

THE ENTROPY OF HYPERFUZZY SETS

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ABSTRACT. This paper deals with studying the entropy of dynamical systems on hyperfuzzy sets. At first we define the entropy of hyperfuzzy partitions and conditional entropy. Then, some basic properties of these notions are established. Furthermore, we introduce the notion of a dynamical systems on hyperfuzzy sets and the entropy of it. Finally, we define a generator of this dynamical system and prove Kolmogorov-Sinai's Theorem is also true for our mentioned generator.

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1. INTRODUCTION

The fundamental concept of fuzzy sets was initiated by Zadeh [10]. The fuzzy algebraic structures play a key role in mathematics and have wide applications in many other branches such as theoretical physics, computer science, control engineering, information science, coding theory, group theory, real analysis, measure theory, etc.

Ghosh and Samanta introduced the concepts of hyperfuzzy sets [3]. Let X be a given set. The mapping $\hat{\mu} : X \longrightarrow P^*([0, 1])$ is called a hyperfuzzy set of X , where $P^*([0, 1])$ denotes the set of all non-empty subset of $[0, 1]$.

The concept of entropy originated in the physical and engineering sciences, but now it plays an important role in all areas of science. In 1865, Rudolf Clausius presented the first mathematical explanation for the concept of entropy. Entropy was first introduced into the theory of dynamical systems by Kolmogorov [4]. Then Kolmogorov's definition was improved by Sinai [8].

Let β be a σ - algebra on X and ρ be a measure on X . The triple (X, β, ρ) is called a probability space if $\rho(X) = 1$. The notion of a fuzzy partition of the probability space (X, β, ρ) was introduced by Ruspini [6] as a finite system of measurable functions $f_i : X \rightarrow [0, 1]$, $i = 1, 2, \dots, n$ such that $\sum_{i=1}^n f_i = 1$, for all $x \in X$. The entropy of this fuzzy partition was given by the classical formula $H(A) = -\sum_{i=1}^n m(f_i) \log(m(f_i))$ whenever $m(f_i) \neq 0$ and $H(A) = 0$ when $m(f_i) = 0$. Several authors have investigated the fuzzy entropy [1, 2, 7].

The rest of the paper is organized as follows: In Sect. 2, we define some concepts about hyperfuzzy sets, a partition on a non-empty set X that is called a hyperfuzzy partition, the entropy with respect to the hyperfuzzy partitions of X , a common refinement of a hyperfuzzy partition and conditional entropy with respect to hyperfuzzy partitions and investigate some properties of them.

In Sect. 3, we introduce the notion of a dynamical system on hyperfuzzy sets and the entropy of it. Also we introduce the generator of a dynamical system and prove Kolmogorov-Sinai's Theorem is also true for our mentioned generator which supplies a tool for calculating the entropy of this dynamical system and shows that it is finite for a special class of dynamical systems.

Notation. In this paper, we use P instead of $P^*([0, 1])$. P^X denotes the set of all hyperfuzzy sets on X . The hyperfuzzy set whose value is the set $\{1\}$ ($\{0\}$) for all $x \in X$, is denoted by $\hat{\mu}_{max}$ ($\hat{\mu}_{min}$).

2. THE ENTROPY OF A HYPERFUZZY PARTITION

In this section we introduce the concept of a hyperfuzzy partition of a non-empty set X and the entropy with respect to this partition. Then we study some properties about them. For this purpose, first we define the following notions:

Definition 2.1. Let X be a non-empty set and $\hat{\mu}$ and $\hat{\nu}$ be two hyperfuzzy sets on X , we define the following statements:

- 1) $\hat{\mu} \leq \hat{\nu}$ iff $\inf \hat{\mu}(x) \leq \inf \hat{\nu}(x)$ and $\sup \hat{\mu}(x) \leq \sup \hat{\nu}(x)$, for all $x \in X$,
- 2) $\hat{\mu} = \hat{\nu}$ iff $\hat{\mu} \leq \hat{\nu}$ and $\hat{\nu} \leq \hat{\mu}$,
- 3) $\hat{\mu}^c(x) = \{1 - a : a \in \hat{\mu}(x)\}$, for all $x \in X$.

