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NORMALLY GENERATED LINE BUNDLE AND LAURENT SERIES SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. The aim of this paper is to demonstrate the rich interaction between the properties of dynamical systems, the geometry of its asymptotic solutions, and the theory of Abelian varieties. We are going to illustrate as well that the nature of many methods for finding solutions of some dynamical systems is determined by the Laurent series for solutions of nonlinear differential equations. We apply the methods to the Kowalewski'top a solid body rotating about a fixed point, the Kirchhoff's equations of motion of a solid in an ideal fluid and the Ramani-Dorizzi-Grammaticos (RDG) series of integrable potentials.

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1. Introduction and general setting

Algebraic complete integrable systems (a.c.i) are integrables systems whose trajectories are straight line motions on Abelian varieties (complex algebraic tori), themselves completions of the level manifolds of the system (here space and time must be thought of as complex). The solutions can be expressed in terms of Abelian integrals, possibly after an algebraic change of variables and in terms of theta functions. The main purpose of this paper is to show how the notions and techniques of normally generated line bundle and Laurent series play a crucial role in the study of algebraically integrable systems. At the bad points, the concept of projective normality plays an important role when studying the convergence of these series solutions. I will discuss some interesting and well known examples : the problem of the rotation of a solid body around a fixed point (integration of the Kowalewski case), the movement of a solid in an ideal fluid (integration of the Clebsch case) and the Ramani-Dorizzi-Grammaticos (RDG) series of integrable potentials.

We will first give a quick survey about a.c.i, while knowing that these facts can be found in detail in [2, 17, 26]. Consider the Hamiltonian vector field X_H,

$$X_H : \dot{x} = J\frac{\partial H}{\partial x} \equiv f(x),\ x \in \mathbb{R}^m, \quad \left(\cdot \equiv \frac{d}{dt} \right) \tag{1}$$

where H is the Hamiltonian and $J = J(x)$ is a skew-symmetric matrix with polynomial entries in x, for which the corresponding Poisson bracket $\{H_i, H_j\} = \left\langle \frac{\partial H_i}{\partial x}, J\frac{\partial H_j}{\partial x} \right\rangle$, satisfies the Jacobi identities. Such a system, with polynomial right hand side, is algebraic complete integrable (a.c.i) [1] when the following conditions hold :

i) The system possesses $n+k$ independent polynomial invariants $H_1, ..., H_{n+k}$ of which k invariants (Casimir functions) lead to zero vector fields $J\frac{\partial H_i}{\partial x}(x) = 0$, $1 \leq i \leq k$, the $n = (m-k)/2$ remaining ones $H_{k+1} = H$,...,H_{k+n} are in involution (i.e., $\{H_i, H_j\} = 0$),which give rise to n commuting vector fields. For generic c_i, the invariant manifolds (level surfaces) $\bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^m : H_i = c_i\}$ are compact and connected and therefore real tori according to the Arnold-Liouville theorem [3].

ii) The invariant manifolds (level surfaces) thought of as lying in $\mathbb{C}^m$,

$$\mathcal{A} = \bigcap_{i=1}^{n+k} \{x \in \mathbb{C}^m : H_i = c_i\},$$

are related, for generic c_i, to Abelian varieties $T^n = \mathbb{C}^n/Lattice$ (complex algebraic tori) as follows :

$$\mathcal{A} = T^n \backslash \mathcal{D},$$

where $\mathcal{D}$ is a (Painlevé) divisor (codimension one subvarieties) in T^n. In the natural coordinates $(t_1, ..., t_n)$ of T^n coming from $\mathbb{C}^n$, the coordinates $x_i = x_i(t_1, ..., t_n)$ are meromorphic and $\mathcal{D}$ is the minimal divisor on T^n where the variables x_i blow up. Moreover, the Hamiltonian flows (1) (run with complex time) are straight-line motions on T^n.

If the Hamiltonian flow (1) is a.c.i., it means that the variables x_i are meromorphic on the torus T^n and by compactness they must blow up along a codimension one subvariety (a divisor) $\mathcal{D} \subset T^n$. By the a.c.i. definition, the flow (1) is a straight line motion in T^n and thus it must hit the divisor $\mathcal{D}$ in at least one place. Moreover through every point of $\mathcal{D}$, there is a straight line motion and therefore a Laurent expansion around that point of intersection. Hence the differential equations must admit Laurent expansions which depend on the $n-1$ parameters defining $\mathcal{D}$ and the $n+k$ constants c_i defining the torus T^n, the total count is therefore $m - 1 = dim\ (phase\ space) - 1$ parameters. Such a necessary condition for algebraic complete integrability can be formulated as follows [1] : If the Hamiltonian system (1) (with invariant tori not containing elliptic curves) is algebraic complete integrable, then each x_i blows up after a finite (complex) time, and for every x_i, there is a family of solutions

$$x_i = \sum_{j=0}^{\infty} x_i^{(j)} t^{j-k_i}, \quad k_i \in \mathbb{Z}, \qquad \text{some } k_i > 0,$$

depending on $dim\ (phase\ space) - 1 = m - 1$ free parameters. Moreover, the system (1) possesses families of Laurent solutions depending on $m-2$, $m-3$,...,$m-n$ free parameters. The coefficients of each one of these Laurent solutions are rational functions on affine algebraic varieties of dimensions $m-1$, $m-2$, $m-3$,...,$m-n$.

The question raised several years ago is whether this criterion is also sufficient. The system (1) with $k+n$ polynomial invariants has a coherent tree of Laurent solutions, when it has families of Laurent solutions in t, depending on $n-1$, $n-2$,...,$m-n$ free parameters. Adler and van Moerbeke [1] have shown that if the system possesses several families of $(n-1)$-dimensional Laurent solutions (principal Painlevé solutions) they must fit together in a coherent way and as we mentioned above, the system must possess $(n-2)$-, $(n-3)$-,... dimensional Laurent solutions (lower Painlevé solutions), which are the gluing agents of the $(n-1)$-dimensional family.

The gluing occurs via a rational change of coordinates in which the lower parameter solutions are seen to be genuine limits of the higher parameter solutions an which in turn appears due to a remarkable propriety of algebraic complete integrable systems; they can be put into quadratic form both in the original variables and in their ratios. As a whole, the full set of Painlevé solutions glue together to form a fiber bundle with singular base. A partial converse to the above result can be formulated as follows [1] : If the Hamiltonian system (1) satisfies the condition i) of algebraic complete integrability and if it possesses a coherent tree of Laurent solutions, then the system is algebraic complete integrable and there are no other $m-1$-dimensional Laurent solutions but those provided by the coherent set.

2. Normally generated line bundle or projectively normal embedding

Following Mumford [22] we call an ample line bundle $\mathcal{L}$ normally generated line bundle, when the natural map

$$\mathrm{Sym}^n H^0(M,\mathcal{L}) \longrightarrow H^0(M,\mathcal{L}^n),$$

is surjective for every $n \geq 2$. In other words, by setting $\mathcal{L} = [\mathcal{D}]$, we say that a divisor $\mathcal{D}$ is normally generated line bundle (or with an abuse of language; projectively normal) when the map

$$\begin{aligned}\mathcal{L}(\mathcal{D})^{\otimes k} \longrightarrow \mathcal{L}(k\mathcal{D}) \quad &= \quad \{\text{meromorphic functions on } T^n \\ &\qquad\qquad \text{with } k-\text{fold poles at worst along } \mathcal{D}\},\end{aligned}$$

is surjective, i.e., every function of $\mathcal{L}(k\mathcal{D})$ can be written as a linear combination of k-fold products of functions of $\mathcal{L}(\mathcal{D})$. It can be shown that a line bundle which is normally generated is very ample, but the converse is not true, in general. So not every very ample divisor $\mathcal{D}$ is normally generated but according to a criterion of Koizumi [12] and Mumford [21], if $\mathcal{D}$ is linearly equivalent to $k\mathcal{D}_0$ for $k \geq 3$ for some divisor $\mathcal{D}_0$, then $\mathcal{D}$ is normally generated. We will encounter concrete example of this when analyzing the algebraic geometry of some problems. We will also say that an embedding of an Abelian variety M into $\mathbb{P}^N(\mathbb{C})$ is projectively normal if the image of M in $\mathbb{P}^N(\mathbb{C})$ is a projectively normal manifold; i.e., if its ring of homogeneous coordinates is completely closed.

