

SYNDETICALLY TRANSITIVE AND SYNDETICALLY SENSITIVE ITERATED FUNCTION SYSTEMS

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ABSTRACT. In this paper, we discuss several stronger forms of sensitivities for iterated function systems (IFSs), such as strong sensitivity and syndetrical sensitivity, and obtain some sufficient conditions for an IFS to have such sensitivities. We pay special attention to IFSs acting on the circle S^1 . We prove that each sensitive IFS acting on the circle S^1 generating by a finite family of circle homeomorphisms is strongly sensitive. However, to obtain syndetrical sensitivity, we impose some extra conditions on the IFS. Finally, we study sensitivity properties for syndetrical transitive IFSs. We show that each syndetically transitive IFS is topologically ergodic.

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1. INTRODUCTION

Sensitivity and chaos are two popular themes in dynamical systems. The sensitivity is widely understood as the central idea in chaos. The first formulate of sensitivity (as far as we know) was given by Guckenheimer on the study of interval maps, [9]. Then Liu and Wang [15] introduced some stronger versions of sensitivities, including thick sensitivity and thickly syndetrical sensitivity. After that Wang, Yin and Yan [18] extended some of these results to semigroup actions. They introduced several stronger forms of sensitivities including thick sensitivity and thickly syndetrical sensitivity. Then they presented some sufficient conditions for a dynamical system of semigroup actions to have these sensitivities. Sensitive systems were also studied by [3, 4, 5, 6, 8, 10, 13, 14]. In the present work we discuss some aspects of this concept in iterated function systems (IFSs).

The paper is organized as follows: In Section 2, we recall some standard definitions about IFSs and their sensitivity properties. In section 3, we discuss several stronger forms of sensitivities for IFSs on the circle S^1 , such as strong sensitivity and syndetrical sensitivity. We show that each sensitive IFS acting on the circle S^1 generating by a finite family of circle homeomorphisms is strongly sensitive. However, to obtain syndetrical sensitivity, some extra condition should be imposed on the IFS. In section 4, we study sensitivity properties for syndetrical transitive IFSs. We assert that each syndetically transitive IFS is topologically ergodic.

2. PRELIMINARIES

In this section, we recall some common notions and definition in the theory of iterated function systems. Throughout this paper, we assume that X is locally connected perfect compact metric space. Consider a finite family of continuous maps $\mathcal{F} = \{f_1, \dots, f_k\}$ defined on a compact metric space X . We denote by $\langle \mathcal{F} \rangle^+$ the semigroup generated by these maps. If $\Gamma = \langle \mathcal{F} \rangle^+$ then the dynamical system (Γ, X) is called an *iterated function system* (IFS) associated to $\mathcal{F}$. We use the usual notations: $\text{IFS}(X; \mathcal{F})$ or $\text{IFS}(\mathcal{F})$. Roughly speaking, an iterated function system (IFS) can be thought of as a finite collection of maps which can be applied successively in any order.

Let us consider the symbol spaces $\Sigma_k^+ = \{1, \dots, k\}^{\mathbb{N}}$ and $\Sigma_k^\ell = \{1, \dots, k\}^\ell$, for some $\ell \in \mathbb{N}$. Further, $\mathcal{P} = (p_1, \dots, p_k)$ is a probability vector which gives us the probability with which we apply f_i . Equip Σ_k^+ and Σ_k^ℓ , with the Bernoulli measures ν^+ and ν^ℓ corresponding to the given probability distributions: $\nu^+ = \nu_0^{\mathbb{N}}$ and $\nu^\ell = \nu_0^\ell$, where $\nu_0(i) = p_i$, $i = 1, \dots, k$. Then this yields to a *random iterated function system* (RIFS) which is denoted by $\text{RIFS}(X; f_1, \dots, f_k; p_1, \dots, p_k)$ or $\text{RIFS}(X; \mathcal{F}; \mathcal{P})$.