Definition 2.2. Let X be a non-empty set and $\hat{\mu}, \hat{\nu}$ be two hyperfuzzy sets of X . Then operations $\oplus, \odot$ and $\circ$ defined by

$$\begin{aligned} (\hat{\mu} \oplus \hat{\nu})(x) &= \{(a + b) \wedge 1 : a \in \hat{\mu}(x), b \in \hat{\nu}(x)\}, \quad \forall x \in X, \\ (\hat{\mu} \odot \hat{\nu})(x) &= \{(a + b - 1) \vee 0 : a \in \hat{\mu}(x), b \in \hat{\nu}(x)\}, \quad \forall x \in X, \\ (\hat{\mu} \circ \hat{\nu})(x) &= \{(a.b) : a \in \hat{\mu}(x), b \in \hat{\nu}(x)\}, \quad \forall x \in X. \end{aligned}$$

Definition 2.3. Let X be a non-empty set. A state on hyperfuzzy sets of X is a mapping $s : P^X \rightarrow [0, 1]$, if it satisfies in the following conditions:

- 1) $s(\hat{\mu}_{max}) = 1$, $s(\hat{\mu}_{min}) = 0$,
- 2) $s(\hat{\mu} \oplus \hat{\nu}) + s(\hat{\mu} \odot \hat{\nu}) = s(\hat{\mu}) + s(\hat{\nu})$,
- 3) if $\hat{\mu}_n \nearrow \hat{\mu}$, then $s(\hat{\mu}_n) \nearrow s(\hat{\mu})$.

Proposition 2.4. Let X be a non-empty set. For each $\hat{\mu}, \hat{\nu}$ and $\hat{\lambda} \in P^X$, the operations $\oplus, \odot$ and $\circ$, that are defined above, have the following properties:

- 1) $\hat{\mu} \oplus \hat{\nu} = \hat{\nu} \oplus \hat{\mu}$, and $\hat{\mu} \circ \hat{\nu} = \hat{\nu} \circ \hat{\mu}$,
- 2) $(\hat{\mu} \oplus \hat{\nu}) \oplus \hat{\lambda} = \hat{\mu} \oplus (\hat{\nu} \oplus \hat{\lambda})$, and $(\hat{\mu} \circ \hat{\nu}) \circ \hat{\lambda} = \hat{\mu} \circ (\hat{\nu} \circ \hat{\lambda})$,
- 3) if $\hat{\mu} \leq \hat{\nu}$, then $\hat{\mu} \odot \hat{\lambda} \leq \hat{\nu} \odot \hat{\lambda}$ and $\hat{\mu} \oplus \hat{\lambda} \leq \hat{\nu} \oplus \hat{\lambda}$,
- 4) if $\hat{\nu} \odot \hat{\lambda} = \hat{\mu}_{min}$, then $(\hat{\mu} \circ \hat{\nu}) \odot (\hat{\mu} \circ \hat{\lambda}) = \hat{\mu}_{min}$, and $\hat{\mu} \circ (\hat{\nu} \oplus \hat{\lambda}) = (\hat{\mu} \circ \hat{\nu}) \oplus (\hat{\mu} \circ \hat{\lambda})$.

Proof. We only prove 4, Since $\hat{\nu} \odot \hat{\lambda} = \hat{\mu}_{min}$, thus $(\hat{\mu} \circ \hat{\nu}) \odot (\hat{\mu} \circ \hat{\lambda}) \leq \hat{\nu} \odot \hat{\lambda} = \hat{\mu}_{min}$, therefore $(\hat{\mu} \circ \hat{\nu}) \odot (\hat{\mu} \circ \hat{\lambda}) = \hat{\mu}_{min}$. Let $a + b \leq 1$, for all $a \in \hat{\nu}(x), b \in \hat{\lambda}(x)$. Then $c.(a + b) = c.a + c.b \leq 1$ for all $c \in \hat{\mu}(x), a \in \hat{\nu}(x), b \in \hat{\lambda}(x)$, therefore $\hat{\mu} \circ (\hat{\nu} \oplus \hat{\lambda}) = (\hat{\mu} \circ \hat{\nu}) \oplus (\hat{\mu} \circ \hat{\lambda})$. $\square$