Let $\mathcal{L}$ be an ample line bundle on an Abelian variety M of dimension g and of type $(\delta_1, ..., \delta_g)$. The map $\psi_{\mathcal{L}^k} : M \longrightarrow \mathbb{P}^N(\mathbb{C})$, associated with the bundle

$\mathcal{L}^k \equiv \mathcal{L}^{\otimes k}$, determines a projectively normal embedding for any $k \geq 3$ and also for $k = 2$ when the points of $K(\mathcal{L}^2))$ are not basic points of $\mathcal{L}$ where the group $K(\mathcal{L}) = \ker f \simeq (\mathbb{Z}/\delta_1\mathbb{Z} \times ... \times \mathbb{Z}/\delta_g\mathbb{Z})^2$, is the kernel of isogeny

$$f : M \longrightarrow M^\vee = Pic^0(M), \quad p \longmapsto \tau_p^*\mathcal{L} \otimes \mathcal{L}^{-1},$$

with $\tau_p^* M \longrightarrow M$, $x \longmapsto x + p$, the translation by p. In the case where M is an Abelian surface (this case is of particular interest to the systems studied further), the rational mapping $\psi_\mathcal{L} : M \longrightarrow \mathbb{P}^{\delta_1\delta_2-1}(\mathbb{C})$, is a projectively normal embedding if the following conditions are satisfied :

(i) When $\delta_1 \geq 3$, then $\psi_\mathcal{L}$ is a projectively normal embedding.

(ii) When $\delta_1 = 2$, then $\psi_\mathcal{L}$ is a projectively normal embedding if and only if the points of $K(\mathcal{L}^2)$ are not basic points of $\mathcal{L}$.

(iii) When $\mathcal{L}$ is primitive, i.e., $(\delta_1, \delta_2) = (1, \delta_2)$ and $\delta_2 \geq 7$, then $\psi_\mathcal{L}$ is a projectively normal embedding if and only if $\mathcal{L}$ is very ample.

(iv) When $\mathcal{L}$ is primitive, $\delta_2 \geq 7$ and M generic, then $\psi_\mathcal{L}$ is a projectively normal embedding.

As noted above, a necessary condition for $\psi_\mathcal{L}$ to be a projectively normal embedding is that $\mathcal{L}$ is very ample. When $\delta_1\delta_2 \leq 4$, $\mathcal{L}$ is not even very ample. Similarly, if $\delta_1\delta_2 \leq 7$, then

$$\dim \mathrm{Sym}^2 H^2(M, \mathcal{L}) < \dim \mathrm{Sym}\ H^0(M, \mathcal{L}^2),$$

and therefore the embedding in question can never be projectively normal.

For an algebraically completely integrable system, the original set of variables $x_1, ..., x_m$ can be replaced by a new set $y_0, y_1, ..., y_N$ having the property that $\left(\frac{y_i}{y_j}\right)^{\cdot}$ is a quadratic polynomial in $(1/y_j, y_1/y_j, \ldots, y_N/y_j)$, $0 \leq i \leq N$, for any fixed but arbitrary choice of y_j in the linear span of the $y_i's$; in particular this property holds for $y_0 = 1$. Indeed it hinges on the projectively normal property of the divisor $\mathcal{D}$. We assume that the divisor is very ample and in addition projectively normal. Consider a point $p \in \mathcal{D}$, a chart U_j around p on the torus and a function y_j in $\mathcal{L}(\mathcal{D})$ having a pole of maximal order at p. Then the vector $(1/y_j, y_1/y_j, \ldots, y_N/y_j)$ provides a good system of coordinates in U_j. Then taking the derivative with regard to one of the flows

$$\left(\frac{y_i}{y_j}\right)^{\cdot} = \frac{\dot{y}_i y_j - y_i \dot{y}_j}{y_j^2}, \quad 1 \leq j \leq N,$$

are finite on U_j as well. Therefore, since y_j^2 has a double pole along $\mathcal{D}$, the numerator must also have a double pole (at worst), i.e., $\dot{y}_i y_j - y_i \dot{y}_j \in \mathcal{L}(2\mathcal{D})$. Hence, when $\mathcal{D}$ is projectively normal, we have that

$$\left(\frac{y_i}{y_j}\right)^{\cdot} = \sum_{k,l} a_{k,l} \left(\frac{y_k}{y_j}\right)\left(\frac{y_l}{y_j}\right),$$

i.e., the ratios y_i/y_j form a closed system of coordinates under differentiation. This remarkable fact hinges on the notion of projective normality of a divisor on an Abelian variety. It plays an important role in the following matter : the successive coefficients in the Laurent series solutions are meromorphic on the divisor $\mathcal{D}$. Therefore at the points where some of these coefficients blow up, the Laurent series cess to exist, unless they are somehow rearranged. In order to circumvent this difficult, one shows that near these points the local coordinates, which the above rational functions, form as above a closed system of ordinary differential equations, whose solutions evolve, as a function of time, in projective space in a nice holomorphic fashion. So at the bad points, the concept of projective normality play an important role : this enables one to show that y_i/y_j is a bona fide Taylor series starting from every point in a neighborhood of the point in question.

3. Applications

3.1. **The problem of the rotation of a solid body around a fixed point. Integration of the Kowalewski case.** The Kowalewski'top [15] is a solid body rotating about a fixed point, whose principal moments of inertia $\lambda = \text{diag}(A, B, C)$ (with regard to the fixed point) satisfy the relation $A = B = 2C$ and its center of mass belongs to the equatorial plane (AB plane) through the fixed point. The motion of this body is governed by the equations :

$$\begin{aligned} \dot{m} &= m \wedge \lambda m + \mu g \gamma \wedge l, \\ \dot{\gamma} &= \gamma \wedge \lambda m, \end{aligned}$$

where m, l and γ denote respectively the angular momentum, the center of mass and the unit vector in the direction of gravity, μ is the mass of the solid and g is the acceleration of gravity. By properly picking the axes of inertia in the equatorial plane and after some rescaling and normalization may be taken as $l = (1, 0, 0)$,

$\lambda m = \left(\frac{m_1}{2}, \frac{m_2}{2}, m_3\right)$. The equations of motion take on the form :

$$\begin{aligned} \dot{m}_1 &= m_2 m_3, & \dot{\gamma}_1 &= 2m_3\gamma_2 - m_2\gamma_3, \\ \dot{m}_2 &= -m_1 m_3 + 2\gamma_3, & \dot{\gamma}_2 &= m_1\gamma_3 - 2m_3\gamma_1, \\ \dot{m}_3 &= -2\gamma_2, & \dot{\gamma}_3 &= m_2\gamma_1 - m_1\gamma_2, \end{aligned} \tag{2}$$

where, without restricting generality, we chose $\mu g = 1$ and we used the substitution $t \longrightarrow 2t$. This system has four first integrals (constants of motion) :

$$H_1 = \frac{1}{2}\left(m_1^2 + m_2^2\right) + m_3^2 + 2\gamma_1,$$

$$H_2 = m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3, \qquad H_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2,$$

$$H_4 = \left(\left(\frac{m_1 + im_2}{2}\right)^2 - (\gamma_1 + i\gamma_2)\right)\left(\left(\frac{m_1 - im_2}{2}\right)^2 - (\gamma_1 - i\gamma_2)\right),$$

and is written in the form of the Hamiltonian vector field :

$$\dot{x} = J\frac{\partial H}{\partial x}, \quad H \equiv H_1, \quad x = (m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3)^\intercal, \tag{3}$$

where

$$J = \begin{pmatrix} 0 & -m_3 & m_2 & 0 & -\gamma_3 & \gamma_2 \\ m_3 & 0 & -m_1 & \gamma_3 & 0 & -\gamma_1 \\ -m_2 & m_1 & 0 & -\gamma_2 & \gamma_1 & 0 \\ 0 & -\gamma_3 & \gamma_2 & 0 & 0 & 0 \\ \gamma_3 & 0 & -\gamma_1 & 0 & 0 & 0 \\ -\gamma_2 & \gamma_1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

