Journal of Dynamical Systems & Geomatric Theories
ISSN : 1726-037X (Print) 2169-0057 (Online)
Vol. 16(2) November 2018, pp. 101–127
DOI : 10.1080/1726037X.2018.1436272

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GRADED VECTOR FIELDS AND INVOLUTIVE DISTRIBUTIONS ON GRADED MANIFOLDS

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(Received: 13 June 2017, Accepted: 16 November 2017)

Abstract. Suppose that$\mathcal{M} = (M, \mathcal{A}_M)$ is a graded manifold and consider a direct subsheaf$\mathcal{D}$of $Der\mathcal{A}_M$ and a graded vector field Γ on$\mathcal{M}$,both satisfying certain conditions. We attach to$\mathcal{D}$a distribution$\mathcal{D}+[\Gamma,\mathcal{D}]$ and characterize its maximal rank with respect to $dim\mathcal{M}$. $\mathcal{D}$ is used to characterize the local expression of Γ.

AMS Classification: 58A50, 58A30, 53C15, 53B25
Keywords: Supermanifold; Involutive distribution; Second-order differential equation field

1. Introduction

It is well known that associated to systems of second order ordinary differential equations (SODE) on a manifold $E = \mathbb{R} \times TM$, where$M$ is a manifold, one can introduce a spray which is useful tool for the investigation of many problems, such as, the inverse problem of the calculus of variations, differential geometric modelling of mechanical systems, and so on. An important geometric concept induced by a spray is the notion of a nonlinear connection (also called a dynamical connection),

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a subbundle $\mathcal{H}$ of TE, which is complementary to the vertical bundle [5, 11] (see [9] for the definition of a dynamical connection). Any non-linear connection generates a covariant derivative, which is called the Berwald connection. This Berwald derivative is an important tool in the study of the linearizability properties of a second-order vector field [10] and for characterizing SODE which are second order in the velocities [12]. Also, it is shown in [7] that if the Berwald curvature vanishes, the system (SODE) is of mechanical type.

Due to the interest aroused by the theory of sprays and connections, many papers have been devoted to these subjects. On the other hand, the graded geometrization of these subjects and their application has been less developed, we cite here the pioneer works with applications in theoretical physics by Vacaru [15], [16, 17], Ibort et al [6] and [1, 4].

There are two approaches to provide supermechanics with a geometrical base. The configuration of each case is a graded manifold of dimension (m, n), but the difference in the approaches is related to a generalization of the tangent bundle in the graded case. In the first approach, supermechanical systems can be considered on the tangent supermanifold such that its dimension is $(2m, 2n)$. The work of Ibort et al [6], is an interesting study of the inverse problem of Lagrangian supermechanics. In the second approach supermechanical systems considered on the tangent superbundle with dimension $(2m + n, 2n + m)$. Related references are [1, 4].

In this paper we start with a graded vector field Γ on a graded manifold $\mathcal{M} = (M, \mathcal{A}_M)$ of dimension (m, n) and a direct subsheaf $\mathcal{D}$ of $Der\mathcal{A}_M$ of rank (r, s), both satisfying certain conditions. By using the Lie bracket of Γ and graded vector fields of $\mathcal{D}$, we construct another direct subsheaf $\mathcal{C} := \mathcal{D} + [\Gamma, \mathcal{D}]$ of $Der\mathcal{A}_M$. There is a beautiful link between the direct subsheaf $\mathcal{D}$ and so-called nonlinear connection (associated with the direct subsheaf $\mathcal{D}$) which will be important for understanding how to define a linear connection on a graded manifold $\mathcal{M}$. To reach this, we define the almost tangent structure J on $\mathcal{C}$ and show that if $[\Gamma, \mathcal{C}] \subset \mathcal{C}$, the graded vector field Γ defines the tensor field $-\mathcal{L}_\Gamma J$ such that the associated horizontal graded subsheaf $\mathcal{H}$ is a complement of $\mathcal{D}$ in $\mathcal{C}$. We define a $\mathcal{C}$-connection on $\mathcal{D}$ which is useful to characterize the local expression of the graded vector field Γ.

One can see that, for any even graded vector field Γ on $\mathbb{R}^{1,1} \times (TM', \mathcal{A}_{TM'})$, where $(TM', \mathcal{A}_{TM'})$ is a tangent supermanifold of dimension $(2m, 2n)$, there is a

local coordinates $(t, x, y; \tau, \eta, \zeta)$ such that $\Gamma = \frac{\partial}{\partial t} + y_i \frac{\partial}{\partial x_i} + \Gamma_i(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial y_i} + \tau \frac{\partial}{\partial \tau} + \zeta_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma_\mu(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial \zeta_\mu}$ and the coordinates t and τ play the role of supertime.

2. Preliminaries

Let $\mathcal{M} = (M, \mathcal{A}_M)$ be an $(m, n)-$dimensional graded manifold, the sheaf of left $\mathcal{A}_M-$modules of derivations of a graded manifold $\mathcal{M}$ is the subsheaf of $End\mathcal{A}_M$ whose sections are linear graded derivations and denoted by $Der\mathcal{A}_M$ the sheaf of graded derivations of $\mathcal{A}_M$. Let $\mathcal{D}$ be a locally free sheaf of $\mathcal{A}_M-$modules. $\mathcal{D}$ is a direct subsheaf of $Der\mathcal{A}_M$ of rank (r, s), if for each point $p \in M$ there is an open subset U over which any set of generators $\{D_i, D\mu | 1 \leq i \leq r, 1 \leq \mu \leq s\}$ of the module $\mathcal{D}(U)$ can be enlarged to a set

$$\left\{ C_u, D_i, D_\mu, C_\alpha \;\middle|\; \begin{matrix} 1 \leq i \leq r \\ r+1 \leq u \leq m \end{matrix} \; and \; \begin{matrix} 1 \leq \mu \leq s \\ s+1 \leq \alpha \leq n \end{matrix} \; \begin{matrix} |C_u| = 0 \\ |D_i| = 0 \end{matrix} \; and \; \begin{matrix} |D_\mu| = 1 \\ |C_\alpha| = 1 \end{matrix} \right\}$$

of free generators of $Der\mathcal{A}_M$. A direct subsheaf $\mathcal{D}$ of $Der\mathcal{A}_M$ is involutive if $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$ (see [13]).

Definition 2.1. *Suppose that $\{X_a | a \in \{1, \cdots, r\} \sqcup \{1, \cdots, s\}\}$ is a set of smooth graded vector fields on an open subset $\mathcal{U}$ of a graded manifold $\mathcal{M}$, and that is a set of generators of $\mathcal{D}$. Then we say that $\{X_a\}$ is a graded commutative set of graded vector fields if we have*

$$[X_a, X_b] = 0, \forall X_a, X_b \in \{D_i, D_\mu\}, i \in \{1, \cdots, r\}, \mu \in \{1, \cdots, s\}.$$

For the following see [13]:

Lemma 2.1. *Let $\mathcal{D} \subset Der\mathcal{A}_M$ be a direct subsheaf of rank (r, s). If $\mathcal{D}$ is involutive, then, there exist even generators $\{D_i | 1 \leq i \leq r\}$, and odd generators $\{D_\mu | 1 \leq \mu \leq s\}$, for $\mathcal{D}$, such that $\{X_a\}$ form a graded commutative set in the sense of Definition 2.1.*

Theorem 2.1. *(Frobenius) Let $\mathcal{D} \subset Der\mathcal{A}_M$ be a direct subsheaf of rank (r,s). Then, $\mathcal{D}$ is involutive, if and only if for each $p \in M$ there exist a coordinate system $\{(y_i; \zeta_\mu) | 1 \leq i \leq m, 1 \leq \mu \leq n\}$, defined in a neighborhood $\mathcal{U} = (U, \mathcal{A}_U)$ of p, such that,*

$$\mathcal{D} = \left\langle \frac{\partial}{\partial y_i}; \frac{\partial}{\partial \zeta_\mu} \right\rangle, 1 \leq i \leq r, \quad 1 \leq \mu \leq s.$$

Definition 2.2. *Let $f : (M, \mathcal{A}_M) \to (M', \mathcal{A}_{M'})$ be a morphism of graded manifolds. Two graded vector fields $Z_1 \in Der\mathcal{A}_M$ and $Z_2 \in Der\mathcal{A}_{M'}$ are said to be f-related (and denoted by $Z_2 = f_*Z_1$) if for each $g \in \mathcal{A}_{M'}$ we have $f^*(Z_2 g) = Z_1(f^*g)$. Then Z_1 is called a f_*-projectable graded vector field and f_*Z_1 is also its projection by means of f. The graded vector field Z_1 will be called vertical derivation if it is f_*-projectable and $f_*Z_1 = 0$. In case f is an isomorphism, we have $f_*Z_1 = (f^*)^{-1} \circ Z_1 \circ f^*$ (see* [14]*, Def. 2.3).*

It is clear that if Z_1 and Z_2 are f-related, then we have $|Z_1| = |Z_2|$, where $|\cdot|$ denotes the degree of graded vector fields.

Lemma 2.2. *If Z_1, Z_2 are two f-related graded vector fields and W_1, W_2 are two f-related graded vector fields then so are $[Z_1, W_1]$ and $[Z_2, W_2]$.*

Proof. For each $g \in \mathcal{A}_{M'}$, we have $Z_1(f^*g) = f^*(Z_2(g))$ and $W_1(f^*g) = f^*(W_2(g))$, then

$$\begin{aligned}
[Z_1, W_1](f^*g) &= \left(Z_1W_1 - (-1)^{|Z_1||W_1|}W_1Z_1\right)(f^*g) \\
&= Z_1W_1(f^*g) - (-1)^{|Z_1||W_1|}W_1Z_1(f^*g) \\
&= Z_1\left(f^*(W_2g)\right) - (-1)^{|Z_1||W_1|}W_1\left(f^*(Z_2g)\right) \\
&= f^*\left(Z_2(W_2g)\right) - (-1)^{|Z_1||W_1|}f^*\left(W_2(Z_2g)\right) \\
&= f^*\left(Z_2W_2(g) - (-1)^{|Z_1||W_1|}W_2Z_2(g)\right) \\
&= f^*\left([Z_2, W_2](g)\right),
\end{aligned}$$

where the last equation comes from $|Z_1| = |Z_2|, |W_1| = |W_2|$. □

Theorem 2.2. *Let $f : \mathcal{M} = (M, \mathcal{A}_M) \to \mathcal{M}' = (M', \mathcal{A}_{M'})$ be an epimorphism of graded manifolds. If $\mathcal{D}$ is an involutive graded direct subsheaf of f_*-projectable graded vector fields on $\mathcal{M}$ then $\bar{\mathcal{D}} := f_*\mathcal{D}$ is involutive graded direct subsheaf of $Der\mathcal{A}_{M'}$.*