Definition 2.5. Let X be a non-empty set. A collection $A = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ of hyperfuzzy sets of X is said to be a partition if it satisfies in the following conditions:

- 1) $\hat{\mu}_{k+1} \odot (\oplus_{i=1}^k \hat{\mu}_i) = \hat{\mu}_{min}$, for any $k = 1, \dots, n - 1$,
- 2) $\oplus_{i=1}^n \hat{\mu}_i = \hat{\mu}_{max}$.

This partition is called a hyperfuzzy partition of X .

Lemma 2.6. *Let $A = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ be a hyperfuzzy partition of X . Then*

$$s(\oplus_{i=1}^n \hat{\mu}_i) = \sum_{i=1}^n s(\hat{\mu}_i).$$

Proof. Since $\hat{\mu}_1 \odot \hat{\mu}_2 = 0$, therefore by Definition 2.3 we have

$$s(\hat{\mu}_1 \oplus \hat{\mu}_2) = s(\hat{\mu}_1) + s(\hat{\mu}_2),$$

then by induction we have

$$s(\oplus_{i=1}^n \hat{\mu}_i) = \sum_{i=1}^n s(\hat{\mu}_i).$$

□

Definition 2.7. Let $A = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ and $B = \{\hat{\nu}_1, \dots, \hat{\nu}_m\}$ be two hyperfuzzy partitions of X . The common refinement of these partitions is defined as follows:

$$A \vee B = \{\hat{\mu}_i \circ \hat{\nu}_j : i = 1, \dots, n, j = 1, \dots, m\}.$$

Definition 2.8. If $A = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ and $B = \{\hat{\nu}_1, \dots, \hat{\nu}_m\}$ are two hyperfuzzy partitions of X . Then B is called a refinement of A , written as $A \leq B$, if there exists a partition $I(1), \dots, I(n)$ of the set $\{1, \dots, m\}$ such that

$$s(\hat{\mu}_i) = \sum_{j \in I(i)} s(\hat{\nu}_j),$$

for every $i = 1, \dots, n$.

Remark 2.9. The hyperfuzzy partition $A = \{\hat{\mu}_{max}\}$ of X is called trivial hyperfuzzy partition. One can easily check that any hyperfuzzy partition B is the refinement of A .

Proposition 2.10. *If $A = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ and $B = \{\hat{\nu}_1, \dots, \hat{\nu}_m\}$ are two hyperfuzzy partitions of X , then $A \vee B$ is a hyperfuzzy partition. Furthermore $A \leq A \vee B$.*

Proof. Since $(\hat{\mu}_{t_1} \circ \hat{\nu}_{s_1}) \odot (\hat{\mu}_{t_2} \circ \hat{\nu}_{s_2}) \leq \hat{\mu}_{t_1} \odot \hat{\mu}_{t_2} = \hat{\mu}_{min}$, thus $(\hat{\mu}_{t_1} \circ \hat{\nu}_{s_1}) \odot (\hat{\mu}_{t_2} \circ \hat{\nu}_{s_2}) = \hat{\mu}_{min}$, and $(\hat{\mu}_{t_3} \circ \hat{\nu}_{s_3}) \odot ((\hat{\mu}_{t_1} \circ \hat{\nu}_{s_1}) \oplus (\hat{\mu}_{t_2} \circ \hat{\nu}_{s_2})) \leq \hat{\mu}_{t_3} \odot (\hat{\mu}_{t_1} \oplus \hat{\mu}_{t_2}) = \hat{\mu}_{min}$, thus $(\hat{\mu}_{t_3} \circ \hat{\nu}_{s_3}) \odot ((\hat{\mu}_{t_1} \circ \hat{\nu}_{s_1}) \oplus (\hat{\mu}_{t_2} \circ \hat{\nu}_{s_2})) = \hat{\mu}_{min}$, and also

$$\oplus_{i=1, j=1}^{n, m} \hat{\mu}_i \circ \hat{\nu}_j = (\oplus_{i=1}^n \hat{\mu}_i) \circ (\oplus_{j=1}^m \hat{\nu}_j) = \hat{\mu}_{max} \circ \hat{\mu}_{max} = \hat{\mu}_{max}.$$