A second flow commuting with the first is regulated by the equations :

$$\dot{x} = J\frac{\partial H_4}{\partial x}, \quad x = (m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3)^\intercal, \tag{4}$$

The first integrals H_1 and H_4 are in involution, while H_2 and H_3 are trivial. Let

$$M_c = \bigcap_{k=1}^{4}\left\{x \in \mathbb{C}^6 : H_k(x) = c_k\right\}, \quad c_3 = 1,$$

be the affine variety defined by the 4 constants of motion. The invariant variety M_c is a smooth affine surface for generic values of c_1, c_2 and c_3. So, the question I address is how does one find the compactification of M_c into an Abelian surface? Following the methods in Adler-van Moerbeke [1], the idea of the direct proof is closely related to the geometric spirit of the (real) Arnold-Liouville theorem [3].

Namely, a compact complex n-dimensional variety on which there exist n holomorphic commuting vector fields which are independent at every point is analytically isomorphic to a n-dimensional complex torus $\mathbf{C}^n/Lattice$ and the complex flows generated by the vector fields are straight lines on this complex torus. Now, the main problem will be to complete M_c into a non singular compact complex algebraic variety $\widetilde{M}_c = M_c \cup \mathcal{D}$, in such a way that the vector fields X_{H_1} and X_{H_4} generated respectively by H_1 and H_4, extend holomorphically along a divisor $\mathcal{D}$ and remain independent there. If this is possible, $\widetilde{M}_c$ is an algebraic complex torus (an Abelian variety) and the coordinates restricted to M_c are Abelian functions. A naive guess would be to take the natural compactification $\overline{M}_c$ of M_c by projectivizing the equations in $\mathbb{P}^6(\mathbb{C})$. Indeed, this can never work for a general reason : an Abelian variety $\widetilde{M}_c$ of dimension bigger or equal than two is never a complete smooth intersection, that is it can never be described in some projective space $\mathbb{P}^n(\mathbb{C})$ by n–dim $\widetilde{M}_c$ global polynomial homogeneous equations. In other words, if M_c is to be the affine part of an Abelian surface, $\overline{M}_c$ must have a singularity somewhere along the locus at infinity $\overline{M}_c \cap \{Z = 0\}$. In fact, what happens in this specific case? Let Z, $M_j = \frac{m_j}{Z}$, $\Gamma_j = \frac{\gamma_j}{Z}$, be the projective coordinates and consider the transformation

$$\overline{M}_c \longrightarrow \overline{M}_c, \quad (Z, M_1, M_2, M_3, \Gamma_1, \Gamma_2, \Gamma_3) \longmapsto (Z, X_1, X_2, M_3, U_1, U_2, \Gamma_3),$$

where $2X_1 = M_1 + iM_2$, $2X_2 = M_1 - iM_2$, $U_1 = \Gamma_1 + i\Gamma_2$, $U_2 = \Gamma_1 - i\Gamma_2$. In these new variables, the equations defining $\overline{M}_c$ are written

$$\begin{aligned}
F_1 &\equiv 2X_1X_2 + M_3^2 + (U_1 + U_2)Z - c_1Z^2 = 0,\\
F_2 &\equiv X_1U_2 + X_2U_1 + M_3\Gamma_3 - c_2Z^2 = 0,\\
F_3 &\equiv U_1U_2 + \Gamma_3^2 - Z^2 = 0,\\
F_4 &\equiv (X_1^2 - U_1Z)(X_2^2 - U_2Z) - c_4Z^4.
\end{aligned}$$

Proposition 1. *Let $\overline{M}_c$ be the projective variety on $\mathbb{P}^6(\mathbb{C})$ defined by the above equations. Then $\overline{M}_c$ is not an Abelian surface. The variety $\overline{M}_c$ intersects the hyperplane at infinity $Z = 0$ according to the two straight lines $d_1 = (0, X_1, 0, 0, U_1, 0, 0)$, $d_2 = (0, 0, X_2, 0, 0, U_2, 0)$ and the circle $S^1 = (0, 0, 0, 0, U_1, U_2, \Gamma_3)$, $U_1U_2 + \Gamma_3^2 = 0$. Moreover, $\overline{M}_c$ is singular along the lines d_1, d_2 (singularity of type $Y^4 + U^3Z^3 = 0$, Y, Z small) and along the circle S^1 (singularity of type $\Gamma_3^4X^2 = (\Gamma_3^4 + \Gamma_3^2 - 1)Z^2$, X, Z small).*

Proof. Suppose that $\overline{M}_c$ is an Abelian surface. Thus this surface is isomorphic to a 2-dimensional complex algebraic torus $\mathbb{C}^2/lattice$ where the lattice is generated by 4 independent vectors in $\mathbb{C}^2$. Let (φ_1, φ_2) be the natural coordinates of this torus. The canonical divisor of this Abelian surface is empty as, up to scalars, the only 2-form is given by $d\varphi_1 \wedge d\varphi_2$, which clearly does not vanish anywhere on $\overline{M}_c$. The canonical bundle $K_{\overline{M}_c}$ of $\overline{M}_c$ is given in terms of the canonical bundle $K_{\mathbb{P}^6(\mathbb{C})}$ of $\mathbb{P}^6(\mathbb{C})$ by the adjunction formula

$$K_{\overline{M}_c} = \left(K_{\mathbb{P}^6(\mathbb{C})} \otimes \left[\overline{M}_c\right]\right)_{\overline{M}_c},$$

where $\left[\overline{M}_c\right]$ denote the line bundle associated to the divisor $\overline{M}_c$ in $\mathbb{P}^6(\mathbb{C})$. Let

$$x_1 = \frac{M_1}{Z}, \quad x_2 = \frac{M_2}{Z}, \quad x_3 = \frac{M_3}{Z}, \quad y_1 = \frac{\Gamma_1}{Z}, \quad y_2 = \frac{\Gamma_2}{Z}, \quad y_3 = \frac{\Gamma_3}{Z},$$

be affine coordinates on $\mathbb{P}^6(\mathbb{C}) \setminus \{Z = 0\}$ and consider the form

$$\omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dy_1 \wedge dy_2 \wedge dy_3.$$

This form has no zeroes or poles in $\{Z \neq 0\}$. Let

$$u = \frac{Z}{\Gamma_3}, \quad u_1 = \frac{M_1}{\Gamma_3}, \quad u_2 = \frac{M_2}{\Gamma_3}, \quad u_3 = \frac{M_3}{\Gamma_3}, \quad v_1 = \frac{\Gamma_1}{\Gamma_3}, \quad v_2 = \frac{\Gamma_2}{\Gamma_3},$$

be affine coordinates on $\mathbb{P}^6(\mathbb{C}) \setminus \{Z = 0\}$. Then

$$u = \frac{1}{y_3}, \quad u_1 = \frac{x_1}{y_3}, \quad u_2 = \frac{x_2}{y_3}, \quad u_3 = \frac{x_3}{y_3}, \quad v_1 = \frac{y_1}{y_3}, \quad v_2 = \frac{y_2}{y_3}$$

and so

$$\omega = \frac{1}{u^7} du \wedge du_1 \wedge du_2 \wedge du_3 \wedge dv_1 \wedge dv_2.$$

