude that

$$|\cup_{j=0}^{k-1} (k\mathbb{E}_n - j)| = k|\mathbb{E}_n|.$$

This implies that for every $n \geq b_m$

$$\frac{1}{nk} |\Lambda_{nk}^c(z, \eta, f, \hat{D})| \geq \frac{1}{n} |\mathbb{E}_n| \geq \epsilon - \frac{1}{m}.$$

Consequently, for any $r \geq kb_m$ and $s \geq b_m$ with $sk \leq r \leq (s+1)k$,

$$\begin{aligned} \frac{1}{r} |\Lambda_r^c(z, \eta, f, \hat{D})| &\geq \frac{1}{(s+1)k} |\Lambda_{sk}^c(z, \eta, f, \hat{D})| \\ &\geq \frac{1}{(s+1)k} (|\Lambda_{(s+1)k}^c(z, \eta, f, \hat{D})| - k) \\ &\geq \frac{1}{(s+1)k} |\Lambda_{(s+1)k}^c(z, \eta, f, \hat{D})| - \frac{1}{s+1} \\ &\geq \epsilon - \frac{1}{m} - \frac{k}{r}. \end{aligned}$$

Thus

$$\liminf_{n \rightarrow \infty} \frac{1}{n} |\Lambda_n^c(z, \eta, f, \hat{D})| \geq \epsilon - \frac{1}{m}.$$

Choosing m large enough, we obtain

$$\liminf_{n \rightarrow \infty} \frac{1}{n} |\Lambda_n^c(z, \eta, f, \hat{D})| \geq \epsilon \geq 1 - \alpha.$$

Hence

$$\limsup_{n \rightarrow \infty} \frac{1}{n} |\Lambda_n^c(z, \eta, f, \hat{D})| = 1 - \liminf_{n \rightarrow \infty} \frac{1}{n} |\Lambda_n^c(z, \eta, f, \hat{D})| \leq 1 - \epsilon \leq \alpha.$$

which is a contradiction, This completes the proof.

(2 $\Rightarrow$ 3) is obvious.

(3 $\Rightarrow$ 1) Suppose that E is an entourage and choose an entourage $\hat{E}$ such that $\hat{E}^k \subset E$. Obviously $\hat{E} \circ \hat{E} \subset E$. By the uniform continuity of f there exists an entourage $G \subset \hat{E}$ such that $\cup_{i=0}^k (f^i(x), f^i(y)) \in \hat{E}$ for all $(x, y) \in G$. For $x_0, x_1, \dots, x_k, z \in X$, if $\cup_{i=0}^k (f(x_i), x_{i+1}) \cup \{(z, x_0)\} \subset G$ then $\cup_{i=0}^k (f^i(z), x_i) \subset E$. Indeed

$$\begin{aligned} (z, x_0) &\in G \subset E \\ (f(z), f(x_0)), (f(x_0), x_1) &\in \hat{E} \Rightarrow (f(z), x_1) \in \hat{E}^2 \subset E \\ &\dots \\ (f^k(z), f^k(x_0)), (f^k(x_0), f^{k-1}(x_1)), \dots, (f(x_{k-1}), x_k) &\in \hat{E} \\ \Rightarrow (f^k(z), x_k) &\in \hat{E}^k \subset E. \end{aligned}$$

Choose an entourage $\hat{G}$ such that $\hat{G}^4 \subset G$. Since the map f^k has the $\overline{D}(\alpha)$ -shadowing property, there exists an entourage D such that $D \subset \hat{G}$ and for every ergodic D -pseudo-orbit ξ of f^k there exists $z \in X$ such that $\Lambda(\xi, z, f^k, \hat{G}) \in \overline{D}(\alpha)$. Similar to the last paragraph we can find an entourage $\hat{D}$ with $\hat{D} \circ \hat{D} \subset D$ such

that if $\cup_{i=0}^k (f(x_i), x_{i+1}) \cup \{(z, x_0)\} \in \hat{D}$, then $\cup_{i=0}^k (f^i(z), x_i) \subset D$. Suppose that $\xi = \{x_i\}_{i=0}^\infty$ is an ergodic $\hat{D}$ -pseudo-orbit for f and define $z_i = x_{ik}$ for $i = 0, 1, 2, \dots$. Denote

$$\begin{aligned} \mathbb{I}_n &= \bigcap_{j=0}^k \{0 \leq i < n : (f(x_{ik+j}), x_{ik+j+1}) \in \hat{D}\} \\ \mathbb{I} &= \bigcup_{n \in \mathbb{N}} \mathbb{I}_n. \end{aligned}$$

So

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} |\Lambda_n(\xi, f, \hat{D})| &= 1, \\ \lim_{n \rightarrow \infty} \frac{1}{n} |0 \leq i < n : (f(x_{ik+j}), x_{ik+j+1})| &= 1. \end{aligned}$$