So $A \vee B$ is a hyperfuzzy partition. Let us mention that $A \vee B$ is indexed by $\{(i, j) : i = 1, \dots, n; j = 1, \dots, m\}$. Now we show that $A \leq A \vee B$.

Let $I(i) = \{(i, 1), \dots, (i, m)\}$. Then

$$s(\widehat{\mu}_i) = s(\oplus_{j=1}^m (\widehat{\mu}_i \circ \widehat{\nu}_j)) = \sum_{j=1}^m s(\widehat{\mu}_i \circ \widehat{\nu}_j) = \sum_{(i,j) \in I(i)} s(\widehat{\mu}_i \circ \widehat{\nu}_j),$$

for every $i = 1, \dots, n$. It follows that $A \leq A \vee B$. $\square$

Proposition 2.11. *If A, B and C be hyperfuzzy partitions of X such that $A \leq B$ and $B \leq C$, then $A \leq C$.*

Proof. Let $A = \{\widehat{\mu}_1, \dots, \widehat{\mu}_n\}$, $B = \{\widehat{\nu}_1, \dots, \widehat{\nu}_m\}$ and $C = \{\widehat{\lambda}_1, \dots, \widehat{\lambda}_p\}$. Since $A \leq B$ and $B \leq C$, there exists a partition $I(1), \dots, I(n)$ of the set $\{1, \dots, m\}$ such that

$$s(\widehat{\mu}_i) = \sum_{j \in I(i)} s(\widehat{\nu}_j),$$

for every $i = 1, \dots, n$, and also there exists a partition $I'(1), \dots, I'(n)$ of the set $\{1, \dots, p\}$ such that

$$s(\widehat{\nu}_j) = \sum_{k \in I'(j)} s(\widehat{\lambda}_k),$$

for every $j = 1, \dots, m$. Thus, there exists a partition $\cup_{j \in I(1)} I'(j), \dots, \cup_{j \in I(n)} I'(j)$ of the set $\{1, \dots, p\}$ such that

$$s(\widehat{\mu}_i) = \sum_{k \in \cup_{j \in I(i)} I'(j)} s(\widehat{\lambda}_k),$$

for every $i = 1, \dots, n$. It follows that $A \leq C$. $\square$

Corollary 2.12. *Let A, B and C be hyperfuzzy partitions of X , and $A \leq B$. Then $A \leq B \vee C$.*

Definition 2.13. If $A = \{\widehat{\mu}_1, \dots, \widehat{\mu}_n\}$ is a hyperfuzzy partition of X , we define the entropy of A by

$$H(A) = \sum_{i=1}^n f(s(\widehat{\mu}_i)),$$

where $f(x) = -x \log x$, for all $x > 0$ and $f(0) = 0$.

Definition 2.14. If $A = \{\widehat{\mu}_1, \dots, \widehat{\mu}_n\}$ and $B = \{\widehat{\nu}_1, \dots, \widehat{\nu}_m\}$ are two hyperfuzzy partitions of X , the conditional entropy of A given B is defined as follows:

- 1) If $A \neq B$, $H(A|B) = \sum_{i=1}^n \sum_{j=1}^m s(\hat{\nu}_j) f(\frac{s(\hat{\mu}_i \circ \hat{\nu}_j)}{s(\hat{\nu}_j)})$, where the j -terms with $s(\hat{\nu}_j) = 0$ are omitted.
- 2) If $A = B$, $H(A|B) = 0$.

Remark 2.15. Suppose that A is the trivial hyperfuzzy partition and B is an arbitrary hperfuzzy partition of X . One can easily check that $H(B|A) = H(B)$.