Therefore, the divisor of ω is $(\omega) = -7\mathbf{H}$ where $\mathbf{H}$ is a hyperplane in $\mathbb{P}^6(\mathbb{C})$ and consequently $K_{\mathbb{P}^6(\mathbb{C})} = [-7\mathbf{H}]$. Since every line bundle on projective space is a multiple of $[\mathbf{H}]$, we have $K_{\overline{M}_c} = [\mathbf{H}]_{\overline{M}_c}$ and this implies that every holomorphic 2-form on $\overline{M}_c$ would have a zero on $\overline{M}_c$, a contradiction. The second part of the proposition is straightforward. We will describe the singularities of $\overline{M}_c$ at infinity. First let us analyze the locus at infinity $C = \overline{M}_c \cap \{Z = 0\}$. It is shown that $C = d_1 \cup d_2 \cup S^1$, where $d_1 = (0, X_1, 0, 0, U_1, 0, 0)$ and $d_2 = (0, 0, X_2, 0, 0, U_2, 0)$ are two straight lines tangent to the circle $S^1 = (0, 0, 0, 0, U_1, U_2, \Gamma_3)$, $U_1U_2 + \Gamma_3^2 = 0$. In the neighborhoods of these lines and of this circle, we have

$$\operatorname{rang} \left.\frac{\partial(F_1, F_2, F_3, F_4)}{\partial(Z, X_1, X_2, M_3, U_1, U_2, \Gamma_3)}\right|_{Z=0} = 3,$$

and the variety $\overline{M}_c$ is singular along the straight lines d_1, d_2 (singularity of type $Y^4 + U^3Z^3 = 0$, Y, Z small) and along the circle S^1 (singularity of type $\Gamma_3^4 X^2 = (\Gamma_3^4 + \Gamma_3^2 - 1)Z^2$, X, Z small). □

The solutions to the system of differential equations (2) or (3) intersect d_1 and d_2, however these solutions ignore S^1 and the same facts holds for the other vector field (4) commuting with the first. To regularize (i.e., make the trajectories parallel) the flow at infinity, it is necessary to solve the singularities of the projective variety $\overline{M}_c$. We have just seen that these singularities are complicated and we must (probably) blow up $\overline{M}_c$ many times along the lines and blow down $\overline{M}_c$ along the circle. Generally, the method of blowing up and blowing down sub-varieties is considered very delicate and its use in the present case is difficult. In fact, we shall show that the existence of meromorphic solutions to the differential equations (2) depending on 4 free parameters can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down. Information about the tori can then be gathered from the divisor. The study of convergence of these Laurent series solutions will be carried out via the majorant method and the concept of projective normality.

Proposition 2. *The differential equations (2) admit two distinct families of Laurent solutions depending on five free parameters. The first meromorphic solution, with the Laurent series, is*

$$\begin{aligned}
m_1(t) &= \frac{\alpha}{t} + i\left(\alpha^2 - 2\right)\beta + o(t), \qquad \gamma_1(t) = \frac{1}{2t^2} + o(t),\\
m_2(t) &= \frac{i\alpha}{t} - \alpha^2\beta + o(t), \qquad \gamma_2(t) = \frac{i}{2t^2} + o(t),\\
m_3(t) &= \frac{i}{t} + \alpha\beta + o(t), \qquad \gamma_3(t) = \frac{\beta}{t} + o(t).
\end{aligned}$$

The second meromorphic solution, with the Laurent series, is

$$\begin{aligned}
m_1(t) &= \frac{\alpha}{t} - i\left(\alpha^2 - 2\right)\beta + o(t), \qquad \gamma_1(t) = \frac{1}{2t^2} + o(t),\\
m_2(t) &= -\frac{i\alpha}{t} - \alpha^2\beta + o(t), \qquad \gamma_2(t) = -\frac{i}{2t^2} + o(t),\\
m_3(t) &= -\frac{i}{t} + \alpha\beta + o(t), \qquad \gamma_3(t) = \frac{\beta}{t} + o(t).
\end{aligned}$$

Proof. The proof of this proposition is a straight forward computation.

As we have pointed out, the problem of convergence of these series as well as that of other series used in this paper will be discussed later by using the majorant method (around all points where $\alpha, \beta \neq 0$)) and the techniques of projective normality (around bad points). Consider the set of Laurent solution which remain confined to a fixed affine invariant surface, i.e.,

$$\begin{aligned}\mathcal{D}_\epsilon &= \left\{ \begin{array}{c} \text{the Laurent solutions } m_j(t), \gamma_j(t), 1 \le j \le 3, \text{ such that } : \\ H_k(m_j(t), \gamma_j(t)) = c_k + \text{Taylor part }, 1 \le k \le 4 \end{array} \right\}, \\ &= \left\{ \begin{array}{c} (\alpha, \beta, \epsilon) \quad \text{such that } : (\alpha^2 - 1)((\alpha^2 - 1)\beta^4 - P(\beta) + c_4 = 0, \\ P(\beta) \equiv c_1\beta^2 - 2\epsilon c_2 \beta - 1, \quad \epsilon \equiv \pm i \end{array} \right\},\end{aligned}$$

where $c_3 = 1$ and c_1, c_2, c_4 are generic constants. Note that the application

$$\sigma_\epsilon : \mathcal{D}_\epsilon \longrightarrow \mathcal{D}_\epsilon, \quad (\alpha, \beta, \epsilon) \longmapsto (-\alpha, \beta, \epsilon),$$

is an involution on $\mathcal{D}_\epsilon$ and the quotient $\mathcal{D}_\epsilon^0 = \mathcal{D}_\epsilon/\sigma_\epsilon$ by this involution is an elliptic curve defined by $u^2 = P^2(\beta) - 4c_4\beta^4$. The curve $\mathcal{D}_\epsilon$ is a 2-sheeted ramified covering of $\mathcal{D}_\epsilon^0$,

$$\mathcal{D}_\epsilon \longrightarrow \mathcal{D}_\epsilon^0, (\alpha, u, \beta, \epsilon) \longmapsto (u, \beta, \epsilon), \quad \mathcal{D}_\epsilon : \left\{ \begin{array}{l} \alpha^2 = \dfrac{2\beta^4 + P(\beta) + u}{2\beta^4} \\ u^2 = P^2(\beta) - 4c_4\beta^4. \end{array} \right.$$

The curve $\mathcal{D}_\epsilon$ has four points at infinity and four branch points on the elliptic curve $\mathcal{D}_\epsilon^0$ and according to the Riemann-Hurwitz formula [8, 19], the genus of $\mathcal{D}_\epsilon$ is $g(\mathcal{D}_\epsilon) = 3$.

Proposition 3. *The divisor on which the solutions of (2) blow up consists of two irreducible components $\mathcal{D}_{\epsilon=i}$ and $\mathcal{D}_{\epsilon=-i}$, which are both non-singular curves of genus 3. Each of these curves $\mathcal{D}_\epsilon$ is a double cover of an elliptic curve $\mathcal{D}_\epsilon^0$ ramified at 4 points.*

Let $\mathcal{D} \equiv \mathcal{D}_{\epsilon=i} + \mathcal{D}_{\epsilon=-i}$. For positive integers k, define the space $\mathcal{L}(k\mathcal{D})$ of meromorphic functions having a k-fold pole at worst along $\mathcal{D}$.