This implies that $\lim_{n \rightarrow \infty} \frac{1}{n} |\mathbb{I}_n| = 1$. If $i \in \mathbb{I}$ then $(f(x_{ik+j}), x_{ik+j+1}) \in \mathbb{D}$ for all $j = 0, 1, \dots, k$. By the choice of $\hat{D}$ we obtain $(f^k(z_i), z_{i+1}) \in \mathbb{D}$. Thus $\eta = \{z_i\}_{i=0}^\infty$ is an ergodic D -pseudo-orbit of f^k and consequently there exists $z \in X$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} |\Lambda_n(\eta, z, f^k, \hat{G})| > \alpha.$$

In particular, there exist a number $\epsilon > \alpha$ and a strictly increasing sequence $\{n_s\}_{s=1}^\infty$ of natural numbers such that

$$\frac{1}{n_s} |\Lambda_{n_s}(\eta, z, f^k, \hat{G})| \geq \epsilon.$$

Define

$$\begin{aligned} \mathbb{J}_{n_s} &:= \bigcap_{i=0}^{k-1} [\Lambda(\xi, f, \hat{D}) - i] \cap k\Lambda_{n_s}(\eta, z, f^k, \hat{G}) \\ \mathbb{K}_s &:= \left(\bigcap_{i=0}^{k-1} [\Lambda(\xi, f, \hat{D}) - i] \right) \cap \{0, 1, \dots, s-1\}. \end{aligned}$$

Hence

$$(1) \quad \mathbb{J}_{n_s} = \mathbb{K}_{n_s} \cap k\Lambda_{n_s}(\eta, z, f^k, \hat{G}).$$

If $j \in \mathbb{J}_{n_s}$ then $j/k \in \Lambda_{n_s}(\eta, z, f^k, \hat{G})$ and $i+j \in \Lambda(\xi, f, \hat{D})$. So $(f^j(z), x_j) = ((f^k)^{j/k}(z), z_{j/k}) \in \hat{G}$ and $(f(x_{i+j}, x_{i+j+1})) \in \hat{D} \subset \hat{E}$ for all $i = 0, 1, \dots, k-1$. By the choice of $\hat{E}$ we have $(f^{i+j}(z), x_{i+j}) \in E$ for $i = 0, 1, \dots, k-1$, and so

$$(2) \quad \bigcup_{i=0}^{k-1} [\mathbb{J}_{n_s} + i] \subset \Lambda_{(n_s+1)k}(\xi, z, f, E)$$

By 1 and the fact $\lim_{s \rightarrow \infty} \frac{1}{s} |\mathbb{K}_s| = 1$ we obtain that for all $i = 0, 1, \dots, k-1$

$$\begin{aligned} \liminf_{s \rightarrow \infty} \frac{1}{(n_s + 1)k} |\mathbb{J}_{n_s+i}| &= \liminf_{s \rightarrow \infty} \frac{1}{(n_s + 1)k} |k\Lambda_{n_s}(\eta, z, f^k, \hat{G}) + i| \\ &\geq \liminf_{s \rightarrow \infty} \frac{n_s \epsilon}{(n_s + 1)k} = \frac{\epsilon}{k}. \end{aligned}$$

Clearly $[\mathbb{J}_{n_s} + p] \cap [\mathbb{J}_{n_s} + q] = \emptyset$ for all $0 \leq p < q < k$, so 2 implies that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} |\Lambda_n(\xi, z, f, E)| &\geq \liminf_{n \rightarrow \infty} \frac{1}{(n_s + 1)k} |\Lambda_{(n_s+1)k}(\xi, z, f, E)| \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{(n_s + 1)k} \sum_{i=0}^{k-1} |\mathbb{J}_{n_s+i}| \\ &\geq \sum_{i=0}^{k-1} \liminf_{n \rightarrow \infty} \frac{1}{(n_s + 1)k} |\mathbb{J}_{n_s+i}| \\ &\geq \epsilon > \alpha. \end{aligned}$$

Therefore (X, f) has the $\overline{\mathcal{D}}(\alpha)$ -shadowing property. $\square$

A similar proof works for the following theorem.

Theorem 3.6. *Let (X, f) be a dynamical system and $\alpha \in [0, 1)$. Then the following statements are equivalent:*

- (1) (X, f) has the $\underline{\mathcal{D}}(\alpha)$ -shadowing property;
- (2) (X, f^k) has the $\underline{\mathcal{D}}(\alpha)$ -shadowing property for any positive integer k ;
- (3) (X, f^k) has the $\underline{\mathcal{D}}(\alpha)$ -shadowing property for some positive integer k .

CONCLUSION

In this note we use the Furstenbug families to measure the weight of times where tracing holds. Using this method to measure the times where other dynamical properties hold will be a topic for further research.

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