Proposition 2.16. *Let A , B and C be hyperfuzzy partitions of X and $B \leq C$ then $H(A|C) \leq H(A|B)$.*

Proof. Let $A = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$, $B = \{\hat{\nu}_1, \dots, \hat{\nu}_m\}$, $C = \{\hat{\lambda}_1, \dots, \hat{\lambda}_p\}$ and $\hat{\nu}_j = \oplus_{t \in I(j)} \hat{\lambda}_t$, where $\{I(1), \dots, I(k)\}$ is the corresponding partition and put (for fixed j) $\alpha_t = \frac{s(\hat{\lambda}_t)}{s(\hat{\nu}_j)}$. Then

$$\sum_{t \in I(j)} \alpha_t = \frac{1}{s(\hat{\nu}_j)} s(\oplus_{t \in I(j)} \hat{\lambda}_t) = 1;$$

hence by the concavness of f ,

$$\sum_t \alpha_t f(x_t) \leq f(\sum_t \alpha_t x_t).$$

Therefore,

$$\begin{aligned} H(A|C) &= \sum_i \sum_t s(\hat{\lambda}_t) f\left(\frac{s(\hat{\mu}_i \circ \hat{\lambda}_t)}{s(\hat{\lambda}_t)}\right) \\ &= \sum_i \sum_j s(\hat{\nu}_j) \sum_{t \in I(j)} \frac{s(\hat{\lambda}_t)}{s(\hat{\nu}_j)} f\left(\frac{s(\hat{\mu}_i \circ \hat{\lambda}_t)}{s(\hat{\lambda}_t)}\right) \\ &\leq \sum_i \sum_j s(\hat{\nu}_j) f\left(\sum_{t \in I(j)} \frac{s(\hat{\mu}_i \circ \hat{\lambda}_t)}{s(\hat{\nu}_j)}\right) \\ &= \sum_i \sum_j s(\hat{\nu}_j) f\left(\frac{s(\hat{\mu}_i \circ (\oplus_{t \in I(j)} \hat{\lambda}_t))}{s(\hat{\nu}_j)}\right) \\ &= \sum_i \sum_j s(\hat{\nu}_j) f\left(\frac{s(\hat{\mu}_i \circ \hat{\nu}_j)}{s(\hat{\nu}_j)}\right) = H(A|B). \end{aligned}$$

□

Corollary 2.17. *Let A and B be hyperfuzzy partitions of X , and $A \leq B$, then $H(A|B) = 0$.*

Corollary 2.18. *Let A and B be hyperfuzzy partitions of X . Then $H(B|A) \leq H(B)$*

Proof. Set $C = \{\widehat{\mu}_{max}\}$. According to Remark 2.15 and Proposition 2.16, we have

$$H(B|A) \leq H(B|C) = H(B).$$

□

Proposition 2.19. *Let A , B and C be hyperfuzzy partitions of X . Then $H(B \vee C|A) = H(B|A) + H(C|B \vee A)$.*

Proof. Let $A = \{\widehat{\mu}_1, \dots, \widehat{\mu}_n\}$, $B = \{\widehat{\nu}_1, \dots, \widehat{\nu}_m\}$ and $C = \{\widehat{\lambda}_1, \dots, \widehat{\lambda}_p\}$. By Definition 2.14, we have

$$\begin{aligned} H(B \vee C|A) &= \sum_{i,j,k} s(\widehat{\mu}_i) f\left(\frac{s(\widehat{\nu}_j \circ \widehat{\lambda}_k \circ \widehat{\mu}_i) s(\widehat{\nu}_j \circ \widehat{\mu}_i)}{s(\widehat{\mu}_i) s(\widehat{\nu}_j \circ \widehat{\mu}_i)}\right) \\ &= - \sum s(\widehat{\nu}_j \circ \widehat{\lambda}_k \circ \widehat{\mu}_i) \log \frac{s(\widehat{\nu}_j \circ \widehat{\lambda}_k \circ \widehat{\mu}_i)}{s(\widehat{\nu}_j \circ \widehat{\mu}_i)} \\ &\quad - \sum s(\widehat{\nu}_j \circ \widehat{\lambda}_k \circ \widehat{\mu}_i) \log \frac{s(\widehat{\nu}_j \circ \widehat{\mu}_i)}{s(\widehat{\mu}_i)} \\ &= H(C|B \vee A) + H(B|A). \end{aligned}$$