An iterated function system $\text{IFS}(X; \mathcal{F})$ is called *forward minimal* or $\langle \mathcal{F} \rangle^+$ acts *forward minimally* on X , if $\overline{\mathcal{O}_{\mathcal{F}}^+(x)} = X$, for every $x \in X$. We remark that for a $x \in X$ the *forward orbit* of x is defined by

$$\mathcal{O}_{\mathcal{F}}^+(x) = \{h(x) : h \in \mathcal{F}^+\}.$$

We say that $\text{IFS}(X; \mathcal{F})$ is *topologically transitive*, if for any two non-empty open sets U and V in X , there exists $h \in \langle \mathcal{F} \rangle^+$ such that $h(U) \cap V \neq \emptyset$.

Following [13], an *M-system* means that the set of almost periodic points is dense in X (the Bronstein condition) and, in addition, the system is topologically transitive. A very particular case of the Bronstein condition is that $\text{IFS}(\mathcal{F})$ has dense periodic points. We recall that a point $x \in X$ is almost periodic if and only if $\overline{\mathcal{O}_{\mathcal{F}}^+(x)}$ is a minimal set.

A point $x \in X$ is called an *equicontinuity point* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(h(x), h(y)) < \varepsilon$, for every $h \in \langle \mathcal{F} \rangle^+$. An iterated function system $\text{IFS}(\mathcal{F})$ is called *equicontinuous* if the subset of equicontinuity points is the whole of the space X .

Symbolic dynamic is a way to represent the elements of $\langle \mathcal{F} \rangle^+$. For any sequence $\omega = (\omega_1 \omega_2 \dots \omega_n \dots) \in \Sigma_k^+$, take $f_\omega^0 := Id$ and

$$(1) \quad f_\omega^n(x) = f_{\omega_1 \dots \omega_n}^n(x) = f_{\omega_n} \circ f_\omega^{n-1}(x); \quad \forall n \in \mathbb{N}.$$

Obviously, $f_\omega^n = f_{\omega_1 \dots \omega_n}^n = f_{\omega_n} \circ f_\omega^{n-1} \in \langle \mathcal{F} \rangle^+$, for every $n \in \mathbb{N}$.

For nonempty open subsets U and V of X , $\delta > 0$ and $\omega \in \Sigma_k^+$, we let

$$(2) \quad N_{\mathcal{F}}(\omega, U, V) = \{n \in \mathbb{Z}^+ : f_\omega^n(U) \cap V \neq \emptyset\},$$

$$(3) \quad N_{\mathcal{F}}(\omega, U, \delta) = \{n \in \mathbb{Z}^+ : \exists y, z \in U, \text{ such that } d(f_\omega^n(y), f_\omega^n(z)) > \delta\},$$

$$(4) \quad N_{\mathcal{F}}(U, \delta) = \{n \in \mathbb{Z}^+ : \exists \omega \in \Sigma_k^+ \text{ and } \exists y, z \in U \text{ such that } d(f_\omega^n(y), f_\omega^n(z)) > \delta\}.$$

A *syndetic set* is a subset of the natural numbers, having the property of bounded gaps, that the sizes of the gaps in the sequence of natural numbers is bounded. An $\text{IFS}(\mathcal{F})$ is called *syndetically transitive* if for any two nonempty open subsets U and V of X , there exists $\omega \in \Sigma_k^+$ such that $N_{\mathcal{F}}(\omega, U, V)$ is a syndetic set of the natural numbers.

A subset S of natural numbers has *positive upper density* if

$$(5) \quad \limsup_{k \rightarrow \infty} \frac{1}{k} \text{card}\{1 \leq j \leq k : j \in S\} > 0.$$

The $\text{IFS}(\mathcal{F})$ is *topologically ergodic* if for every two open subsets U and V of X there exists $\omega \in \Sigma_k^+$ such that $N_{\mathcal{F}}(\omega, U, V)$ has positive upper density.