Proof. Let $Z_1, W_1 \in \mathcal{D}$ and $Z_2 = f_*Z_1, W_2 = f_*W_1 \in Der\mathcal{A}_{M'}$. Since $\mathcal{D}$ is involutive, $[Z_1, W_1] \in \mathcal{D}$. Clearly Z_1, Z_2 and W_1, W_2 are f-related, then $[Z_1, W_1]$ and $[Z_2, W_2]$ are f-related, i.e., for each $g \in \mathcal{A}_{M'}$, $f^*[Z_2, W_2](g) = [Z_1, W_1](f^*g) = f_*[Z_1, W_1]g = f^*(f_*[Z_1, W_1])g$. Thus the uniqueness property of Definition 2.2 implies that $f_*[Z_1, W_1] = [Z_2, W_2]$. Thus $[Z_2, W_2] \in \bar{\mathcal{D}}$, and therefore $\bar{\mathcal{D}}$ is involutive. □

3. Geometric structures on graded manifolds

Suppose that $\mathcal{M}$ is an $(m,n)-$dimensional graded manifold. Let $\mathcal{D}$ be an involutive direct subsheaf of $Der\mathcal{A}_M$ of rank (r,s) such that $2r \leq m$, $2s \leq n$ and Γ a homogeneous graded vector field on $\mathcal{M}$ such that $\Gamma \notin \mathcal{D}$. Set

$$\mathcal{C} := \mathcal{D} + [\Gamma, \mathcal{D}] = \{D_1 + [\Gamma, D_2] : D_1, D_2 \in \mathcal{D}\}.$$

For each $p \in M$, since $\mathcal{D}$ is involutive, according to Theorem 2.1, there is a coordinate neighborhood $\mathcal{U}$ of p and coordinates $(q_u, y_i; \theta_\alpha, \zeta_\mu)$, for $u = 1, 2, \cdots, m-r$, $i = 1, 2, \cdots, r$, $\quad \alpha = 1, 2, \cdots, n-s$ and $\mu = 1, 2, \cdots, s$ such that $\mathcal{D}|_{\mathcal{U}} = \langle \frac{\partial}{\partial y_i}; \frac{\partial}{\partial \zeta_\mu} \rangle$. In this coordinate system, the local representation of Γ is

$$\Gamma|_{\mathcal{U}} = \Gamma_u \frac{\partial}{\partial q_u} + \Gamma_i \frac{\partial}{\partial y_i} + \Gamma_\alpha \frac{\partial}{\partial \theta_\alpha} + \Gamma_\mu \frac{\partial}{\partial \zeta_\mu}, \tag{1}$$

where $\Gamma_u, \Gamma_i, \Gamma_\mu$, and Γ_α are smooth superfunctions on $\mathcal{U}$. We want to find the conditions under which two graded vector fields $[\Gamma|_{\mathcal{U}}, \frac{\partial}{\partial y_i}]$ and $[\Gamma|_{\mathcal{U}}, \frac{\partial}{\partial \zeta_\mu}]$ are linearly independent. The local representation of these graded vector fields are

$$\begin{aligned} [\Gamma|_{\mathcal{U}}, \frac{\partial}{\partial y_i}] &\equiv -\frac{\partial \Gamma_u}{\partial y_i}\frac{\partial}{\partial q_u} - \frac{\partial \Gamma_\alpha}{\partial y_i}\frac{\partial}{\partial \theta_\alpha} \qquad (mod\ \mathcal{D}), \\ [\Gamma|_{\mathcal{U}}, \frac{\partial}{\partial \zeta_\mu}] &\equiv -(-1)^{|\Gamma_u|}\frac{\partial \Gamma_u}{\partial \zeta_\mu}\frac{\partial}{\partial q_u} + (-1)^{|\Gamma_\alpha|}\frac{\partial \Gamma_\alpha}{\partial \zeta_\mu}\frac{\partial}{\partial \theta_\alpha} \qquad (mod\ \mathcal{D}). \end{aligned}$$

If the local coefficients of these graded vector fields are zero, then we have $[\Gamma, \mathcal{D}] \subset \mathcal{D}$, and they are dependent vector fields. Therefore we suppose that $[\Gamma, \mathcal{D}] \cap \mathcal{D} = \{0\}$, i.e., if $D \in \mathcal{D}$ and $[\Gamma, D] \in \mathcal{D}$ then $D = 0$, and in the next theorem we show that they are linearly independent.

Theorem 3.1. *Suppose that the graded vector field Γ on $\mathcal{M}$ is such that $[\Gamma, \mathcal{D}] \cap \mathcal{D} = \{0\}$.*
(1) If $|\Gamma| = 0$, $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have maximal ranks respectively r and s. Then $rank_p(\mathcal{C}) = (2r, 2s)$ and $\Xi := \{p \in M | rank_p(\mathcal{C}) = (2r, 2s)\}$ is open and dense.
(2) If $|\Gamma| = 1$, then

- *for $m = 2r, n = 2s$,*
  - *if $r \leq s$, both $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have the same maximal rank r,*
  - *if $r > s$ both $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have the same maximal rank s,*
- *for $m = 2r + 1, n = 2s$,*

  - *if* $r < s$, $[\Gamma, \mathcal{D}]_0$ *and* $[\Gamma, \mathcal{D}]_1$ *have maximal ranks respectively* $r+1$ *and* r,
  - *if* $r \geq s$ *both* $[\Gamma, \mathcal{D}]_0$ *and* $[\Gamma, \mathcal{D}]_1$ *have the same maximal rank* s,
- *for* $m = 2r, n = 2s+1$,
  - *if* $r \leq s$, *both* $[\Gamma, \mathcal{D}]_0$ *and* $[\Gamma, \mathcal{D}]_1$ *have the same maximal rank* r,
  - *if* $r > s$, $[\Gamma, \mathcal{D}]_0$ *and* $[\Gamma, \mathcal{D}]_1$ *have maximal ranks respectively* s *and* $s+1$,
- *for* $m = 2r+1, n = 2s+1$,
  - *if* $r = s$, *both* $[\Gamma, \mathcal{D}]_0$ *and* $[\Gamma, \mathcal{D}]_1$ *have the same maximal rank* r,
  - *if* $r < s$, $[\Gamma, \mathcal{D}]_0$ *and* $[\Gamma, \mathcal{D}]_1$ *have maximal ranks respectively* $r+1$ *and* r,
  - *if* $r > s$, $[\Gamma, \mathcal{D}]_0$ *and* $[\Gamma, \mathcal{D}]_1$ *have maximal ranks respectively* s *and* $s+1$.

In each of these cases, $rank_p(\mathcal{C}) = (r + rank[\Gamma, \mathcal{D}]_0, s + rank[\Gamma, \mathcal{D}]_1)$.

Proof. Let $p \in M$ and $\mathcal{U}$ a coordinate neighborhood of p with local coordinates $(q_u, y_i; \theta_\alpha, \zeta_\mu)$, as above. Let $D^i(p)[\Gamma, \frac{\partial}{\partial y_i}](p) + D^\mu(p)[\Gamma, \frac{\partial}{\partial \zeta_\mu}](p) = 0$. A simple computation will shows that we have

$$\begin{cases} D^i \frac{\partial \Gamma_u}{\partial y_i} + (-1)^{|\Gamma_u|} D^\mu \frac{\partial \Gamma_u}{\partial \zeta_\mu} \equiv 0 \quad (mod\ \mathcal{D}), \\ D^i \frac{\partial \Gamma_\alpha}{\partial y_i} - (-1)^{|\Gamma_\alpha|} D^\mu \frac{\partial \Gamma_\alpha}{\partial \zeta_\mu} \equiv 0 \quad (mod\ \mathcal{D}). \end{cases} \tag{2}$$

Let

$$J_\Gamma := \begin{pmatrix} \frac{\partial \Gamma_u}{\partial y_i} & \frac{\partial \Gamma_\alpha}{\partial y_i} \\ (-1)^{|\Gamma_u|} \frac{\partial \Gamma_u}{\partial \zeta_\mu} & -(-1)^{|\Gamma_\alpha|} \frac{\partial \Gamma_\alpha}{\partial \zeta_\mu} \end{pmatrix}.$$

If $|\Gamma| = 0$ we see that the matrices

$$\left(\frac{\partial \Gamma_u}{\partial y_i}\right)_{1 \leq i \leq r, 1 \leq u \leq m-r}, \left(\frac{\partial \Gamma_\alpha}{\partial \zeta_\mu}\right)_{1 \leq \mu \leq s, 1 \leq \alpha \leq n-s},$$

have maximal ranks respectively r and s. Let $rank_p J_\Gamma = (r, s)$. By permuting the $\Gamma_{u'}$ and $\Gamma_{\alpha'}$, we may therefore assume that the matrices

$$\left(\frac{\partial \Gamma_{u'}}{\partial y_i}\right)_{1 \leq i \leq r, 1 \leq u' \leq r}, \left(\frac{\partial \Gamma_{\alpha'}}{\partial \zeta_\mu}\right)_{1 \leq \mu \leq s, 1 \leq \alpha' \leq s},$$

are invertible at p. Then from (2) we conclude that $D^i(p) = 0 = D^\mu(p)$. This means that the graded vector fields $[\Gamma, \frac{\partial}{\partial y_i}]$ and $[\Gamma, \frac{\partial}{\partial \zeta_\mu}]$, $(1 \leq i \leq r$, and $1 \leq \mu \leq s)$ are linearly independent at p, thus $rank_p(\mathcal{C}) = (2r, 2s)$, i.e. $\Xi \neq \emptyset$. Now we show that

Ξ is an open subset of M as follows: Let p' be a point of M such that $rank_{p'}(\mathcal{C}) \neq (2r, 2s)$ then there exists a graded vector field $0 \neq D \in \mathcal{D}$ on a neighborhood of p such that $[\Gamma|_U, D] \in \mathcal{D}$, which is a contradiction, therefore Ξ is open and in the similar method Ξ is dense.