□

Proposition 2.20. *Let A , B , C be hyperfuzzy partitions of X . Then*

- 1) $H(B \vee C|A) \leq H(B|A) + H(C|A)$,
- 2) $H(B \vee C) = H(B) + H(C|B)$,
- 3) $H(B \vee C) \leq H(B) + H(C)$.

Proof. (1) By Propositions 2.16 and 2.19, this proof is clear.

(2) Let $A = \{\widehat{\mu}_{max}\}$; then $H(B|A) = H(B)$; and by Propositions 2.19, we have

$$\begin{aligned} H(B \vee C) &= H(B \vee C|A) \\ &= H(B|A) + H(C|B \vee A) \\ &= H(B) + H(C|B). \end{aligned}$$

(3) By the part (2) and Proposition 2.16, we obtain

$$H(B \vee C) = H(B) + H(C|B) \leq H(B) + H(C).$$

□

3. ENTROPY OF DYNAMICAL SYSTEMS ON HYPERFUZZY SETS

In this section, we introduce the notion of a dynamical system on hyperfuzzy sets and the entropy of it. Also we introduce the generator of the dynamical system and prove Kolmogorov-Sinai's Theorem is also true for our mentioned generator.

Definition 3.1. Let P^X is the set of all hyperfuzzy sets on X . A triple (P^X, s, ϕ) , where $s : P^X \rightarrow [0, 1]$ is a state on P^X and $\phi : P^X \rightarrow P^X$ is a mapping, is called a dynamical system on P^X , if it satisfies in the following conditions:

1. If $\hat{\mu} \odot \hat{\nu} = \hat{\mu}_{min}$ then $\phi(\hat{\mu}) \odot \phi(\hat{\nu}) = \hat{\mu}_{min}$ and $\phi(\hat{\mu} \oplus \hat{\nu}) = \phi(\hat{\mu}) \oplus \phi(\hat{\nu})$;
2. $\phi(\hat{\mu} \circ \hat{\nu}) = \phi(\hat{\mu}) \circ \phi(\hat{\nu})$;
3. $\phi(\hat{\mu}_{max}) = \hat{\mu}_{max}$;
4. $s(\phi(\hat{\mu})) = s(\hat{\mu})$ for all $\hat{\mu} \in P^X$.

Proposition 3.2. If (P^X, s, ϕ) is a dynamical system on P^X and $A = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ is a hyperfuzzy partition of X , then $\phi(A)$ is a hyperfuzzy partition of X , too. If $B = \{\hat{\nu}_1, \dots, \hat{\nu}_m\}$ is another hyperfuzzy partition of X , then $H(\phi(A)|\phi(B)) = H(A|B)$.

Proof. It is straightforward. □

Lemma 3.3. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative numbers such that $a_{r+t} \leq a_r + a_t$, for any $r, t \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} \frac{a_n}{n}$ exists.

Proof. See [5, 9]. □

Proposition 3.4. Let (P^X, s, ϕ) be a dynamical system on P^X . Then $\lim_{n \rightarrow \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} \phi^i(A))$, exists, for any hyperfuzzy partition $A = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$.

Proof. The proof is immediate from the part (3) of Proposition 2.20 and Lemma 3.3. □

Note that in the following results (P^X, s, ϕ) is a dynamical system on P^X .