The $\text{IFS}(\mathcal{F})$ is *sensitive* on X if there exists $\delta > 0$ (sensitivity constant) such that for every point $x \in X$ and every open neighborhood U of x there are $y \in U$, $\omega \in \Sigma_k^+$ and $n \in \mathbb{N}$ with $d(f_{\omega}^n(x), f_{\omega}^n(y)) > \delta$. The iterated function system $\text{IFS}(\mathcal{F})$ is *syndetically sensitive* on X if $\text{IFS}(\mathcal{F})$ is sensitive with sensitivity constant $\delta > 0$ and for each open set U , the subset $N_{\mathcal{F}}(U, \delta)$ is syndetic. Finally, $\text{IFS}(\mathcal{F})$ is *strongly sensitive* on X if for each $x \in X$ there exists $\Omega(x) \subset \Sigma_k^+$ with $\nu^+(\Omega(x)) = 1$ such that for each neighborhood U of x and each $\omega \in \Omega(x)$ one has that

$$N_{\mathcal{F}}(\omega, x, U, \delta) = \{n \in \mathbb{N} : \exists y \in U \text{ such that } d(f_{\omega}^n(x), f_{\omega}^n(y)) > \delta\} \neq \emptyset.$$

We say that $\text{IFS}(\mathcal{F})$ is *contractible at* $x \in X$ if there exists a neighborhood U of x and a sequence $\omega \in \Sigma_k^+$ such that $\text{diam}(f_{\omega}^n(U))$ tends to zero whenever $n \rightarrow +\infty$.

3. SYNETICALLY SENSITIVE AND STRONGLY SENSITIVE IFSs ON THE CIRCLE S^1

In this section, we discuss several stronger forms of sensitivities for IFSs on the circle S^1 , such as strong sensitivity and syndetical sensitivity.

Proposition 3.1. *Let $\text{IFS}(\mathcal{F})$ be an IFS acting on the circle S^1 generating by a finite family of circle homeomorphisms $\{f_i : i = 1, \dots, k\}$. If $\text{IFS}(\mathcal{F})$ is sensitive with sensitivity constant $\delta > 0$ and it is not contractible at some point $x \in X$ then $N_{\mathcal{F}}(U, \delta)$ is syndetic for each small open neighborhood U of x .*

Proof. Assume that $\text{IFS}(\mathcal{F})$ is sensitive with sensitivity constant δ and it is not contractible at some point $x \in X$. Suppose to the contrary that $N_{\mathcal{F}}(U, \delta)$ is not syndetic for some neighborhood U of x . Then there exist a sequence $(a_i)_{i \in \mathbb{N}}$, finite

words $(\omega_1, \dots, \omega_{a_i})$ of the alphabets $\{1, \dots, k\}$ and a sequence $(k_i)_{i \in \mathbb{N}}$ of positive integers with $k_i \rightarrow \infty$, as $i \rightarrow \infty$, such that

$$\text{diam}(f_{(\omega_1, \dots, \omega_{a_i}, \rho_1, \dots, \rho_j)}(U)) = \text{diam}(f_{(\rho_1, \dots, \rho_j)} \circ f_{(\omega_1, \dots, \omega_{a_i})}(U)) \leq \delta$$

for each finite word $(\rho_1, \dots, \rho_j)$, $j = 1, \dots, k_i$.