(2) Let $|\Gamma| = 1$. We may choose $m = 2r + 1, n = 2s + 1, \quad r < s$. Then the matrices

$$\left(\frac{\partial \Gamma_\alpha}{\partial y_i}\right)_{1\leq i\leq r, 1\leq \alpha\leq n-s}, \left(\frac{\partial \Gamma_u}{\partial \zeta_\mu}\right)_{1\leq \mu\leq s, 1\leq u\leq m-r},$$

are even and have maximal ranks respectively r and $r+1$. A computation similar to part (1) shows that the odd graded vector fields $[\Gamma, \frac{\partial}{\partial y_i}]$ and the even graded vector fields $[\Gamma, \frac{\partial}{\partial \zeta_\mu}], \quad (1 \leq i \leq r, \text{ and } 1 \leq \mu \leq r + 1)$ are linearly independent at p, so $[\Gamma, \mathcal{D}]_0$ and $[\Gamma, \mathcal{D}]_1$ have maximal ranks respectively $r + 1$ and r, and $rank_p(\mathcal{C}) = (2r + 1, s + r)$. Similarly, one may choose $m = 2r + 1, n = 2s + 1, \quad r > s$, etc. We will therefore have twelve types of possibilities for m, n, r and s. In each of these cases, the matrices $\left(\frac{\partial \Gamma_\alpha}{\partial y_i}\right), \left(\frac{\partial \Gamma_u}{\partial \zeta_\mu}\right)$ are even and have maximal ranks and we have a number of odd graded vector fields $[\Gamma, \frac{\partial}{\partial y_i}]$ and a number of the even graded vector fields $[\Gamma, \frac{\partial}{\partial \zeta_\mu}]$ which are linearly independent at p. □

Hereafter, unless otherwise stated, we will assume that the graded vector field Γ satisfies the conditions of Theorem 3.1, and $[\mathcal{C}, \mathcal{C}] \subset \mathcal{C}$. From the above theorem, we see that if $\{D_i, D_\mu\}$ is a local basis of $\mathcal{D}$ consisting of coordinate fileds $\partial/\partial y_i$ and $\partial/\partial \zeta_\mu$ of a local coordinate system $(q_u, y_i; \theta_\alpha, \zeta_\mu)$ and if we set $C_a = [\Gamma, D_a]$ (for $|D_a| = 0$) and $C_b = [\Gamma, D_b]$ (for $|D_b| = 1$), then $\{C_a, D_i, D_\mu, C_b\}$ is a local basis for $\mathcal{C}$, where $\{C_a, C_b\}$ are generators of $[\Gamma, \mathcal{D}]$. Thus $\mathcal{C}$ is a direct subsheaf of $Der\mathcal{A}_M$. Moreover, $C_a = [\Gamma, D_a]$ and $C_b = [\Gamma, D_b]$ are respectively odd and even graded vector fields whenever Γ is odd and $|D_a| = 0$, $|D_b| = 1$.

Now we consider a graded tensor field on $\mathcal{C}$ which can be extended to a tensor field on $\mathcal{M}$ as a nonlinear connection similar to the classical case. Consider the tensor field $J : \mathcal{C} \to \mathcal{C}$ of type (1,1) by

$$(3) \qquad J(X_a) = 0 \quad and \quad J(Y_a) = -X_a, \quad X_a \in \{D_i, D_\mu\}, Y_a = [\Gamma, X_a].$$

It is called pseudo almost tangent structure (see also [3, 6]). Clearly $J^2 = 0$ and $|J| = |\Gamma|$. If $|\Gamma| = 0$ then $Im\ J = Ker\ J = \mathcal{D}$. Since $\mathcal{D}$ is involutive, for any $D_1, D_2 \in \mathcal{D}$ we have $[D_1, D_2] \in \mathcal{D}$, then if $|\Gamma| = 0$, $N_J(D_1, D_2) = 0$ and $N_J(D_1, [\Gamma, D_2]) = 0$,

where N_J is the formal Nijenhuis torsion (see [6] for the definition). Also from the Jacobi identity, we have

$$\begin{aligned} N_J([\Gamma, D_1], [\Gamma, D_2]) &= [D_1, D_2] + J[D_1, [\Gamma, D_2]] + J[[\Gamma, D_1], D_2] \\ &= [D_1, D_2] + J[\Gamma, [D_1, D_2]] = [D_1, D_2] - [D_1, D_2] = 0. \end{aligned}$$

Let $\Pi : \mathcal{C} \to \mathcal{C}$ be a graded tensor field of type $(1, 1)$, satisfying

$$\Pi \circ J = -J, \ J \circ \Pi = J. \tag{4}$$

A simple computation shows that, there are superfunctions π_{ab} belong to $\mathcal{A}_M$ such that

$$\Pi(X_a) = -X_a, \quad \Pi(Y_a) = Y_a - 2\pi_{ab}X_b,$$

where "-2" is taken for convenience. Now, we can associate to Π two canonical operators h and v given by

$$h = \frac{1}{2}(Id + \Pi), \ v = \frac{1}{2}(Id - \Pi).$$

We deduce that h and v are complementary projections and they are given by

$$h(X_a) = 0, \quad v(X_a) = X_a, \quad h(Y_a) = Y_a - \pi_{ab}X_b, \quad v(Y_a) = \pi_{ab}X_b.$$

If we set $\mathcal{H} = Im\ h$ and $\mathcal{D} = Im\ v$ then $\mathcal{H}$ and $\mathcal{D}$ are complementary direct subsheavs and

$$\mathcal{C} = \mathcal{H} \oplus \mathcal{D}. \tag{5}$$

$\mathcal{H}$ is a subsheaf of $Der\mathcal{A}_M$ and the sections of $\mathcal{H}$ will be called horizontal graded vector fields. If $\{D_i, D_\mu\}$ is a local basis of $\mathcal{D}$ and $\{C_a, D_i, D_\mu, C_b\}$ is a local basis of $\mathcal{C}$ then $\{h(C_a), h(C_b)\}$ is a local basis for $\mathcal{H}$.

Lemma 3.1. *Let* $[X_a, X_b] = \lambda^c_{ab}X_c$, $[X_a, Y_b] = \alpha^c_{ab}X_c$ *and* $[Y_a, Y_b] = \epsilon^c_{ab}X_c + \xi^c_{ab}Y_c$, *where* $X_a, X_b, X_c \in \{D_i, D_\mu\}$ *and* $Y_a, Y_b, Y_c \in \{C_i, C_\mu\}$. *From (5), a necessary and sufficient condition for involutivity of* $\mathcal{H}$ *is that*

$$\begin{aligned} \Phi^a_{bc} &:= \xi^d_{ab}\pi_{dc} + \epsilon^c_{ab} - Y_a(\pi_{bc}) - (-1)^{|Y_a||Y_b|}\pi_{bd}\alpha^c_{da} - \pi_{ad}\alpha^c_{db} \\ &\quad + (-1)^{|Y_b||Y_a|}Y_b(\pi_{ac}) + (-1)^{|X_e|(|Y_b|+|X_d|)}\pi_{ae}\pi_{bd}\lambda^c_{ed} + \pi_{ad}X_d(\pi_{bc}) \\ &\quad - (-1)^{(|Y_a|+|X_c|)(|Y_b|+|X_d|)}\pi_{bd}X_d(\pi_{ac}) \\ &= 0. \end{aligned}$$

Theorem 3.2. *A necessary and sufficient condition for involutivity of $\mathcal{H}$ is that $N_\Pi = 0$, where N_Π is the Nijenhuis tensor associated to Π.*

Proof. Clearly, $\Pi^2 = Id$, $\Pi(\mathcal{H}) = \mathcal{H}$ and

$$N_\Pi(\mathcal{D}, \mathcal{D}) = N_\Pi(\mathcal{D}, \mathcal{H}) = N_\Pi(\mathcal{H}, \mathcal{D}) = 0,$$

therefore it suffices to show the theorem only for horizontal graded vector fields. Suppose $H_a = Y_a - \pi_{ac}X_c$ and $H'_b = Y_b - \pi_{bd}X_d$ are elements of $\mathcal{H}$, where $X_\bullet \in \{D_i, D_\mu\}$ and $Y_\bullet \in \{C_i, C_\mu\}$. Then

$$N_\Pi(H_a, H'_b) = 2[H_a, H'_b] - 2\Pi[H_a, H'_b].$$

If $\mathcal{H}$ is an involutive graded direct subsheaf, then $\Pi[H_a, H'_b] = [H_a, H'_b]$, and hence $N_\Pi(H_a, H'_b) = 0$.

Conversely, since $0 = N_\Pi(H_a, H'_b) = 2[H_a, H'_b] - 2\Pi[H_a, H'_b] = 4\Phi^c_{ab}X_c$ then the proof is finished in view of the Lemma 3.1. □

Given a graded vector field Γ such that $[\Gamma, \mathcal{C}] \subset \mathcal{C}$ and consider the morphism $-\mathcal{L}_\Gamma J$. For any $C \in \mathcal{C}$ and $D \in \mathcal{D}$, we have

$$(\mathcal{L}_\Gamma J)(C) := [\Gamma, J(C)] - (-1)^{|J||\Gamma|}J[\Gamma, C],$$

and

$$(-\mathcal{L}_\Gamma J)(D) = -[\Gamma, J(D)] + (-1)^{|J||\Gamma|}J[\Gamma, D] = -(-1)^{|J||\Gamma|}D,$$

thus $(-\mathcal{L}_\Gamma J)^2(D) = D$. Also,

$$\begin{aligned}(-\mathcal{L}_\Gamma J)([\Gamma, D]) &= -[\Gamma, J([\Gamma, D])] + (-1)^{|J||\Gamma|}J[\Gamma, [\Gamma, D]] \\ &= [\Gamma, D] + (-1)^{|J||\Gamma|}J[\Gamma, [\Gamma, D]].\end{aligned}$$

Since $J[\Gamma, [\Gamma, D]] \in \mathcal{D}$, we have $(-\mathcal{L}_\Gamma J)(J[\Gamma, [\Gamma, D]]) = -(-1)^{|J||\Gamma|}J[\Gamma, [\Gamma, D]]$, and therefore

$$(-\mathcal{L}_\Gamma J)^2\Big([\Gamma, D]\Big) = \Big([\Gamma, D] + ((-1)^{|J||\Gamma|} - 1)J\Big[\Gamma, [\Gamma, D]\Big]\Big).$$

In this case, for $h = \frac{1}{2}(Id - \mathcal{L}_\Gamma J)$ and $v = \frac{1}{2}(Id + \mathcal{L}_\Gamma J)$, we have

$$h(D) = 0, \quad v(D) = D, \quad h(C) = C + \frac{1}{2}J[\Gamma, C], \quad v(C) = -\frac{1}{2}J[\Gamma, C].$$

If $|\Gamma| = 0$, it is clear that $(\mathcal{L}_\Gamma J)^2 = Id$. We have shown the following.