Definition 3.5. For every hyperfuzzy partition A of X , we define

$$h(\phi, A) = \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \phi^i(A)\right).$$

Furthermore we define the entropy of the dynamical system by

$$h(\phi) = \sup\{h(\phi, A); A \text{ is a partition}\}.$$

Proposition 3.6. *Let A be a hyperfuzzy partition of X . Then*

$$h(\phi, A) = h(\phi, \bigvee_{j=0}^k \phi^j(A)), \quad \forall k \in \mathbb{N}.$$

Proof. By Definition 3.5, we obtain

$$\begin{aligned} h(\phi, \bigvee_{j=0}^k \phi^j(A)) &= \lim_{n \rightarrow \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} \phi^i(\bigvee_{j=0}^k \phi^j(A))) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} H(\bigvee_{t=0}^{n+k-1} \phi^t(A)) \\ &= \lim_{p \rightarrow \infty} \frac{p}{p-k} \frac{1}{p} H(\bigvee_{t=0}^{p-1} \phi^t(A)) \\ &= h(\phi, A). \end{aligned}$$

□

Proposition 3.7. *Let A be a hyperfuzzy partition of X . Then*

$$h(\phi, A) = \lim_{n \rightarrow \infty} H(A | \bigvee_{i=1}^n \phi^i(A)).$$

Proof. By Proposition 2.20, we have

$$\begin{aligned} H(\bigvee_{i=0}^k \phi^i(A)) &= H(A \vee \phi(\bigvee_{i=0}^{k-1} \phi^i(A))) \\ &= H(\phi(\bigvee_{i=0}^{k-1} \phi^i(A))) + H(A | \phi(\bigvee_{i=0}^{k-1} \phi^i(A))) \\ &= H(\bigvee_{i=0}^{k-1} \phi^i(A)) + H(A | \bigvee_{i=1}^k \phi^i(A)). \end{aligned}$$

Now, by induction we obtain

$$(1) \quad H(\bigvee_{i=0}^{n-1} \phi^i(A)) = H(A) + \sum_{k=1}^{n-1} H(A | \bigvee_{i=1}^k \phi^i(A)).$$

According to Proposition 2.16, the sequence $\{H(A | \bigvee_{i=1}^n \phi^i(A))\}$ is decreasing, so that

$$\lim_{n \rightarrow \infty} H(A | \bigvee_{i=1}^n \phi^i(A)),$$

exists. Also the limit of the Cesaro means

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n H(A | \bigvee_{i=1}^k \phi^i(A)),$$

exists. So by equation (1), we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} H(A | \bigvee_{i=1}^n \phi^i(A)) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n H(A | \bigvee_{i=1}^k \phi^i(A)) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} [H(\bigvee_{i=1}^n \phi^i(A)) - H(A)] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} H(\bigvee_{j=0}^{n-1} \phi^j(A)) = h(\phi, A). \end{aligned}$$

□

Proposition 3.8. *Let A be a hyperfuzzy partition of X . Then*

$$h(\phi^k) = kh(\phi) \quad \forall k \in \mathbb{N}.$$

Proof. By Proposition 3.6, we have

$$\begin{aligned} h(\phi^k, A) &= h(\phi^k, \bigvee_{i=0}^{k-1} \phi^i(A)) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} H(\bigvee_{j=0}^{n-1} \phi^{jk}(\bigvee_{i=0}^{k-1} \phi^i(A))) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} H(\bigvee_{i=0}^{nk-1} \phi^i(A)) \\ &= k \lim_{n \rightarrow \infty} \frac{1}{nk} H(\bigvee_{i=0}^{nk-1} \phi^i(A)) \\ &= kh(\phi, A). \end{aligned}$$

□

Proposition 3.9. *Let A, C be hyperfuzzy partitions of X . Then*

$$h(\phi, A) \leq h(\phi, C) + H(A|C).$$

Proof. Since $H(B \vee D) = H(B) + H(D|B)$, so

$$\begin{aligned}
 H\left(\bigvee_{i=0}^{n-1} \phi^i(A)\right) &\leq H\left[\left(\bigvee_{i=0}^{n-1} \phi^i(A)\right) \vee \left(\bigvee_{j=0}^{n-1} \phi^j(C)\right)\right] \\
 (2) \qquad &= H\left(\bigvee_{j=0}^{n-1} \phi^j(C)\right) + H\left(\bigvee_{i=0}^{n-1} \phi^i(A) \middle| \bigvee_{j=0}^{n-1} \phi^j(C)\right).
 \end{aligned}$$