Now, by passing to a subsequence, let $f_{(\omega_1, \dots, \omega_{a_i})}(x) \rightarrow y$. Since $\text{IFS}(\mathcal{F})$ is not contractible at x hence for each sequence $\omega \in \Sigma_k^+$ and every open neighborhood U of x , $\text{diam}(f_\omega^n(U))$ is bounded away from zero. This means that there are $\varepsilon > 0$ and $i_0 \in \mathbb{N}$ such that $f_{(\omega_1, \dots, \omega_{a_i})}(U) \supseteq B_\varepsilon(y)$, for each $i \geq i_0$. Take $V := B_\varepsilon(y)$. Since $\text{IFS}(\mathcal{F})$ is sensitive with sensitivity constant δ , there exists finite word $(\rho_1, \dots, \rho_j)$ such that $\text{diam}(f_{(\rho_1, \dots, \rho_j)})(V) > \delta$. This fact implies that

$$\text{diam}(f_{(\omega_1, \dots, \omega_{a_i}, \rho_1, \dots, \rho_j)}(U)) \geq \text{diam}(f_{(\rho_1, \dots, \rho_j)})(V) > \delta,$$

which is a contradiction. $\square$

As a consequence, we get the next result.

Corollary 3.2. *Let $\text{IFS}(\mathcal{F})$ be an IFS acting on the circle S^1 generating by a finite family of circle homeomorphisms $\{f_i : i = 1, \dots, k\}$. If $\text{IFS}(\mathcal{F})$ is sensitive and it has not any contractible point then $\text{IFS}(\mathcal{F})$ is syndetically sensitive.*

In the rest of this section, we prove that each sensitive IFS acting on the circle S^1 generating by a finite family of circle homeomorphisms is strongly sensitive. First, let us recall some notations and several definitions from [7].

Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be two relations on X . Denote by $a\mathcal{E}b$ (resp. $a\tilde{\mathcal{E}}b$) for two elements a and b of X with $(a, b) \in \mathcal{E}$ (resp. $(a, b) \in \tilde{\mathcal{E}}$). With respect to the relation $\mathcal{E}$, we define a general form of a chain. Given $\omega \in \Sigma_k^+$, a sequence $\{x_i\}_{i=0}^n$ with $x_0 = a$, $x_n = b$ is called a *chain in direction of ω* with respect to the relation $\mathcal{E}$ from a to b if

$$f_{\omega_{i+1}}(x_i)\mathcal{E}x_{i+1}; \quad \forall i = 0, 1, \dots, n-1.$$

In this case, we use the notation $a\mathcal{E}_\omega^n b$ (or $a\mathcal{E}_{\omega_1 \dots \omega_n}^n b$).

Definition 3.3. *Let A and $\mathcal{Q}$ be two subsets of X and let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be two relations on X . We say that the set A has chain connection to $\mathcal{Q}$ with respect to the pair $(\mathcal{E}, \tilde{\mathcal{E}})$ if there exist points $a \in A$, $q \in \mathcal{Q}$, $\omega \in \Sigma_k^+$ and a sequence of points $a = x_0, x_1, \dots, x_{n+1} = q$ so that $\{x_i\}_{i=0}^n$ is a chain in direction of ω with respect to the*

relation $\mathcal{E}$ from $x_0 = a$ to x_n and $\{x_{n+j}\}_{j=0}^1$ is a chain in direction of $\sigma^n(\omega)$ with respect to the relation $\tilde{\mathcal{E}}$ from x_n to $x_{n+1} = b$, where $\sigma : \Sigma_k^+ \rightarrow \Sigma_k^+$ is the Bernoulli shift map.

We denote this composition of chains by $a\mathcal{E}_\omega^n \tilde{\mathcal{E}}_{\sigma^n(\omega)} q$ (or $a\mathcal{E}_{\omega_1 \dots \omega_n} \tilde{\mathcal{E}}_{\omega_{n+1}} q$).

Definition 3.4. Let X be a locally connected compact metric space and A and $\mathcal{Q}$ be two subsets of X . Also let $a\mathcal{E}_\omega^n \tilde{\mathcal{E}}_{\sigma^n(\omega)} q$ for some points $a \in A$, $q \in \mathcal{Q}$. If there exists an open set U of the point a so that for all points $u \in U$, $u\mathcal{E}_\omega^n \tilde{\mathcal{E}}_{\sigma^n(\omega)} q$, then we say that the point a has a stable chain connection to $\mathcal{Q}$ with respect to the pair $(\mathcal{E}, \tilde{\mathcal{E}})$.