Proposition 4. *The space $\mathcal{L}(\mathcal{D})$ is spanned by the following functions*

$$f_0 = 1, \; f_1 = m_1, \; f_2 = m_2, \; f_3 = m_3, \; f_4 = \gamma_3, \; f_5 = f_1^2 + f_2^2,$$

$$f_6 = 4f_1f_4 - f_3f_5, \; f_7 = (f_2\gamma_1 - f_1\gamma_2) f_3 + 2f_4\gamma_2.$$

In addition, $\mathcal{D}$ is embedded into $\mathbb{P}^7(\mathbb{C})$ according to

$$p=(\alpha,u,\beta)\longmapsto \lim_{t\to 0} t(1,f_1(p),...,f_7(p))=\left(0,f_1^{(0)}(p),...,f_7^{(0)}(p)\right),$$

In a way that $\mathcal{D}_{\epsilon=i}$ and $\mathcal{D}_{\epsilon=-i}$ intersect each other transversally in four points at infinity $\left(\alpha=\pm 1,\ u=\pm\beta^2\sqrt{c_1^2-4c_4},\ \beta=\infty\right)$ and that the geometric genus of $\mathcal{D}$ is 9.

Proof. Let $L^{(r)}$ be the set of polynomial functions $f=f(x(t))$ of degree $\leq r$ modulo the constants $H_k=c_k, 1\leq k\leq 4$ and such that $f(x(t))=t^{-1}(x^{(0)}+x^{(1)}t+\cdots)$, $x^{(0)}\neq 0$ on $\mathcal{D}$ with $x(t)=(m_1,m_2,m_3,\gamma_1,\gamma_2,\gamma_3)$ given explicitly above. We look for r such that : geometric genus of $\mathcal{D}^{(r)}=N_r+2$, $\mathcal{D}^{(r)}\subset\mathbb{P}^{N_r}(\mathbb{C})$ and we shall show that it is unnecessary to go beyond $r=4$. Indeed, using the Laurent series obtained previously, one obtains that the spaces $L^{(r)}$, nested according to weighted degree, are generated as follows $L^{(1)}=\{f_0=1,f_1=m_1,f_2=m_2,f_3=m_3\}, L^{(2)}=L^{(1)}\oplus\left\{f_4=\gamma_3,f_5=-\frac{1}{4}(f_1^2+f_2^2)\right\}$, $L^{(3)}=L^{(2)}\oplus\{f_6=f_3f_5+f_1f_4\}$, $L^{(4)}=L^{(3)}\oplus\{f_7=(f_2\gamma_1-f_1\gamma_2)f_3+2f_4\gamma_2\}$. The spaces $L^{(1)}$, $L^{(2)}$ and $L^{(3)}$ do not provide the embedding because $g(\mathcal{D}^{(r)})\neq\dim L^{(r)}+1$, $r=1,2,3$. More precisely, since the embedding in $\mathbb{P}^3(\mathbb{C})$ via $L^{(1)}$ does not separate the sheets, we then pass to a $L^{(2)}$ and we also find that the embedding in $\mathbb{P}^6(\mathbb{C})$ is invalid because $g\left(\mathcal{D}^{(2)}\text{ embedded in }\mathbb{P}^5(\mathbb{C})\right)-2>5$, which contradicts the fact that $N_r+1=g(\mathcal{D}^{(2)})-1$. In the same way, using space $L^{(3)}$ we obtain $g\left(\mathcal{D}^{(3)}\text{ embedded in }\mathbb{P}^6(\mathbb{C})\right)-2>6$, and the contradiction persists. We must therefore consider $L^{(4)}$ and study the embedding of $\mathcal{D}$ into $\mathbb{P}^7(\mathbb{C})$ using the functions of $L^{(4)}$. Let

$$\left(\frac{f_0}{f_4},...,\frac{f_7}{f_4}\right)\equiv F_k=F_k^{(0)}+F_k^{(1)}t+...,\quad 0\leq k\leq 7,$$

and $p_j\equiv\left(\alpha=\pm 1,u=\pm\beta^2\sqrt{c_1^2-4c_4},\beta=\infty\right)$, $1\leq j\leq 4$, be the 4 points at ∞ of $\mathcal{D}_\epsilon$. Therefore

$$\begin{aligned}F_k^{(0)}(p_j) &= \left(0,\frac{\alpha}{\beta},\frac{\epsilon\alpha}{\beta},\frac{\epsilon}{\beta},1,\epsilon\alpha,\epsilon(\alpha^2-1)\beta,-\frac{c_2}{\beta}+\frac{\epsilon(1-u)}{\beta^2}\right)(p_j),\\ &= \left(0,0,0,0,1,\pm\epsilon,0,\mp\epsilon\sqrt{c_1^2-4c_4}\right)=4\text{ distinct points.}\end{aligned}$$

The 4 points at ∞ are separated on each curve $\mathcal{D}_\varepsilon$ but using the transformation

$$\mathcal{D}_{\epsilon=i}\longrightarrow\mathcal{D}_{\epsilon=-i},\quad(\alpha,u,\beta)\longmapsto(-\alpha,-u,\beta),$$

one shows

$$F_k^{(0)}(p_j)\Big|_{\mathcal{D}_{\epsilon=i}} = F_k^{(0)}(p_j)\Big|_{\mathcal{D}_{\epsilon=-i}},$$

i.e., the 4 points at ∞ are identified pairwise with the the 4 points at ∞ on the other curve. Let $s = \frac{1}{\beta}$ be a local parameter for p_j. Since $\frac{\partial F_k^{(0)}}{\partial s}(p_j) = (0, \pm1, \pm\epsilon, \epsilon, 0, -c_2)$, then

$$\frac{\partial F_k^{(0)}}{\partial s}(p_j)\Bigg|_{\mathcal{D}_{\epsilon=i}} \neq \frac{\partial F_k^{(0)}}{\partial s}(p_j)\Bigg|_{\mathcal{D}_{\epsilon=-i}},$$

and consequently the curve $\mathcal{D}_{\epsilon=i}$ intersects transversely the curve $\mathcal{D}_{\epsilon=-i}$ in 4 points at infinity $b_1, ..., b_4 = (\alpha = \infty, \beta = 0)$, $p_1, ..., p_4 = (\alpha = \pm1, u = \pm\beta^2\sqrt{c_1^2 - 4c_4}, \beta = \infty)$. In a neighborhood of the points $b_j = (\alpha = \infty, \beta = 0)$, $1 \leq j \leq 4$, one divides by f_1. By setting,

$$\left(\frac{f_0}{f_1}, ..., \frac{f_7}{f_1}\right) \equiv G_k = G_k^{(0)} + G_k^{(1)}t + ..., \quad 0 \leq k \leq 7,$$

one obtains $G_k^{(0)}(p_j) = (0, 1, \epsilon, 0, 0, 0, \mp1, 0)$ which are the coordinates of 4 different points in $\mathbb{P}^7(\mathbb{C})$. Therefore, $g\left(\mathcal{D}^{(4)}\right)$ embedded in $\mathbb{P}^7(\mathbb{C})\big) - 2 = 7$ and therefore, $g\left(\mathcal{D}^{(4)}\right) = 9$. $\square$

To avoid overburdening the text, we will keep the notation $\mathcal{D}$ instead of $\mathcal{D}^{(4)}$ as being embedded in $\mathbb{P}^7(\mathbb{C})$. At all points where a, $\alpha, \beta \neq 0$, the Laurent solutions are nicely convergent (by the majorant method [16]). Therefore, at most points of $\mathcal{D}$ there is a transversal fiber to the curve. Hence this defines a smooth surface strip around $\mathcal{D}$ except at the bad points. Now we need to construct a surface strip around $\mathcal{D}$ at the bad points as well. For doing that, we must use the concept of normally generated line bundle or projective normality. Ultimately, we wish to prove that in the various charts

$$\left(\frac{f_j}{f_k}\right)^{\cdot} = \text{polynomial}\left(\frac{f_j}{f_k}\right), \quad 1 \leq j \leq 7, k \text{ fixed} \tag{5}$$