Furthermore $H(D \vee E|B) \leq H(D|B) + H(E|B)$, $H(D|B \vee E) \leq H(D|B)$. It implies that

$$\begin{aligned}
 &H\left(\bigvee_{i=0}^{n-1} \phi^i(A) \middle| \bigvee_{j=0}^{n-1} \phi^j(C)\right) \\
 &\leq \sum_{i=0}^{n-1} H(\phi^i(A) \middle| \bigvee_{j=0}^{n-1} \phi^j(C)) \\
 (3) \qquad &\leq \sum_{i=0}^{n-1} H(\phi^i(A) | \phi^i(C)) = nH(A|C).
 \end{aligned}$$

By relations (2) and (3), we have

$$\begin{aligned}
 &\lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \phi^i(A)\right) \\
 &\leq \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} \phi^j(C)\right) + H(A|C),
 \end{aligned}$$

and finally we get

$$h(\phi, A) \leq h(\phi, C) + H(A|C).$$

□

Corollary 3.10. *If $C_n = \bigvee_{i=0}^n \phi^i(C)$, then for any hyperfuzzy partition A of X , we have*

$$h(\phi, A) \leq h(\phi, C) + H(A|C_n).$$

Proof. The proof is immediate from Propositions 3.6 and 3.9. □

Proposition 3.11. *Let A be a hyperfuzzy partition of X , and there exists $k \in \mathbb{N}$ such that $A \leq \bigvee_{i=1}^k \phi^i(A)$. Then $h(\phi, A) = 0$*

Proof. Since there exists $k > 0$ such that $A \leq \bigvee_{i=1}^k \phi^i(A)$, by Corollary 2.12, we have

$$A \leq \bigvee_{i=1}^n \phi^i(A), \quad \forall n \geq k.$$

Thus by Corollary 2.17

$$H(A | \bigvee_{i=1}^n \phi^i(A)) = 0, \quad \forall n \geq k.$$

Therefore

$$h(\phi, A) = \lim_{n \rightarrow \infty} H(A | \bigvee_{i=1}^n \phi^i(A)) = 0.$$

□

Corollary 3.12. *Let $\Phi = (P^X, s, \phi)$ be a dynamical system, and for every hyperfuzzy partition A of X , there exists $k \in \mathbb{N}$ such that $A \leq \bigvee_{i=1}^k \phi^i(A)$. Then $h(\Phi) = 0$*

Definition 3.13. Let $\Phi = (P^X, s, \phi)$ be a dynamical system, A hyperfuzzy partition A of X is said to be a generator of (P^X, s, ϕ) if there exists $r \in \mathbb{N}$ such that

$$B \leq \bigvee_{i=0}^r \phi^i(A)$$

for each hyperfuzzy partition B of (P^X, s, ϕ) .

Theorem 3.14. *If A is a generator of dynamical system $\Phi = (P^X, s, \phi)$, then*

$$h(\Phi) = h(\phi, A).$$

Proof. Let B be an arbitrary partition. Since A is a generator of Φ then there exists $r \in \mathbb{N}$ such that

$$B \leq \bigvee_{i=0}^r \phi^i(A).$$

From Corollary 2.17, it follows that

$$(4) \quad H(B | \bigvee_{i=0}^r \phi^i(A)) = 0.$$

So by Corollary 3.10, we have

$$h(\phi, B) \leq h(\phi, \bigvee_{i=0}^r \phi^i(A)) + H(B | \bigvee_{i=0}^r \phi^i(A)),$$

and also by Proposition 3.6, we get

$$(5) \quad h(\phi, B) \leq h(\phi, A) + H(B | \bigvee_{i=0}^r \phi^i(A)).$$

Thus the relations (4) and (5), imply that

$$\begin{aligned} h(\phi, B) &\leq h(\phi, A), \\ h(\Phi) &= \sup_B h(\phi, B) = h(\phi, A). \end{aligned}$$

□

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