Proposition 3.1. *If $[\Gamma, \mathcal{C}] \subset \mathcal{C}$, the graded vector field Γ defines the tensor field $-\mathcal{L}_\Gamma J$ such that the associated horizontal graded subsheaf $\mathcal{H}$ is a complement of $\mathcal{D}$ in $\mathcal{C}$.*

Definition 3.1. *For any $D \in \mathcal{D}$, denote by D^H the unique element of $\mathcal{H}$ such that $J(D^H) = D$.*

Lemma 3.2. *Let $\Pi = -\mathcal{L}_\Gamma J$ and $|\Gamma| = 0$. For any $X_a \in \{D_i, D_\mu\}$, we have $X_a^H = -C_a - \frac{1}{2}J[\Gamma, C_a]$ and $h(X_a^H) = X_a^H$ and $v(X_a^H) = 0$, where $C_a = [\Gamma, X_a]$.*

Let $\mathcal{F}$ be the set of projectable superfunctions of $\mathcal{A}_M$, i.e. $\mathcal{F} = \{f \in \mathcal{A}_M : \forall D \in \mathcal{D}, D(f) = 0\}$, and $\widetilde{D} := \big\{D \in \mathcal{D} : [D^H, \mathcal{D}] \subset \mathcal{D}\big\}$, i.e. the set of $D \in \mathcal{D}$ for which the corresponding D^H is projectable. Then for each $f \in \mathcal{F}$ and $D \in \widetilde{D}$ since $fD \in \widetilde{D}$ so $\widetilde{D}$ is a $\mathcal{F}$-module. We can now define a $\mathcal{C}$-connection on $\mathcal{D}$ as follows:

Definition 3.2. *Let J be an even morphism with the properties (3). For any graded vector fields $C \in \mathcal{C}$ and $D \in \mathcal{D}$, set*

$$\nabla_C D := v\Big([h(C), D]\Big) + J\Big([v(C), D^H]\Big). \tag{6}$$

It is a covariant derivative, because for $f \in \mathcal{A}_M$

$$\begin{aligned}
\nabla_{fC} D &= v\Big(f[h(C), D] - (-1)^{|fh(C)||D|} D(f) h(C)\Big) \\
&\quad + J\Big(f[v(C), D^H] - (-1)^{|fv(C)||D^H|} D^H(f) v(C)\Big) \\
&= f\Big(v[h(C), D] + (-1)^{|f||J|} J[v(C), D^H]\Big) \\
&\quad - (-1)^{|fh(C)||D|} v\Big(D(f) h(C)\Big) - (-1)^{|fv(C)||D^H|} J\Big(D^H(f) v(C)\Big) \\
&= f\Big(v[h(C), D] + (-1)^{|f||J|} J[v(C), D^H]\Big) \\
&= f \nabla_C D,
\end{aligned}$$

and

$$\begin{aligned}
\nabla_C(fD) &= v\Big([h(C), fD]\Big) + J\Big([v(C), (-1)^{|f||\Gamma|} fD^H]\Big) \\
&= v\Big((-1)^{|f||C|} f[h(C), D] + h(C)(f)D\Big) \\
&\quad +(-1)^{|f||\Gamma|} J\Big((-1)^{|f||C|} f[v(C), D^H] + v(C)(f)D^H\Big) \\
&= (-1)^{|f||C|} f\nabla_C D + v\Big(h(C)(f)D\Big) + (-1)^{|C||\Gamma|} J\Big(v(C)(f)D^H\Big) \\
&= (-1)^{|f||C|} f\nabla_C D + \Big(h(C) + (-1)^{|C||\Gamma|} v(C)\Big)(f)D \\
&= (-1)^{|f||C|} f\nabla_C D + C(f)D.
\end{aligned}$$

Lemma 3.3. *$D_2 \in \widetilde{D}$ if and only if $\nabla_{D_1} D_2 = 0$, for all $D_1 \in \mathcal{D}$.*

Proof. First, let $D_2 \in \widetilde{D}$ and $D_1 \in \mathcal{D}$, since $h(D_1) = 0$ we have

$$\nabla_{D_1} D_2 = v\Big([h(D_1), D_2]\Big) + J\Big([v(D_1), D_2^H]\Big) = J\Big([D_1, D_2^H]\Big) = 0.$$

Conversely, let $D_1, D_2 \in \mathcal{D}$ and $\nabla_{D_1} D_2 = 0$, the first term of $\nabla_{D_1} D_2$ is zero then $J\Big([D_1, D_2^H]\Big) = 0$, and from the identity *ker* $J = \mathcal{D}$, we have $[D_1, D_2^H] \in \mathcal{D}$. Thus $D_2 \in \widetilde{D}$. □

Now let $X = A_a X_a$ be an element of $\mathcal{D}$. The necessary and sufficient condition for $X \in \widetilde{D}$ is that, for every $X' = B_b X_b$ belongs to $\mathcal{D}$, $B_b(X_b(A_c) - (-1)^{|X_b|(|X|+|\Gamma|)} A_a \beta_{ba}^c) = 0$, where β_{ab}^c are coefficients of $Y_\bullet$ in $[X_a, Y_b]$, and we will explain its further properties in the next remark.

It is clear that for each $D_1, D_2 \in \mathcal{D}$ and $D_3 \in \widetilde{\mathcal{D}}$, ∇ has vanishing graded curvature

$$\begin{aligned}
R(D_1, D_2, D_3) &= \nabla_{D_1}\nabla_{D_2} D_3 - (-1)^{|D_1||D_2|}\nabla_{D_2}\nabla_{D_1} D_3 - \nabla_{[D_1, D_2]} D_3 \\
&= 0.
\end{aligned}$$

The covariant derivative ∇, when restricted to the odd part of $\mathcal{D}$, has the torsion tensor given by

$$\begin{aligned} T(D_1,D_2) &= \nabla_{D_1}D_2-(-1)^{|D_1||D_2|}\nabla_{D_2}D_1-[D_1,D_2] \\ &= J\Big(\Big[D_1,-C_2-\frac{1}{2}J[\Gamma,C_2]\Big]\Big)+J\Big(\Big[D_2,-C_1-\frac{1}{2}J[\Gamma,C_1]\Big]\Big) \\ &\quad -[J(C_1),J(C_2)] \\ &= J[J(C_1),C_2]+J[J(C_2),C_1]-[J(C_1),J(C_2)] \\ &= -N_J(C_1,C_2), \end{aligned}$$

where $C_1,C_2\in\mathcal{C}_{odd}$. Then the torsion vanishes if and only if the formal graded Nijenhuis N_J of J vanishes. Thus if J has vanishing Nijenhuis torsion, and we restrict our attention to these special cases, then we have a flat connection.

Let $D_1,D_2\in\widetilde{D}$, from the definition of ∇ we define the operator $\overline{\nabla}$ as follows:

$$\overline{\nabla}_{D_1}D_2:=\nabla_{D_1^H}D_2. \tag{7}$$

Suppose that $D_3\in\mathcal{D}$, and consider the tensor field of type (2,1) on $\mathcal{D}$ as follows:

$$\delta(D_1,D_3,D_2):=\nabla_{D_1^H}\nabla_{D_3}D_2-(-1)^{|D_1||D_3|}\nabla_{D_3}\nabla_{D_1^H}D_2-\nabla_{[D_1^H,D_3]}D_2. \tag{8}$$

Then $\overline{\nabla}_{D_1}D_2=\nabla_{D_1^H}D_2\in\widetilde{D}$ if and only if $\delta=0$: Since $D_1,D_2\in\widetilde{D}$, from $[D_1^H,D_3]\in\mathcal{D}$ and Lemma 3.3, the last term of (8) is zero and the same holds for the first term. Thus for each $D_3\in D,\nabla_{D_3}\nabla_{D_1^H}D_2=0$, this means that $\bar{\nabla}_{D_1}D_2\in\tilde{D}$.

Also if f is a projectable superfunction then

$$\begin{aligned} \overline{\nabla}_{fD_1}D_2 &= \nabla_{(fD_1)^H}D_2=v\Big(\big[h\big((fD_1)^H\big),D_2\big]\Big)+J\Big(\big[v\big((fD_1)^H\big),D_2^H\big]\Big) \\ &= v\Big(\big[(fD_1)^H,D_2\big]\Big) \\ &= (-1)^{|\Gamma||f|}v\Big(f\big[D_1^H,D_2\big]\Big)-(-1)^{|D_2||fD_1^H|}D_2(f)D_1^H \\ &= (-1)^{|\Gamma||f|}fv[D_1^H,D_2]=(-1)^{|\Gamma||f|}fv[h(D_1^H),D_2] \\ &= (-1)^{|\Gamma||f|}f\overline{\nabla}_{D_1}D_2 \end{aligned}$$

and

$$\begin{aligned} \overline{\nabla}_{D_1}fD_2 &= v\Big([h(D_1^H),fD_2]\Big)+J\Big([v(D_1^H),(fD_2)^H]\Big)=v\Big([D_1^H,fD_2]\Big) \\ &= (-1)^{(|\Gamma|+|D_1|)|f|}f\overline{\nabla}_{D_1}D_2+D_1^H(f)D_2. \end{aligned}$$

Since $D_2\left(D_1^H(f)\right) = [D_2, D_1^H](f)+(-1)^{|D_1^H||D_2|}D_1^H D_2(f) = 0$ and f is a projectable superfunction, $D_1^H(f)$ is a projectable superfunction and $\overline{\nabla}_{D_1} f D_2 \in \widetilde{D}$. Thus for even graded vector field Γ, $\overline{\nabla}$ satisfies connection properties.

4. Local representation of Γ

As we saw earlier, for a pair $(\Gamma, \mathcal{D})$, where $\mathcal{D}$ is a direct subsheaf of $Der\mathcal{A}_M$ and Γ is a graded vector field on $\mathcal{M}$, we have a subsheaf $\mathcal{C} := \mathcal{D} + [\Gamma, \mathcal{D}]$ such that if $[\Gamma, \mathcal{C}] \subset \mathcal{C}$, the tensor field $\Pi = -\mathcal{L}_\Gamma J$ defines the graded subsheaf $\mathcal{H}$ which is a complement of $\mathcal{D}$ in $\mathcal{C}$ and $\Pi(\mathcal{H}) = \mathcal{H}$. The technical importance of the pair $(\Gamma, \mathcal{D})$, is in the next results, where we find local supercoordinates on $\mathcal{M}$, needed for understanding of the local expression of the graded vector field Γ.

Remark 4.1. *Let $\mathcal{D} \subset Der\mathcal{A}_M$ be an involutive direct subsheaf with even and odd generators $\{D_i, D_\mu\}$, $(1 \leq i \leq r,$ and $1 \leq \mu \leq s)$ such that $[X_a, X_b] = 0, \forall X_a, X_b \in \{D_i, D_\mu\}$. We want to find the conditions under which for all a, b, $[X_a, Y_b] \in \mathcal{D}$, where $Y_a = [\Gamma, X_a]$.*

If we change the above basis to

$$\widehat{D}_i = A_{ij}D_j + B_{i\nu}D_\nu, \quad \widehat{D}_\mu = E_{\mu j}D_j + F_{\mu\nu}D_\nu,$$

where $A_{\bullet\bullet}, B_{\bullet\bullet}, E_{\bullet\bullet}$ and $F_{\bullet\bullet}$ are superfunctions and

$$G = (G_{ab}) = \begin{pmatrix} A_{ij} & B_{i\nu} \\ E_{\mu j} & F_{\mu\nu} \end{pmatrix},$$

and if we again assume that $\hat{X}_a, \hat{X}_b \in \{\hat{D}_i, \hat{D}_\mu\}$, then the necessary and sufficient conditions for $[\widehat{X}_a, \widehat{X}_b] = 0$, is that

$$\forall d, \quad G_{ac}X_c(G_{bd}) = (-1)^{|G_{bc}X_c||G_{ad}X_d|}G_{bc}X_c(G_{ad}), \tag{9}$$

(we use the Einstein convention, that is, repeated indices denotes summation over their range). For example, $[\widehat{D}_i, \widehat{D}_\mu] = 0$ if and only if