This enables one to show that (f_j/f_k), is a bona fide Taylor series starting from every point in a neighborhood of the point in question $\subseteq \mathbb{P}^7(\mathbb{C})$. As mentioned above, let $\mathcal{L} \equiv L^{(4)}$ and $\mathcal{D} \equiv \mathcal{D}^{(4)}$. In fact, we shall prove a somewhat stronger statement than (5), namely that (5) is satisfied with quadratic polynomials. By inspection one sees that the $f_0, ..., f_7$ do not satisfy that property. For example

$$\left(\frac{f_0}{f_4}\right)^{\cdot} = \frac{f_1\gamma_2 - f_2\gamma_1}{f_4^2} \neq \text{polynomial}\left(\frac{f_j}{f_4}\right), \quad 1 \leq j \leq 7$$

The problem is that $\mathcal{D}$ although very ample, will not be projectively normal (more precisely, the line bundle $[\mathcal{D}]$ is not normally generated) which forces us to look at the divisor $2\mathcal{D}$ or maybe simply $2\mathcal{D}_{\epsilon=i}+\mathcal{D}_{\epsilon=-i}$. Hence we must take functions with higher order poles. Let us consider for instance the space $\mathcal{L}(2\mathcal{D}_{\epsilon=i}+\mathcal{D}_{\epsilon=-i})$ of functions which have double poles at worst along $\mathcal{D}_{\epsilon=i}$ and simple ones at worst along $\mathcal{D}_{\epsilon=-i}$. By the Riemann-Roch theorem, if $2\mathcal{D}_{\epsilon=i}+\mathcal{D}_{\epsilon=-i}$ is a divisor on an Abelian variety, this space must be 18-dimensional. Indeed, let

$$\mathcal{L}(2\mathcal{D}_{\epsilon=i}+\mathcal{D}_{\epsilon=-i}) = L^{(4)}\oplus\{g_8,...,g_{17}\}\equiv\{g_0,...,g_{17}\},$$

where $g_0=1$, $g_1=\frac{1}{2}(m_1+im_2)$, $g_2=\frac{1}{2}(m_1-im_2)$, $g_3=f_3$, $g_4=f_4$, $g_5=f_5$, $g_6=f_6$, $g_7=f_7$, $g_8=g_2^2$, $g_9=g_2g_3$, $g_{10}=g_2g_4$, $g_{11}=g_2^2-\gamma_1-i\gamma_2$, $g_{12}=g_1g_{11}$, $g_{13}=g_1^2g_{11}$, $g_{14}=g_1^3g_{11}$, $g_{15}=g_2g_5$, $g_{16}=g_2g_6$, $g_{17}=g_2g_7$, where one reads off the behavior of the g_k along $\mathcal{D}_\epsilon$ from the Laurent series solutions; here (g_k) denote the divisor restricted to $\mathcal{D}_\epsilon$: $(g_1)=-\mathcal{D}_{\epsilon=-i}$, $(g_2)=-\mathcal{D}_{\epsilon=i}$, $(g_k)=-\mathcal{D}_{\epsilon=i}-\mathcal{D}_{\epsilon=-i}$, $3\leq k\leq 7$, $(g_8)=-2\mathcal{D}_{\epsilon=i}$, $(g_k)=-2\mathcal{D}_{\epsilon=i}-\mathcal{D}_{\epsilon=-i}$, $k=9,10,15,16,17$, $(g_{11})=-2\mathcal{D}_{\epsilon=i}+2\mathcal{D}_{\epsilon=-i}$, $(g_{12})=-2\mathcal{D}_{\epsilon=i}+\mathcal{D}_{\epsilon=-i}$, $(g_k)=-2\mathcal{D}_{\epsilon=-i}$, $k=13,14$. Note that all g_k have the property $(g_k)\geq-2\mathcal{D}_{\epsilon=i}-\mathcal{D}_{\epsilon=-i}$. Consider now the embedding of $2\mathcal{D}_{\epsilon=i}+\mathcal{D}_{\epsilon=-i}$ via these functions

$$2\mathcal{D}_{\epsilon=i}+\mathcal{D}_{\epsilon=-i}\to\mathbb{P}^{17}(\mathbb{C}), p\mapsto\lim_{t\to0}t\left(g_0(p),...,g_{17}(p)\right)=\left(0,g_1^{(0)}(p),...,g_{17}^{(0)}(p)\right).$$

About the points $b_j=(\alpha=\infty,\beta=0)$, it is appropriate to divide by g_8, which implies that $\left(\frac{g_k}{g_8}\right)(b_j)$, $0\leq j\leq 17$, is finite. In a neighborhood of the points at infinity $p_j=\left(\alpha=\pm1,u=\pm\beta^2\sqrt{c_1^2-4c_4},\beta=\infty\right)$, $1\leq j\leq 4$, one divides $g_0,...,g_{17}$ by g_{10}, which makes $\left(\frac{g_k}{g_{10}}\right)(p_j)$, $0\leq j\leq 17$, finite. Next, using the differential equations (2), we show that in a neighborhood of the points p_j and modulo linear combination of the constants of motion

$$\left(\frac{g_k}{g_{10}}\right)^{\cdot}=\frac{\dot{g}_kg_{10}-g_k\dot{g}_{10}}{g_{10}^2}=\text{quadratic polynomial of}\left(\frac{g_0}{g_{10}},...,\frac{g_{17}}{g_{10}}\right).$$

Also, in a neighborhood of the points b_j we have

$$\left(\frac{g_k}{g_8}\right)^{\cdot}=\text{quadratic polynomial of}\left(\frac{g_0}{g_8},...,\frac{g_{17}}{g_8}\right).$$

It follows that at the bad points p_j, b_j the series $\left(\frac{g_k}{g_{10}}\right)(p_j)$, $\left(\frac{g_k}{g_8}\right)(b_j)$, $0 \leq k \leq 17$, converges as a consequence of Picard's theorem applied to the system of ordinary differential equations $\left(\frac{g_k}{g_l}\right)^{\cdot}$, $l = 8, 10$. Therefore, we have the following proposition,

Proposition 5. *The divisor $2\mathcal{D}_{\epsilon=i}+\mathcal{D}_{\epsilon=-i}$ is projectively normal and has a smooth embedding in $\mathbb{P}^{17}(\mathbb{C})$, which shows that in particular the Laurent series solutions converge everywhere.*

In summary, the use of the Laurent series (and the study of convergence of various series intervening in the problem via the majorant method and the concept of normally generated line bundle) show that the orbits of the vector field (3) going through the curve $\mathcal{D}$ form a smooth surface Σ near $\mathcal{D}$ such that $\Sigma\backslash M_c \subseteq \widetilde{M_c}$ and the variety $\widetilde{M_c} = M_c \cup \Sigma$ is smooth, compact and connected. More precisely, for generic c the affine surface M_c completes into an Abelian surface $\widetilde{M_c}$ of polarization $(1, 2)$, by adding a singular divisor $\mathcal{D}$ consisting of two isomorphic genus 3 curves $\mathcal{D}_{\epsilon=i}$ and $\mathcal{D}_{\epsilon=-i}$ intersecting in 4 points. Each $\mathcal{D}_{\epsilon=\pm i}$ is a double cover of an elliptic curve $\mathcal{D}_{\epsilon}^{0}$ ramified at 4 points. It defines a line bundle and a polarization $(1, 2)$ on $\widetilde{M_c}$. The divisors $2\mathcal{D}_{\epsilon=i}$, $2\mathcal{D}_{\epsilon=-i}$ or $\mathcal{D}_{\epsilon=i} + \mathcal{D}_{\epsilon=-i}$ (of geometric genus 9) are all very ample and define a polarization $(2, 4)$; and moreover the 8-dimensional space of sections of the corresponding line bundle embeds the Abelian surface into $\mathbb{P}^7(\mathbb{C})$. In other words, the line bundle $[\mathcal{D}]$ defines a polarization of type $(2, 4)$ on $\widetilde{M_c}$ and leads to an embedding of $\widetilde{M_c}$ in $\mathbb{P}^7(\mathbb{C})$; it is not normally generated, but the line bundle $[2\mathcal{D}_{\epsilon=i}+\mathcal{D}_{\epsilon=-i}]$ is. The Abelian surface $\widetilde{M_c}$ is equipped with two everywhere independent commuting vector fields. Otherwise, the Abelian surface $\widetilde{M_c}$ can also be identified as the dual of the Prym variety $\left(\mathcal{D}_{\epsilon}/\mathcal{D}_{\epsilon}^{0}\right)$.