We need the auxiliary lemma that follows.

Lemma 3.5. ([7, Lemma. 3.1]) Consider $\text{RIFS}(X; f_1, \dots, f_k; p_1, \dots, p_k)$ on a locally connected compact metric space X and a subset $\mathcal{Q}$ of X . If for $x \in X$ there is $0 < \ell = \ell(x)$ so that

$$p = \inf_{y \in \mathcal{O}^+(x)} \nu^\ell(\{\omega_1 \dots \omega_\ell \in \Sigma_k^\ell : y\mathcal{E}_{\omega_1 \dots \omega_\ell} \tilde{\mathcal{E}} q \text{ for some } 1 \leq i \leq \ell \text{ and } q \in \mathcal{Q}\}) \neq 0$$

then for every $x \in X$, there exists $\Omega(x) \subset \Sigma_k^+$ with $\nu^+(\Omega(x)) = 1$ such that x has a chain connection to $\mathcal{Q}$ in direction of ω , for every $\omega \in \Omega(x)$, where ν^ℓ and ν^+ are the Bernoulli measures on Σ_k^ℓ and Σ_k^+ , respectively, corresponding to the given probability distributions.

Theorem 3.6. Let $\text{IFS}(\mathcal{F})$ be an IFS acting on the circle S^1 generating by a finite family of circle homeomorphisms $\mathcal{F}$. If $\text{IFS}(\mathcal{F})$ is sensitive then $\text{IFS}(\mathcal{F})$ is strongly sensitive.

Proof. Consider a sensitive iterated function system $\text{IFS}(\mathcal{F})$ acting on the circle S^1 with sensitivity constant δ . Assume that $x \in X$, U is a small open connected set containing x and take $\mathcal{Q} := \mathcal{O}_{\mathcal{F}}^+(U)$. One can define two relations $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as follows:

- i) $x\mathcal{E}y \iff f_j(x) = y$, for some $1 \leq j \leq k$;
- ii) $x\tilde{\mathcal{E}}y \iff d(x, y) > \delta$.

Since $\text{IFS}(\mathcal{F})$ is sensitive, hence there is a finite word $w = w_x = (\omega_1, \dots, \omega_n)$ contained in Σ_k^n and $y \in U$ such that $d(f_{[w]}(x), f_{[w]}(y)) > \delta$, where $f_{[w]} := f_{\omega_n} \circ$

$\dots \circ f_{\omega_1}$. By continuity, there exist open sets A_x and A_y containing x and y , respectively, satisfy the following condition:

$$(6) \quad \forall (z_1, z_2) \in A_x \times A_y, \quad d(f_{[w]}(z_1), f_{[w]}(z_2)) > \delta.$$

By compactness of $\overline{\mathcal{O}_{\mathcal{F}}^+(x)}$, there are two finite families $\{A_{x_1}, \dots, A_{x_m}\}$ and $\{A_{y_1}, \dots, A_{y_m}\}$ of open subsets of X and finite words w_{x_i} , $1 \leq i \leq m$, such that $\{A_{x_1}, \dots, A_{x_m}\}$ is a finite cover of $\overline{\mathcal{O}_{\mathcal{F}}^+(x)}$ and every pair (A_{x_i}, A_{y_i}) , $1 \leq i \leq m$, with together $f_{[w_{x_i}]}$ satisfying the property (6).

Let us take $\ell = \max\{n_{x_i} : i = 1, \dots, m\}$, where n_{x_i} is the length of w_{x_i} . Clearly, ℓ satisfies the conclusion of the previous lemma, that is there exists $\Omega(x) \subset \Sigma_k^+$ with $\nu^+(\Omega(x)) = 1$ such that x has a chain connection to $\mathcal{Q}$ in direction of ω . Hence, the proof is completed. $\square$

4. SYNDETICALLY TRANSITIVE IFSs

In this section, we restrict ourselves to syndetical transitive IFSs and provide some sensitivity properties for them.