(10)

$$\begin{cases} A_{ij}D_j(E_{\mu k}) + B_{i\nu}D_\nu(E_{\mu k}) &= E_{\mu j}D_j(A_{ik}) + F_{\mu\nu}D_\nu(A_{ik}), \\ A_{ij}D_j(F_{\mu\omega}) + B_{i\nu}D_\nu(F_{\mu\omega}) &= E_{\mu j}D_j(B_{i\omega}) + F_{\mu\nu}D_\nu(B_{i\omega}). \end{cases}$$

Also C_i and C_μ change to

$$\begin{aligned}\widehat{C}_i &= \Gamma(A_{ij})D_j + \Gamma(B_{i\nu})D_\nu + A_{ij}C_j + (-1)^{|\Gamma|}B_{i\nu}C_\nu,\\ \widehat{C}_\mu &= \Gamma(E_{\mu j})D_j + \Gamma(F_{\mu\nu})D_\nu + (-1)^{|\Gamma|}E_{\mu j}C_j + F_{\mu\nu}C_\nu.\end{aligned}$$

On the other hand, $\mathcal{C}$ is involutive, then there are superfunctions $\alpha^c_{ab}, \beta^c_{ab} \in \mathcal{A}_M$ such that

$$[X_a, Y_b] = \alpha^c_{ab}X_c + \beta^c_{ab}Y_c, \quad X_a, X_c \in \{D_i, D_\mu\}, \quad Y_b, Y_c \in \{C_i, C_\mu\}. \tag{11}$$

Therefore $[\Gamma, [X_a, X_b]] = 0$. From the graded Jacobi identity, we have

$$0 = [\Gamma, [X_a, X_b]] = -(-1)^{(|\Gamma|+|X_a|)|X_b|}[X_b, Y_a] + (-1)^{|\Gamma||X_a|}[X_a, Y_b],$$

and the superfunctions $\alpha^c_{ab}, \beta^c_{ab}$ are:

- *symmetric for lower case Latin indexes,*
- *symmetric up to $(-1)^{|\Gamma|}$, i.e., $\beta^c_{ab} = (-1)^{|\Gamma|}\beta^c_{ba}$, if one of the lower indices is Greek,*
- *antisymmetric for lower case Greek indexes.*

If we change the basis of $\mathcal{D}$ to $\widehat{X}_a = G_{ab}X_b$, then we have

$$\begin{aligned}[\widehat{X}_a, \widehat{Y}_c] &= \Big[G_{ab}X_b, \Gamma(G_{cd})X_d + (-1)^{|\Gamma||G_{cd}|}G_{cd}Y_d\Big]\\ &\equiv \left\{(-1)^{|\Gamma||G_{cd}|}G_{ab}X_b(G_{cd}) + (-1)^{|G_{ce}|(|\Gamma|+|X_b|)}G_{ab}G_{ce}\beta^d_{be}\right\}Y_d \quad (mod\ \mathcal{D}).\end{aligned}$$

Thus $[\widehat{X}_a, \widehat{Y}_c] \in \mathcal{D}$ if and only if

$$(-1)^{|\Gamma||G_{cd}|}G_{ab}X_b(G_{cd}) + (-1)^{|G_{ce}|(|\Gamma|+|X_b|)}G_{ab}G_{ce}\beta^d_{be} = 0,$$

for each d.

Theorem 4.1. *Let $\mathcal{D}$ and $\mathcal{C}$ be involutive, as above. For any even graded vector field Γ on $\mathcal{M}$, we can find local supercoordinates $(t_l, x_i, y_i; \tau_\rho, \eta_\mu, \zeta_\mu)$ on $\mathcal{M}$, $l = 1, \cdots, m-2r$, $i = 1, \cdots, r, \rho = 1, \cdots, n-2s$, $\mu = 1, \cdots, s$, such that*

$$D_i = \frac{\partial}{\partial y_i}, \qquad C_i \equiv -\frac{\partial}{\partial x_i} \quad (mod\ \mathcal{D}), \tag{12}$$

$$D_\mu = \frac{\partial}{\partial \zeta_\mu}, \qquad C_\mu \equiv -\frac{\partial}{\partial \eta_\mu} \quad (mod\ \mathcal{D}). \tag{13}$$

Proof. By the Frobenius theorem, we take a coordinate neighborhood $\mathcal{U}$ of $p \in \mathcal{M}$, with supercoordinates $(q_u, y_i; \theta_\alpha, \zeta_\mu), u = 1, \cdots, m-r, i = 1, \cdots, r, \alpha = 1, \cdots, n-s$, $\mu = 1, \cdots, s$, such that $D_i = \partial/\partial y_i$ and $D_\mu = \partial/\partial\zeta_\mu$. Let $\mathcal{U}$ be the image of a product of open subsets $\mathcal{U}_1 \subset \mathcal{R}^{m-r|n-s}$ and $\mathcal{U}_2 \subset \mathcal{R}^{r|s}$, where $0 \in \mathcal{U}_2$ (c.f. [18]). Then $y_i = 0, \zeta_\mu = 0$ define a graded submanifold (in the sense of 3.2.1 of [8]) $\mathcal{N} = (N, \mathcal{A}_N)$ of $\mathcal{U}$ of graded dimension $(m-r, n-s)$. Denote by $pr : \mathcal{U} \to \mathcal{N}$ the corresponding projection morphism, then $pr_*(D_i) = pr_*(D_\mu) = 0$. It is clear that the restrictions of C_i and C_μ to $\mathcal{U}$ are pr_*-projectable to $\mathcal{N}$. We denote pr_*-projections of C_j and C_ν by graded vector fields $\bar{C}_j$ and $\bar{C}_\nu$ on $\mathcal{N}$ respectively. Let us denote by $\bar{\mathcal{C}}$ the graded direct subsheaf on $\mathcal{N}$ spanned by $\bar{C}_i$ and $\bar{C}_\mu$, i.e., $\bar{\mathcal{C}} = \langle \bar{C}_i; \bar{C}_\mu \rangle$. Since $\mathcal{C}$ is involutive, $\bar{\mathcal{C}}$ is an involutive direct subsheaf of rank (r, s), by Theorem 2.2. Now again repeat the Theorem 3.1 for the graded manifold $\mathcal{N}$, the graded vector field Γ, and graded direct subsheaf $\bar{\mathcal{C}}$ of rank (r, s), then we may choose supercoordinates $(t_l, x_i; \tau_\rho, \eta_\mu)$ on $\mathcal{N}$, where $l = 1, \cdots, m-2r, \rho = 1, \cdots, n-2s$, such that the rank of supermatrix $\bar{J}_\Gamma$ in this case is (r, s).Then with respect to the supercoordinates $(t_l, x_i, y_i; \tau_\rho, \eta_\mu, \zeta_\mu)$ on $\mathcal{U}$ we have

$$D_i = \frac{\partial}{\partial y_i}, \qquad C_i \equiv P_{ij}(x_i; \eta_\mu)\frac{\partial}{\partial x_j} + Q_{i\nu}(x_i; \eta_\mu)\frac{\partial}{\partial \eta_\nu} \qquad (mod\ \mathcal{D}),$$
$$D_\mu = \frac{\partial}{\partial \zeta_\mu}, \qquad C_\mu \equiv R_{\mu j}(x_i; \eta_\mu)\frac{\partial}{\partial x_j} + S_{\mu\nu}(x_i; \eta_\mu)\frac{\partial}{\partial \eta_\nu} \qquad (mod\ \mathcal{D}),$$

where the coefficients $P_{ij}, Q_{i\nu}, R_{\mu j}$, and $S_{\mu\nu}$ are the components of a nonsingular supermatrix. Let

$$\begin{pmatrix} A_{ij} & B_{i\nu} \\ E_{\mu j} & F_{\mu\nu} \end{pmatrix}$$

be its inverse. If we set

$$\widehat{D}_i = A_{ij}(t, x; \tau, \eta)D_j + B_{i\nu}(t, x; \tau, \eta)D_\nu,$$
$$\widehat{D}_\mu = E_{\mu j}(t, x; \tau, \eta)D_j + F_{\mu\nu}(t, x; \tau, \eta)D_\nu,$$

then

$$\widehat{C}_i = [\Gamma, \widehat{D}_i] = \Gamma(A_{ij})D_j + (-1)^{|\Gamma|}\Gamma(B_{i\nu})D_\nu + A_{ij}C_j + B_{i\nu}C_\nu,$$
$$\widehat{C}_\mu = [\Gamma, \widehat{D}_\mu] = (-1)^{|\Gamma|}\Gamma(E_{\mu j})D_j + \Gamma(F_{\mu\nu})D_\nu + E_{\mu j}C_j + F_{\mu\nu}C_\nu.$$

A simple computation shows that

$$\widehat{D}_i = A_{ij}\frac{\partial}{\partial y_j} + B_{i\nu}\frac{\partial}{\partial \zeta_\nu}, \qquad \widehat{C}_i \equiv \frac{\partial}{\partial x_i} \qquad (mod\ \mathcal{D}),$$
$$\widehat{D}_\mu = E_{\mu j}\frac{\partial}{\partial y_j} + F_{\mu\nu}\frac{\partial}{\partial \zeta_\nu}, \qquad \widehat{C}_\mu \equiv \frac{\partial}{\partial \eta_\mu} \qquad (mod\ \mathcal{D}).$$

Therefore, a new change of the coordinates

$$\widehat{t}_l = t_l, \quad \widehat{y}_i = P_{ij}(x_i;\eta_\mu)y_j + Q_{i\nu}(x_i;\eta_\mu)\zeta_\nu, \quad \widehat{x}_i = -x_i,$$
$$\widehat{\tau}_\rho = \tau_\rho, \quad \widehat{\zeta}_\mu = R_{\mu j}(x_i;\eta_\mu)y_j + S_{\mu\nu}(x_i;\eta_\mu)\zeta_\nu, \quad \widehat{\eta}_\mu = -\eta_\mu,$$

may be performed to bring the local basis of $Der\mathcal{A}_M$ into the form

$$\frac{\partial}{\partial \widehat{t}_l} \equiv \frac{\partial}{\partial t_l} \quad (mod\ \mathcal{D}), \qquad \frac{\partial}{\partial \widehat{x}_i} \equiv -\frac{\partial}{\partial x_i} \quad (mod\ \mathcal{D}), \qquad \frac{\partial}{\partial \widehat{y}_i} = A_{ij}\frac{\partial}{\partial y_j} + E_{i\nu}\frac{\partial}{\partial \zeta_\nu},$$
$$\frac{\partial}{\partial \widehat{\tau}_\rho} \equiv \frac{\partial}{\partial \tau_\rho} \quad (mod\ \mathcal{D}), \qquad \frac{\partial}{\partial \widehat{\eta}_\mu} \equiv -\frac{\partial}{\partial \eta_\mu} \quad (mod\ \mathcal{D}), \qquad \frac{\partial}{\partial \widehat{\zeta}_\mu} = B_{\mu j}\frac{\partial}{\partial y_j} + F_{\mu\nu}\frac{\partial}{\partial \zeta_\nu},$$

and this completes the proof. □

Theorem 4.2. *Let $\mathcal{D}$ and $\mathcal{C}$ be involutive and assume that $|\Gamma| = 1$. We can find local supercoordinates $(t_l, x_a, y_i; \tau_\rho, \eta_b, \zeta_\mu)$ on $\mathcal{M}$, $l = 1, \cdots, l_1$, $a = 1, \cdots, a_1$, $i = 1, \cdots, r, \rho = 1, \cdots, \rho_1$, $b = 1, \cdots, b_1$, $\mu = 1, \cdots, s$, such that*

$$D_i = \frac{\partial}{\partial y_i}, \qquad C_b = [\Gamma, \frac{\partial}{\partial y_b}] \equiv -\frac{\partial}{\partial \eta_b} \quad (mod\ \mathcal{D}), \tag{14}$$

$$D_\mu = \frac{\partial}{\partial \zeta_\mu}, \qquad C_a = [\Gamma, \frac{\partial}{\partial \zeta_a}] \equiv \frac{\partial}{\partial x_a} \quad (mod\ \mathcal{D}), \tag{15}$$

and l_1, a_1, b_1 and ρ_1 are given as in the following table (Table 1):