3.2. Movement of a solid in an ideal fluid. Integration of the Clebsch case.

The Kirchhoff's equations [7] of motion of a solid in an ideal fluid have the form

$$
\begin{aligned}
\dot{p}_1 &= p_2\frac{\partial H}{\partial l_3} - p_3\frac{\partial H}{\partial l_2}, \qquad \dot{l}_1 = p_2\frac{\partial H}{\partial p_3} - p_3\frac{\partial H}{\partial p_2} + l_2\frac{\partial H}{\partial l_3} - l_3\frac{\partial H}{\partial l_2},\\
\dot{p}_2 &= p_3\frac{\partial H}{\partial l_1} - p_1\frac{\partial H}{\partial l_3}, \qquad \dot{l}_2 = p_3\frac{\partial H}{\partial p_1} - p_1\frac{\partial H}{\partial p_3} + l_3\frac{\partial H}{\partial l_1} - l_1\frac{\partial H}{\partial l_3},\\
\dot{p}_3 &= p_1\frac{\partial H}{\partial l_2} - p_2\frac{\partial H}{\partial l_1}, \qquad \dot{l}_3 = p_1\frac{\partial H}{\partial p_2} - p_2\frac{\partial H}{\partial p_1} + l_1\frac{\partial H}{\partial l_2} - l_2\frac{\partial H}{\partial l_1},
\end{aligned} \tag{6}
$$

where (p_1, p_2, p_3) is the velocity of a point fixed relatively to the solid, (l_1, l_2, l_3) the angular velocity of the body expressed with regard to a frame of reference also fixed relatively to the solid and H is the hamiltonian. In Clebsch's case, equations (6) have the four invariants :

$$H_1 = H = \frac{1}{2}\left(a_1 p_1^2 + a_2 p_2^2 + a_3 p_3^2 + b_1 l_1^2 + b_2 l_2^2 + b_3 l_3^2\right),$$

$$H_2 = p_1^2 + p_2^2 + p_3^2, \qquad H_3 = p_1 l_1 + p_2 l_2 + p_3 l_3,$$

$$H_4 = \frac{1}{2}\left(b_1 p_1^2 + b_2 p_2^2 + b_3 p_3^2 + \varrho\left(l_1^2 + l_2^2 + l_3^2\right)\right),$$

with $\frac{a_2-a_3}{b_1} + \frac{a_3-a_1}{b_2} + \frac{a_1-a_2}{b_3} = 0$, and the constant ϱ satisfies the conditions $\varrho = \frac{b_1(b_2-b_3)}{a_2-a_3} = \frac{b_2(b_3-b_1)}{a_3-a_1} = \frac{b_3(b_1-b_2)}{a_1-a_2}$. We give a sketch of the study of this problem and we refer the reader to [17] for details. Consider points at infinity which are limit points of trajectories of the flow. In fact, there is a Laurent decomposition of such asymptotic solutions,

$$x(t) = t^{-1}\left(x^{(0)} + x^{(1)}t + x^{(2)}t^2 + \cdots\right), \quad x \equiv (p_1, p_2, p_3, l_1, l_2, l_3),$$

which depend on $\dim(\textit{phase space}) - 1 = 5$ free parameters. Taking into account only solutions trajectories lying on the invariant surface

$$M_c = \bigcap_{i=1}^{4} \{x : H_i(x) = c_i\} \subset \mathbb{C}^6,$$

we obtain one-parameter families which are parameterized by a curve :

$$\mathcal{D} : \theta^2 + c_1\beta^2\gamma^2 + c_2\alpha^2\gamma^2 + c_3\alpha^2\beta^2 + c_4\alpha\beta\gamma = 0,$$

where θ is an arbitrary parameter and where $\alpha = x_4^{(0)}, \beta = x_5^{(0)}, \gamma = x_6^{(0)}$ parameterizes the elliptic curve

$$\mathcal{E} : \beta^2 = d_1^2\alpha^2 - 1,\ \gamma^2 = d_2^2\alpha^2 + 1,$$

with d_1, d_2 such that: $d_1^2 + d_2^2 + 1 = 0$. The curve $\mathcal{D}$ is a 2-sheeted ramified covering of the elliptic curve $\mathcal{E}$. The branch points are defined by the 16 zeroes of $c_1\beta^2\gamma^2 + c_2\alpha^2\gamma^2 + c_3\alpha^2\beta^2 + c_4\alpha\beta\gamma$ on $\mathcal{E}$. The curve $\mathcal{D}$ is unramified at infinity and by Hurwitz's formula, the genus of $\mathcal{D}$ is 9. Upon putting $\zeta \equiv \alpha^2$, the curve $\mathcal{D}$ can also be seen as a 4−sheeted unramified covering of the following curve of genus 3 :

$$C : \left(\theta^2 + c_1\beta^2\gamma^2 + \left(c_2\gamma^2 + c_3\beta^2\right)\zeta\right)^2 - c_4^2\zeta\beta^2\gamma^2 = 0.$$

Moreover, the map

$$\tau : C \longrightarrow C, \quad (\theta, \zeta) \longmapsto (-\theta, \zeta),$$

is an involution on C and the quotient $C_0 = C/\tau$ is an elliptic curve defined by

$$C_0 : \eta^2 = c_4^2 \zeta \left(d_1^2 d_2^2 \zeta^2 + \left(d_1^2 - d_2^2 \right) \zeta - 1 \right).$$

The curve C is a double ramified covering of C_0, $C \longrightarrow C_0, (\theta, \eta, \zeta) \longmapsto (\eta, \zeta)$,

$$C : \begin{cases} \theta^2 = -c_1 \beta^2 \gamma^2 - \left(c_2 \gamma^2 + c_3 \beta^2 \right) \zeta + \eta \\ \eta^2 = c_4^2 \zeta \left(d_1^2 d_2^2 \zeta^2 + \left(d_1^2 - d_2^2 \right) \zeta - 1 \right). \end{cases}$$

The flow evolves on an Abelian surface $\widetilde{M_c} \subseteq \mathbb{P}^7(\mathbb{C})$, whose period matrix reads $\begin{pmatrix} 2 & 0 & a & c \\ 0 & 4 & c & b \end{pmatrix}$, $\operatorname{Im} \begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0$. The divisor $\mathcal{D}$ although very ample, will not be projectively normal (i.e., its line bundle $[\mathcal{D}]$ is not normally generated) because the 8 functions spanning $\mathcal{L}(\mathcal{D})$ do not form a closed system of quadratic differential equations, which forces us to look at the divisor $2\mathcal{D}$. Indeed, a direct calculation using the Laurent series shows that the 32 functions spanning $\mathcal{L}(2\mathcal{D})$ form a closed system of quadratic differential equations and the divisor $2\mathcal{D}$ is projectively normal.