Proposition 4.1. *If $\text{IFS}(\mathcal{F})$ is minimal then $\text{IFS}(\mathcal{F})$ is syndetically transitive.*

Proof. Let U and V be two open subsets of X . We must show that there exists $\omega \in \Sigma_k^+$ such that $N_{\mathcal{F}}(\omega, U, V)$ is syndetic. Suppose that for each ω , $N_{\mathcal{F}}(\omega, U, V)$ is not syndetic. Then there exist a sequence $(n_i)_{i \in \mathbb{N}}$, finite words $(\omega_1, \dots, \omega_{n_i})$ of the alphabets $\{1, \dots, k\}$, points $x_i \in U$ and a sequence $(k_i)_{i \in \mathbb{N}}$ of positive integers with $k_i \rightarrow \infty$, as $i \rightarrow \infty$, such that $f_{(\omega_1, \dots, \omega_{n_i}, \rho_1, \dots, \rho_j)}(x_i) \notin V$ for each finite word $(\rho_1, \dots, \rho_j)$, $j = 1, \dots, k_i$.

Since X is compact, by passing to a subsequence, let $f_{(\omega_1, \dots, \omega_{n_i})}(x_i) \rightarrow y$. Then for each finite word $(\rho_1, \dots, \rho_j) \in \Sigma_k^j$, $j = 1, \dots, k_i$, one has $f_{(\omega_1, \dots, \omega_{n_i}, \rho_1, \dots, \rho_j)}(x_i) \rightarrow f_{(\rho_1, \dots, \rho_j)}(y)$.

Since $\text{IFS}(\mathcal{F})$ is minimal hence one can find $j \in \mathbb{N}$ and $(\rho_1, \dots, \rho_j) \in \Sigma_k^j$ such that $f_{(\rho_1, \dots, \rho_j)}(y) \in V$. In the following, we fix such j and $(\rho_1, \dots, \rho_j)$. Therefore, for large enough n_i , $f_{(\omega_1, \dots, \omega_{n_i}, \rho_1, \dots, \rho_j)}(x_i) \in V$ which is a contradiction. $\square$

The next result follows immediately.

Proposition 4.2. *If $\text{IFS}(\mathcal{F})$ is syndetically transitive then $\text{IFS}(\mathcal{F})$ is topologically ergodic. In particular, if $\text{IFS}(\mathcal{F})$ is minimal then $\text{IFS}(\mathcal{F})$ is topologically ergodic.*

By Lemma 4.3 of [13], each syndetically transitive IFS on a compact metric space X is an M-system. By this fact and [18, Thm. 4.10], we provide the next result.

Proposition 4.3. *If $IFS(\mathcal{F})$ is abelian, minimal and sensitive then it is syndetically sensitive.*

The next result follows by [19, Cor. 3.6] and the above observations.

Proposition 4.4. *Let $IFS(\mathcal{F})$ be syndetically transitive. Then, either $IFS(\mathcal{F})$ is minimal or it has infinitely many minimal subsets.*

Definition 4.5. *For a given integer $n \geq 2$, an IFS $IFS(\mathcal{F})$ is said to be n -sensitive if there exists an $\varepsilon > 0$ such that for any non-empty open set U there are distinct points $x_1, x_2, \dots, x_n \in U$ and some $h \in \langle \mathcal{F} \rangle^+$ with*

$$\min\{d(h(x_i), h(x_j)) : 1 \leq i \neq j \leq n\} \geq \varepsilon.$$

By [19, Thm. 3.10] the next result holds.

Proposition 4.6. *Let $IFS(\mathcal{F})$ be an abelian syndetically transitive IFS. If $IFS(\mathcal{F})$ is not minimal then it is n -sensitive, for each $n \geq 2$.*

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