$dim\mathcal{M} = (m, n)$	$r = s$	$r < s$	$r > s$
$(2r, 2s)$	$l_1 = 0, a_1 = r,$ $b_1 = s, \rho_1 = 0$	$l_1 = 0, a_1 = r,$ $b_1 = r, \rho_1 = s - r$	$l_1 = r - s, a_1 = s,$ $b_1 = s, \rho_1 = 0$
$(2r+1, 2s)$	$l_1 = 1, a_1 = s,$ $b_1 = s, \rho_1 = 0$	$l_1 = 0, a_1 = r+1,$ $b_1 = r, \rho_1 = s - r$	$l_1 = r + 1 - s, a_1 = s,$ $b_1 = s, \rho_1 = 0$
$(2r, 2s+1)$	$l_1 = 0, a_1 = s,$ $b_1 = s, \rho_1 = 1$	$l_1 = 0, a_1 = r,$ $b_1 = r, \rho_1 = s + 1 - r$	$l_1 = r - s, a_1 = s,$ $b_1 = s + 1, \rho_1 = 0$
$(2r+1, 2s+1)$	$l_1 = 1, a_1 = s,$ $b_1 = s, \rho_1 = 1$	$l_1 = 0, a_1 = r+1,$ $b_1 = r, \rho_1 = s + 1 - r$	$l_1 = r + 1 - s, a_1 = s,$ $b_1 = s + 1, \rho_1 = 0$

TABLE 1. The range of of indices l_1, a_1, b_1 and ρ_1

Proof. We consider the result in Theorem 4.1 and apply it to the case that $|\Gamma| = 1$. There is a coordinate neighborhood $\mathcal{U}$ of $p \in \mathcal{M}$, with supercoordinates $(q_u, y_i; \theta_\alpha, \zeta_\mu)$, $u = 1, \cdots, m-r$, $i = 1, \cdots, r, \alpha = 1, \cdots, n-s$, $\mu = 1, \cdots, s$, a graded submanifold $\mathcal{N} = (N, \mathcal{A}_N)$ of $\mathcal{U}$ of graded dimension $(m-r, n-s)$ and the corresponding projection morphism $pr : \mathcal{U} \to \mathcal{N}$, such that $pr_*(D_i) = pr_*(D_\mu) = 0$.

As we have seen in Theorem 3.1, we have twelve types of possibilities for m, n, r and s. We may choose $m = 2r+1, n = 2s+1, \quad r < s$ to prove the theorem and a similar proof can also be performed in other cases.

Since $rank_p(\mathcal{C}) = (2r+1, s+r)$, we assume that

$$\mathcal{C} = \left\langle C_a, D_i, D_\mu, C_b \middle| \begin{array}{c} a=1,\cdots,r+1 \\ b=1,\cdots,r \end{array} \text{ and } \begin{array}{c} C_a=[\Gamma, \frac{\partial}{\partial \zeta_a}] \\ C_b=[\Gamma, \frac{\partial}{\partial y_b}] \end{array} \text{ and } \begin{array}{c} |C_a|=0 \\ |C_b|=1 \end{array} \right\rangle$$

Then $\bar{\mathcal{C}} = \langle \bar{C}_a; \bar{C}_b \rangle$ is an involutive direct subsheaf of $Der\mathcal{A}_M$ and $rank_p\bar{\mathcal{C}} = (r+1, r)$, where $\bar{C}_a$ and $\bar{C}_b$ are pr_*-projections of C_a and C_b on $\mathcal{N}$ respectively. We can continue the method discussed in Theorem 3.1, but there is another way to find the local generators of $\mathcal{C}$. By the Frobenius theorem, we take a local coordinate $(x'_a; \tau'_\rho, \eta'_b), \rho = 1, ..., s+1-r$, on $\mathcal{N}$ of $\tilde{pr}(p)$, such that

$$\bar{C}_a = \frac{\partial}{\partial x'_a}, \bar{C}_b = \frac{\partial}{\partial \eta'_b}.$$

Then there exists a coordinate neighborhood $\mathcal{U}'$ of $p \in \mathcal{M}$, with local coordinates

$$\left\{ pr^*x'_a, y'_i; pr^*\tau'_\rho, pr^*\eta'_b, \zeta'_\alpha) \middle| \begin{array}{c} a=1,\cdots,r+1 \\ b=1,\cdots,r \end{array} \text{ and } i = 1, ..., r \text{ and } \begin{array}{c} \alpha=1,...,s \\ \rho=1,...,s+1-r \end{array} \right\}$$

such that $D_i = \frac{\partial}{\partial y'_i}, \quad D_\alpha = \frac{\partial}{\partial \zeta'_\alpha}, C_a \equiv \frac{\partial}{\partial pr^* x'_a}, C_b \equiv \frac{\partial}{\partial pr^* \eta'_b} (mod\, \mathcal{D})$. We shall write these coordinates as $\{x_a, y_i; \tau_\rho, \eta_b, \zeta_\alpha\}$. Thus, a new change of the coordinates $\{x_a, y_i; \tau_\rho, \eta_b, \zeta_\alpha\} \mapsto \{x_a, y_i; \tau_\rho, \eta_b, \zeta_\alpha\}$ may be performed to complete the proof. □

Proposition 4.1. *If both $\mathcal{D}$ and $\mathcal{C}$ are involutive, then there is a graded commuting basis $\{X_a\}$ of $\mathcal{D}$ such that for all a, b we have $[X_a, Y_b] \in \mathcal{D}$, where $X_a \in \{D_i, D_\mu\}$ and $Y_a = [\Gamma, X_a]$.*

Proof. This is an immediate consequence of Remark 4.1 and Theorems 4.1 and 4.2. □

Now we want to write the local form of Γ in a local supercoordinate system. There are two cases to consider, $\Gamma \in \mathcal{C}$ and $\Gamma \notin \mathcal{C}$.

Theorem 4.3. *Let $\mathcal{D}$ and $\mathcal{C}$ be involutive and $\Gamma \in \mathcal{C}$. Assume that the set $N = \{z \in M : \Gamma(z) \in \mathcal{D}(z)\} \subset M$, is nonempty.*
(1) If $|\Gamma| = 0$, then we may choose supercoordinates $(t_l, x_i, y_i; \tau_\rho, \eta_\mu, \zeta_\mu), i = 1, ..., r$, $l = 1, ..., m - 2r, \mu = 1, ...s, \rho = 1, ..., n - 2s$, with respect to which

$$\Gamma = y_i \frac{\partial}{\partial x_i} + \Gamma_i(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial y_i} + \zeta_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma_\mu(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial \zeta_\mu}.$$

(2) If $|\Gamma| = 1$, then we may choose supercoordinates $(t_l, x_a, y_i; \tau_\rho, \eta_b, \zeta_\mu)$, as described in Theorem 4.2, with respect to which

$$\Gamma = \zeta_a \frac{\partial}{\partial x_a} + \Gamma_i(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial y_i} + y_b \frac{\partial}{\partial \eta_b} + \Gamma_\mu(t, x, y; \tau, \eta, \zeta) \frac{\partial}{\partial \zeta_\mu}.$$

Proof. Since $\Gamma \in \mathcal{C}$, with respect to the basis $\{C_i, D_i; D_\mu, C_\mu\}$ for $\mathcal{C}$ (see Proposition 4.1), we have

$$\Gamma = f_i D_i + \bar{f}_\mu D_\mu + g_i C_i + \bar{g}_\mu C_\mu, \tag{16}$$

where $f_i, \bar{f}_\mu, g_i$, and $\bar{g}_\mu$ are superfunctions on $\mathcal{M}$. Then from

$$\begin{aligned} C_i &= [\Gamma, D_i] \equiv -D_i(g_j)C_j - D_i(\bar{g}_\mu)C_\mu \quad (mod\ \mathcal{D}), \\ C_\mu &= [\Gamma, D_\mu] \equiv -(-1)^{|\Gamma|} D_\mu(g_i)C_i - (-1)^{|\Gamma|} D_\mu(\bar{g}_\nu)C_\nu \quad (mod\ \mathcal{D}), \end{aligned}$$

we conclude that $D_i(g_j) = -\delta_i^j, D_i(\bar{g}_\mu) = 0 = D_\mu(g_i)$, and $D_\mu(\bar{g}_\nu) = -(-1)^{|\Gamma|}\delta_\mu^\nu$, so

$$\begin{pmatrix} D_i(g_j) & D_i(\bar{g}_\nu) \\ D_\mu(g_j) & D_\mu(\bar{g}_\nu) \end{pmatrix}$$

is nonsingular. Note that in this computation we use $[X_a, Y_b] \in \mathcal{D}$, where $X_a \in \{D_i, D_\mu\}$ and $Y_a = [\Gamma, X_a]$. We sketch the proof of the theorem for the case (1). By the Frobenius theorem we take a coordinate neighborhood $\mathcal{U}$ in $\mathcal{M}$, with supercoordinates $(q_u, y_i; \theta_\alpha, \zeta_\mu)$ such that $D_i = \partial/\partial y_i$ and $D_\mu = \partial/\partial \zeta_\mu$, (see Theorem 4.1). Now consider the Jacobian matrix of the superfunctions g_i and $\bar{g}_\mu$ with respect to D_i and D_μ, it is nonsingular of rank (r, s). Then for each $p \in N$, $\Gamma(p) \in \mathcal{D}$ if and only if $g_i(p) = 0$ and $\bar{g}_\mu(p) = 0$. Thus we have a closed embedded supermanifold $\mathcal{N}$ as a subsupermanifold of $\mathcal{M}$ with the base manifold given by $\{\tilde{g}_i^{-1}(0) | i = 1, ..., r\} \cap \{\tilde{g}_\mu^{-1}(0) | i = 1, ..., s\}$. From Theorem 4.1 we can take supercoordinates $(t_l, x_i, y_i; \tau_\rho, \eta_\mu, \zeta_\mu)$ such that

$$C_i \equiv -\frac{\partial}{\partial x_i} \quad (mod\ \mathcal{D}), \qquad C_\mu \equiv -\frac{\partial}{\partial \eta_\mu} \quad (mod\ \mathcal{D}).$$