Proposition 6. *The system of differential equations (6) has one single* 5*-parameter family of Laurent. The invariant surface M_c completes into an Abelian surface $\widetilde{M_c}$ by adjoining a smooth curve $\mathcal{D}$ of genus* 9*, which is a* 2*-sheeted ramified covering of the elliptic curve $\mathcal{E}$ and can also be seen as a* 4*-sheeted unramified covering of a curve C of genus* 3*. The latter is a double ramified covering of an elliptic curve C_0 and therefore $\widetilde{M_c}$ can be identified as $Prym(C/C_0)$. The functions of $\mathcal{L}(\mathcal{D})$ embeds $\widetilde{M_c}$ smoothly into $\mathbb{P}^7(\mathbb{C})$ and the divisor $\mathcal{D}$ defines on $\widetilde{M_c}$ a polarization* $(2,4)$*. The divisor $2\mathcal{D}$ is projectively normal (i.e., the line bundle $[2\mathcal{D}]$ is normally generated) and has a smooth embedding in $\mathbb{P}^{31}(\mathbb{C})$, and in particular the Laurent series solutions converge everywhere.*

The Kirchhoff's equations of motion of a solid in an ideal fluid can be regarded as the equations of the geodesics of the right-invariant metric on the group $E(3) = SO(3) \times \mathbb{R}^3$ of motions of 3-dimensional euclidean space $\mathbb{R}^3$, generated by rotations and translations. Hence the motion has the trivial coadjoint orbit invariants $\langle p, p \rangle$ and $\langle p, l \rangle$. As it turns out, this is a special case of a more general system of equations written as

$$\dot{x} = x \wedge \frac{\partial H}{\partial x} + y \wedge \frac{\partial H}{\partial y}, \quad \dot{y} = y \wedge \frac{\partial H}{\partial x} + x \wedge \frac{\partial H}{\partial y},$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. The first set can be obtained from the second by putting $(x, y) = (l, p/\varepsilon)$ and letting $\varepsilon \to 0$. The latter set of equations is the geodesic flow on $SO(4)$ for a left invariant metric defined by the quadratic form H. For further information see for example [2].

3.3. **The Ramani-Dorizzi-Grammaticos (RDG) series of integrable potentials.** There are many examples of differential equations which have the weak Painlevé property that all movable singularities of the general solution have only a finite number of branches and some interesting integrable systems appear as coverings of algebraic completely integrable systems. The invariant varieties are coverings of Abelian varieties and these systems are called algebraic completely integrable in the generalized sense. These systems are Liouville integrable and by the Arnold-Liouville theorem, the compact connected manifolds invariant by the real flows are tori; the real parts of complex affine coverings of Abelian varieties. Most of these systems of differential equations possess solutions which are Laurent series of $t^{1/n}$ (t being complex time) and whose coefficients depend rationally on certain algebraic parameters. In other words, for these systems just replace in the definition of the algebraic completely integrable systems (see section 1) condition *ii*) by the following: (*iii*) the invariant manifolds $\mathcal{A}$ are related to an l-fold cover $\widetilde{T}^n$ of T^n ramified along a divisor $\mathcal{D}$ in T^n as follows : $\mathcal{A} = \widetilde{T}^n \backslash \mathcal{D}$.

Let us consider the Ramani-Dorizzi-Grammaticos (RDG) series of integrable potentials [24, 11] :

$$V(x, y) = \sum_{k=0}^{[m/2]} 2^{m-2i} \begin{pmatrix} m-i \\ i \end{pmatrix} x^{2i} y^{m-2i}, \quad m = 1, 2, ...$$

It can be straightforwardly proven that a Hamiltonian H :

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \alpha_m V_m,, \quad m = 1, 2, ...$$

containing V is Liouville integrable, with an additional first integral :

$$F = p_x(xp_y - yp_x) + \alpha_m x^2 V_{m-1}, \quad m = 1, 2, ...$$

The study of cases $m = 1$ and $m = 2$ is easy. The study of other cases is not obvious. For the case $m = 3$, one obtains the Hénon-Heiles system. The latter admits admits Laurent solutions in $\sqrt{t}$, depending on three free parameters : α, β, γ and they are explicitly given as follows (see [18] for details): :

$$\begin{aligned}
y_1 &= \frac{\alpha}{\sqrt{t}} + \beta t\sqrt{t} - \frac{\alpha}{18}t^2\sqrt{t} + \frac{\alpha A_1^2}{10}t^3\sqrt{t} - \frac{\alpha^2\beta}{18}t^4\sqrt{t} + \cdots ,\\
y_2 &= -\frac{3}{8t^2} - \frac{A}{2} + \frac{\alpha^2}{12}t - \frac{2A^2}{5}t^2 + \frac{\alpha\beta}{3}t^3 - \gamma t^4 + \cdots ,\\
x_1 &= -\frac{1}{2}\frac{\alpha}{t\sqrt{t}} + \frac{3}{2}\beta\sqrt{t} - \frac{5}{36}\alpha t\sqrt{t} + \frac{7}{20}\alpha A^2 t^2\sqrt{t} - \frac{1}{4}\alpha^2\beta t^3\sqrt{t} + \cdots ,\\
x_2 &= \frac{3}{4t^3} + \frac{1}{12}\alpha^2 - \frac{4}{5}A^2 t + \alpha\beta t^2 - 4\gamma t^3 + \cdots
\end{aligned}$$

where A is a constant. These formal series solutions are convergent as a consequence of the majorant method and the concept of projective normality. The case $m = 4$, corresponds to the system whose Laurent series solutions are of the form

$$(y_1, y_2, x_1, x_2) = (t^{-1/2}, t^{-1}, t^{-3/2}, t^{-2}) \times \text{ a Taylor series in } t.$$

More precisely, the system in this case possesses 3-dimensional family of Laurent solutions (principal balances) depending on three free parameters u, v and w. There are precisely two such families, labelled by $\varepsilon = \pm 1$, and they are explicitly given as follows (see [17] for details):

$$\begin{aligned}
y_1 &= \frac{1}{\sqrt{t}}(u - \frac{1}{4}u^3 t + vt^2 - \frac{5}{128}u^7 t^3 + \frac{1}{8}u(\frac{3}{4}u^3 v - \frac{7}{256}u^8 + 3\varepsilon w)t^4 + \cdots),\\
y_2 &= \frac{1}{t}(\frac{1}{2}\varepsilon - \frac{1}{4}\varepsilon u^2 t + \frac{1}{8}\varepsilon u^4 t^2 + \frac{1}{4}\varepsilon u(\frac{1}{32}u^5 - 3v)t^3 + wt^4 + \cdots),\\
x_1 &= \frac{1}{2t\sqrt{t}}(-u - \frac{1}{4}u^3 t + 3vt^2 - \frac{25}{128}t^3 u^7 +\\
&\qquad \frac{7}{8}u(\frac{3}{4}u^3 v - \frac{7}{256}u^8 + 3\varepsilon w)t^4 + \cdots),\\
x_2 &= \frac{1}{t^2}(-\frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon u^4 t^2 + \frac{1}{2}\varepsilon u(\frac{1}{32}u^5 - 3v)t^3 + 3wt^4 + \cdots).
\end{aligned}$$

Here too, the study of convergence of these Laurent series solutions will be carried out via the majorant method and the concept of projective normality.

However, the case $m = 5$, corresponds to a system with an Hamiltonian of the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + y^5 + x^2 y^3 + \frac{3}{16}x^4 y.$$

The corresponding Hamiltonian system admits a second first integral :

$$F = -p_x^2 y + p_x p_y x - \frac{1}{2}x^2 y^4 + \frac{3}{8}x^4 y^2 + \frac{1}{32}x^6,$$

and admits three 3-dimensional families solutions x, y, which are Laurent series of $t^{1/3}$:

$$x = at^{-\frac{1}{3}}, \qquad x = bt^{-\frac{2}{3}}, \qquad b^3 = -\frac{2}{9},$$

but for which there are no polynomial P such that $P(x(t), y(t), \dot{x}(t), \dot{y}(t))$ is Laurent serie in t. This problem needs to be studied and understood.

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