Then $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \eta_\mu}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial \zeta_\mu}\}$ evaluated at p, is another basis for $\mathcal{C}$, and in this super-coordinates

$$\Gamma = \gamma_i \frac{\partial}{\partial x_i} + \gamma_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma_i \frac{\partial}{\partial y_i} + \Gamma_\mu \frac{\partial}{\partial \zeta_\mu},$$

where $\gamma_i, \gamma_\mu, \Gamma_i$, and Γ_μ are superfunctions of $(t, x, y; \tau, \eta, \zeta)$. Note that $\Gamma \in \mathcal{C}$, so in the local form of Γ, the coefficients of $\frac{\partial}{\partial t_l}$ and $\frac{\partial}{\partial \tau_\rho}$ are zero. We will find the coefficients γ_i and γ_μ by using $C_i = [\Gamma, \frac{\partial}{\partial y_i}]$ and $C_\mu = [\Gamma, \frac{\partial}{\partial \zeta_\mu}]$. Thus we have

$$C_i \equiv -\frac{\partial \gamma_j}{\partial y_i}\frac{\partial}{\partial x_j} - \frac{\partial \gamma_\mu}{\partial y_i}\frac{\partial}{\partial \eta_\mu} \ (mod\ \mathcal{D}), \quad C_\mu \equiv -\left(\frac{\partial \gamma_i}{\partial \zeta_\mu}\frac{\partial}{\partial x_i} + \frac{\partial \gamma_\nu}{\partial \zeta_\mu}\frac{\partial}{\partial \eta_\nu}\right) \ (mod\ \mathcal{D}).$$

For each $p \in N$, $\gamma_i(p) = 0$ and $\gamma_\mu(p) = 0$, then from

$$\frac{\partial \gamma_j}{\partial y_i} = \delta_i^j, \quad \frac{\partial \gamma_\nu}{\partial y_i} = 0 = \frac{\partial \gamma_j}{\partial \zeta_\mu}, \quad \frac{\partial \gamma_\nu}{\partial \zeta_\mu} = \delta_\mu^\nu,$$

we have $\gamma_i(t, x, y; \tau, \eta, \zeta) = y_i, \qquad \gamma_\mu(t, x, y; \tau, \eta, \zeta) = \zeta_\mu$.

(2) Let $|\Gamma| = 1$, we consider again Γ as a linear combination of $\{C_i, D_i; D_\mu, C_\mu\}$ as (16). Then $|g_i| = 0$ and $|\bar{g}_\mu| = 1$ and the Jacobian matrix of the superfunctions g_i and $\bar{g}_\mu$ with respect to D_i and D_μ, is nonsingular of rank (r, s). Repeat the above analysis for this case and using Theorem 4.2, then there exist a coordinate system $(t_l, x_a, y_i; \tau_\rho, \eta_b, \zeta_\mu)$ on $\mathcal{M}$, $l = 1, \cdots, l_1$, $a = 1, \cdots, a_1$, $i = 1, \cdots, r, \rho = 1, \cdots, \rho_1$, $b = 1, \cdots, b_1$, $\mu = 1, \cdots, s$, such that

$$D_i = \frac{\partial}{\partial y_i}, \qquad C_a = [\Gamma, \frac{\partial}{\partial y_a}] \equiv -\frac{\partial}{\partial \eta_a} \quad (mod\ \mathcal{D}), \tag{17}$$

$$D_\mu = \frac{\partial}{\partial \zeta_\mu}, \qquad C_\mu = [\Gamma, \frac{\partial}{\partial \zeta_\mu}] \equiv \frac{\partial}{\partial x_\mu} \quad (mod\ \mathcal{D}), \tag{18}$$

and l_1, a_1, b_1 and ρ_1 are given in the Table **??**. According to this table, we have twelve types of possibilities for m, n, r, s. In each of these cases we may write Γ in the form

$$\Gamma = \gamma_a \frac{\partial}{\partial x_a} + \gamma_b \frac{\partial}{\partial \eta_b} + \Gamma_i \frac{\partial}{\partial y_i} + \Gamma_\mu \frac{\partial}{\partial \zeta_\mu},$$

and it is a straightforward matter to verify that

$$\frac{\partial \gamma_a}{\partial \zeta_\mu} = \delta_\mu^a, \quad \frac{\partial \gamma_\alpha}{\partial \zeta_\mu} = 0 = \frac{\partial \gamma_b}{\partial y_a}, \quad \frac{\partial \gamma_\alpha}{\partial y_a} = \delta_a^\alpha.$$

For each $p \in N$, $\gamma_a(p) = 0$ and $\gamma_\mu(p) = 0$, then $\gamma_a = \zeta_a$ and $\gamma_b = y_b$ and this completes the proof. □

Remark 4.2. *Suppose that* $dim\mathcal{M} = (2r+1, 2s+1), r < s$ *and* Γ *is an odd graded graded vector field on* $\mathcal{M}$. *Let* $\mathcal{D}$ *be an involutive direct subsheaf of* $Der\mathcal{A}_M$ *of rank* (r, s). *Then according to Theorem 3.1, there is a coordinate neighborhood* $\mathcal{U}$ *around each point* $p \in M$ *with local coordinates* $(q_u, y_i; \theta_\alpha, \zeta_\mu)$, *such that the submatrices* $(\frac{\partial \Gamma_\alpha}{\partial y_i})_{r\times(s+1)}$ *and* $(\frac{\partial \Gamma_u}{\partial \zeta_\mu})_{s\times(r+1)}$ *are even and have maximal ranks respectively* r *and* $r+1$. *This means that the odd graded vector fields* $[\Gamma, \frac{\partial}{\partial y_i}]$ *and the even graded vector fields* $[\Gamma, \frac{\partial}{\partial \zeta_\mu}]$, $(1 \leq i \leq r,$ *and* $1 \leq \mu \leq r+1)$ *are linearly independent at* p, *so* $rank_p(\mathcal{C}) = (2r+1, s+r)$. *Thus there exist (see Theorem 4.2) local supercoordinates* $(x_a, y_i; \tau_\rho, \eta_b, \zeta_\mu)$ *on* $\mathcal{M}$, $a = 1, \cdots, r+1$, $i = 1, \cdots, r, \rho = 1, \cdots, s+1-r$, $b = 1, \cdots, r$, $\mu = 1, \cdots, s$, *such that*

$$D_i = \frac{\partial}{\partial y_i}, \qquad C_b = [\Gamma, \frac{\partial}{\partial y_b}] \equiv -\frac{\partial}{\partial \eta_b} \quad (mod\ \mathcal{D}), \tag{19}$$

$$D_\mu = \frac{\partial}{\partial \zeta_\mu}, \qquad C_a = [\Gamma, \frac{\partial}{\partial \zeta_a}] \equiv \frac{\partial}{\partial x_a} \quad (mod\ \mathcal{D}). \tag{20}$$

Therefore $\{\frac{\partial}{\partial x_a}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial \eta_b}, \frac{\partial}{\partial \zeta_\mu}\}$ *evaluated at* p, *is another basis for* $\mathcal{C}$. *Now let* $\Gamma \in \mathcal{C}$, *because the coefficients of the* $\frac{\partial}{\partial \tau_\rho}$ *are zero and from (19) and (20), we see that*

$$\Gamma = \zeta_a \frac{\partial}{\partial x_a} + \Gamma_i(x, y; \tau, \eta, \zeta)\frac{\partial}{\partial y_i} + y_b \frac{\partial}{\partial \eta_b} + \Gamma_\mu(x, y; \tau, \eta, \zeta)\frac{\partial}{\partial \zeta_\mu}.$$

Theorem 4.4. *Let* $\mathcal{D}$ *and* $\mathcal{C}$ *be involutive. Let* Γ *be everywhere independent of* $\mathcal{C}$ *and* $[\Gamma, \mathcal{C}] \subset \mathcal{C}$.

(1) If $|\Gamma| = 0$, *then we may choose supercoordinates* $(t_1, x_i, y_i; \tau_1, \eta_\mu, \zeta_\mu), i = 1, ..., r$, $\mu = 1, ...s$, *such that*

- *(a)* $\Gamma = \frac{\partial}{\partial t_1} + y_i \frac{\partial}{\partial x_i} + \Gamma_i \frac{\partial}{\partial y_i} + \zeta_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma_\mu \frac{\partial}{\partial \zeta_\mu}$, *if the coefficient of* $\frac{\partial}{\partial \tau_1}$ *in* Γ *is zero,*
- *(b)* $\Gamma = \tau_1 \frac{\partial}{\partial \tau_1} + y_i \frac{\partial}{\partial x_i} + \Gamma_i \frac{\partial}{\partial y_i} + \zeta_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma_\mu \frac{\partial}{\partial \zeta_\mu}$, *if the coefficient of* $\frac{\partial}{\partial t_1}$ *in* Γ *is zero,*
- *(c)* $\Gamma = \frac{\partial}{\partial t_1} + \tau_1 \frac{\partial}{\partial \tau_1} + y_i \frac{\partial}{\partial x_i} + \Gamma_i \frac{\partial}{\partial y_i} + \zeta_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma_\mu \frac{\partial}{\partial \zeta_\mu}$, *if the coefficients of* $\frac{\partial}{\partial t_1}$ *and* $\frac{\partial}{\partial \tau_1}$ *in* Γ *are nonzero, where* Γ_i *and* Γ_μ *are superfunctions on* $\mathcal{M}$.

(2) If $|\Gamma| = 1$, *then we may choose supercoordinates* $(t_l, x_a, y_i; \tau_\rho, \eta_b, \zeta_\mu)$, *as described in Theorem 4.2, with respect to which*

$$\Gamma = \phi_l \frac{\partial}{\partial t_l} + \zeta_a \frac{\partial}{\partial x_a} + \Gamma_i(t, x, y; \tau, \eta, \zeta)\frac{\partial}{\partial y_i} + \varphi_\rho \frac{\partial}{\partial \tau_\rho} + y_b \frac{\partial}{\partial \eta_b} + \Gamma_\mu(t, x, y; \tau, \eta, \zeta)\frac{\partial}{\partial \zeta_\mu}.$$

where ϕ_l *and* φ_ρ *are independent of* x, y, η *and* ζ *and* Γ_i *and* Γ_μ *are superfunctions on* $\mathcal{M}$.

Proof. Using the same arguments we used in the proof of Theorem 4.1 we can take supercoordinates $(t_l, x_i, y_i; \tau_\rho, \eta_\mu, \zeta_\mu)$ such that

$$\begin{aligned}\Gamma &\equiv \phi_l(t,x,y;\tau,\eta,\zeta)\frac{\partial}{\partial t_l} + \varphi_\rho(t,x,y;\tau,\eta,\zeta)\frac{\partial}{\partial \tau_\rho} \quad (mod\ \mathcal{C}).\\ &= \phi_l\frac{\partial}{\partial t_l} + \varphi_\rho\frac{\partial}{\partial \tau_\rho} + \gamma_i\frac{\partial}{\partial x_i} + \gamma_\mu\frac{\partial}{\partial \eta_\mu} + \Gamma_i\frac{\partial}{\partial y_i} + \Gamma_\mu\frac{\partial}{\partial \zeta_\mu}.\end{aligned}$$

From $[\Gamma, \mathcal{C}] \subset \mathcal{C}$ we conclude that the derivative of ϕ and φ with respect to x, y, η, and ζ are zero, then ϕ and φ depend only on the t and τ. According to the procedure given in the previous theorem, we have

$$\frac{\partial\gamma_j}{\partial y_i} = \delta_i^j, \quad \frac{\partial\gamma_\nu}{\partial y_i} = 0 = \frac{\partial\gamma_j}{\partial\zeta_\mu}, \quad \frac{\partial\gamma_\nu}{\partial\zeta_\mu} = \delta_\mu^\nu.$$

Then

$$\gamma_i(t,x,y;\tau,\eta,\zeta) = y_i + h_i(t,x;\tau,\eta), \qquad \gamma_\mu(t,x,y;\tau,\eta,\zeta) = \zeta_\mu + k_\mu(t,x;\tau,\eta).$$

Now if $\Gamma \neq 0$, since ϕ and φ are arbitrary, we may choose $\phi \neq 0$ and $\varphi = 0$ to finding the local expression for Γ. Similarly, one may choose $\phi = 0$ and $\varphi \neq 0$, etc. So by a transformation of the coordinates t_l we may take $\Gamma \equiv \frac{\partial}{\partial t_1} \quad (mod\ \mathcal{C})$.

Consider the new change of supercoordinates

$$\begin{aligned}&\widehat{t_1} = t_1, \quad \widehat{y_i} = y_i + h_i(t,x;\tau,\eta), \quad \widehat{x_i} = x_i,\\ &\widehat{\tau_1} = \tau_1, \quad \widehat{\zeta_\mu} = \zeta_\mu + k_\mu(t,x;\tau,\eta), \quad \widehat{\eta_\mu} = \eta_\mu.\end{aligned}$$

Therefore

$$\begin{aligned}\Gamma &= \frac{\partial}{\partial\widehat{t_1}} + \widehat{y}_i\frac{\partial}{\partial\widehat{x_i}} + \left(\Gamma_i + \frac{\partial h_i}{\partial t_1} + \widehat{y}_j\frac{\partial h_i}{\partial x_j} + \widehat{\zeta}_\mu\frac{\partial h_i}{\partial\eta_\mu}\right)\frac{\partial}{\partial\widehat{y_i}}\\ &\quad + \widehat{\zeta}_\mu\frac{\partial}{\partial\widehat{\eta}_\mu} + \left(\Gamma_\mu + \frac{\partial k_\mu}{\partial t_1} + \widehat{y}_i\frac{\partial k_\mu}{\partial x_i} + \widehat{\zeta}_\nu\frac{\partial k_\mu}{\partial\eta_\nu}\right)\frac{\partial}{\partial\widehat{\zeta_\mu}}.\end{aligned}$$

Now let $\widehat{\Gamma}_i = \Gamma_i + \frac{\partial h_i}{\partial t_1} + \widehat{y}_j\frac{\partial h_i}{\partial x_j} + \widehat{\zeta}_\mu\frac{\partial h_i}{\partial\eta_\mu}$ and $\widehat{\Gamma}_\mu = \Gamma_\mu + \frac{\partial k_\mu}{\partial t_1} + \widehat{y}_i\frac{\partial k_\mu}{\partial x_i} + \widehat{\zeta}_\nu\frac{\partial k_\mu}{\partial\eta_\nu}$, this completes the proof.

(2) The proof of this part follows simply from the above discussion and Theorems 4.2 and 4.3. □

Remark 4.3. *In the case when $\mathcal{M}$ is a tangent supermanifold $(TM', \mathcal{A}_{TM'})$, i.e. we assume that $m = 2r$ and $n = 2s$, from Theorem 4.3, we see that the local expression for Γ is the same as Ibort's one* [6]. *Another case when we talk about the time-dependent systems is the case when $\mathcal{M} = R^{1,1} \times (TM', \mathcal{A}_{TM'})$ where $m = 2r + 1$*

and $n = 2s+1$. According to Theorem 4.4, Γ is presented by the $(t, x, y; \tau, \eta, \zeta)$ where $\{t, \tau\}$ is a set of local coordinates on $R^{1,1}$ which can be considered as a supertime.

We now turn our attention to the coefficients of the covariant derivative defined in (7). We will find the local representation of these coefficients in the local coordinates $(t_l, x_a, y_i; \tau_\rho, \eta_b, \zeta_\mu)$ defined in Theorems 4.3 and 4.4. With respect to these supercoordinates we have

$$\begin{aligned}
J(\frac{\partial}{\partial y_i}) &= 0, \quad J(\frac{\partial}{\partial x_a}) = \frac{\partial}{\partial y_a}, \quad J(\frac{\partial}{\partial \zeta_\mu}) = 0, \quad J(\frac{\partial}{\partial \eta_b}) = \frac{\partial}{\partial \zeta_b}, \\
h(\frac{\partial}{\partial y_i}) &= 0, \quad h(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i} + \left(\frac{\partial \Gamma_j}{\partial y_i} + \pi_{ij}\right)\frac{\partial}{\partial y_j} + \left(\frac{\partial \Gamma_\mu}{\partial y_i} + \pi_{i\mu}\right)\frac{\partial}{\partial \zeta_\mu}, \\
h(\frac{\partial}{\partial \zeta_\mu}) &= 0, \quad h(\frac{\partial}{\partial \eta_\mu}) = \frac{\partial}{\partial \eta_\mu} + \left(\frac{\partial \Gamma_j}{\partial \zeta_\mu} + \pi_{\mu j}\right)\frac{\partial}{\partial y_j} + \left(\frac{\partial \Gamma_\nu}{\partial \zeta_\mu} + \pi_{\mu\nu}\right)\frac{\partial}{\partial \zeta_\nu},
\end{aligned}$$

and

$$(\frac{\partial}{\partial y_i})^H = -h(\frac{\partial}{\partial x_i}), \quad (\frac{\partial}{\partial \zeta_\mu})^H = -h(\frac{\partial}{\partial \eta_\mu}).$$

As a result we now see that

$$\begin{aligned}
\mathcal{H} = \quad & \left\langle \frac{\partial}{\partial x_i} + \left(\frac{\partial \Gamma_j}{\partial y_i} + \pi_{ij}\right)\frac{\partial}{\partial y_j} + \left(\frac{\partial \Gamma_\mu}{\partial y_i} + \pi_{i\mu}\right)\frac{\partial}{\partial \zeta_\mu} \right\rangle \oplus \\
& \left\langle \frac{\partial}{\partial \eta_\mu} + \left(\frac{\partial \Gamma_j}{\partial \zeta_\mu} + \pi_{\mu j}\right)\frac{\partial}{\partial y_j} + \left(\frac{\partial \Gamma_\nu}{\partial \zeta_\mu} + \pi_{\mu\nu}\right)\frac{\partial}{\partial \zeta_\nu} \right\rangle .
\end{aligned} \tag{21}$$

Now consider the operator δ defined in (8). It is useful to provide a criterion for the existence of a quadratic second-order superdifferential equation field.

Proposition 4.2. *Let $\mathcal{D}$ and $\mathcal{C}$ be involutive and $[\Gamma, \mathcal{C}] \subset \mathcal{C}$. If $\delta = 0$, then for each $p \in M$ there exists a coordinate system defined in a neighborhood $\mathcal{U}$ of p, such that Γ takes the form of a quadratic second-order superdifferential equation field.*

Proof. Let $\delta = 0$. In the supercoordinates as defined in Theorems 4.3 and 4.4 the connection coefficients for $X_a, X_b, X_c \in \{\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \zeta_\mu}\}$ are given by

$$\overline{\nabla}_{X_a} X_b = (-1)^{|X_a||X_b|} \Gamma^c_{ab} X_c,$$

where $\Gamma_{ab} = -\frac{1}{2} X_b(\Gamma_a)$, and $\Gamma^c_{ab} = X_c(\Gamma_{ab})$. For example

$$\begin{aligned}
\overline{\nabla}_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial \zeta_\mu} &= \nabla_{(\frac{\partial}{\partial y_i})^H} \frac{\partial}{\partial \zeta_\mu} = -\frac{1}{2}\frac{\partial}{\partial \zeta_\mu}(\frac{\partial \Gamma_k}{\partial y_i})\frac{\partial}{\partial y_k} - \frac{1}{2}\frac{\partial}{\partial \zeta_\mu}(\frac{\partial \Gamma_\nu}{\partial y_i})\frac{\partial}{\partial \zeta_\nu} \\
&= \Gamma^k_{i\mu}\frac{\partial}{\partial y_k} + \Gamma^\nu_{i\mu}\frac{\partial}{\partial \zeta_\nu}.
\end{aligned}$$

Then $\Gamma^c_{ab}X_c \in \widetilde{D}$, this means that for each $X_d \in \mathcal{D}$, $[\Gamma^c_{ab}(X_c)^H, X_d] \in \mathcal{D}$. From Theorems 4.3 and 4.4, we have $[Y_c, X_d] \in \mathcal{D}$, so $X_d(\Gamma^c_{ab}) = 0$. Thus there are projectable superfunctions $V_{ai}(t,x,\tau,\eta), W_{a\mu}(t,x,\tau,\eta)$, and $Z_a(t,x,\tau,\eta)$ such that

$$\Gamma_a = -\Gamma^a_{ij}y_iy_j - \Gamma^a_{\mu\nu}\xi_\mu\xi_\nu - 2\Gamma^a_{i\mu}y_i\xi_\mu + V_{ai}y_i + W_{a\mu}\xi_\mu + Z_a.$$

□

5. Involutive direct subsheaves and dynamical connections

Again return to the Theorem 4.4 and suppose that $\mathcal{D}$ and $\mathcal{C}$ are involutive and Γ is everywhere independent of $\mathcal{C}$ such that $[\Gamma, \mathcal{C}] \subset \mathcal{C}$. Then in local coordinate system $(t, x_i, y_i; \tau, \eta_\mu, \zeta_\mu)$ we may define a new graded tensor field $\widetilde{J}$ of type (1,1) on $\mathcal{M}$

$$\widetilde{J} = J - (y_i\frac{\partial}{\partial y_i} + \zeta_\mu\frac{\partial}{\partial\zeta_\mu}) \otimes (dt - \tau d\tau), \tag{22}$$

where J is defined by (3). $\widetilde{J}$ is locally characterized by

$$\widetilde{J}(\frac{\partial}{\partial t}) = -(y_i\frac{\partial}{\partial y_i} + \zeta_\mu\frac{\partial}{\partial\zeta_\mu}), \quad \widetilde{J}(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \quad \widetilde{J}(\frac{\partial}{\partial y_i}) = 0,$$
$$\widetilde{J}(\frac{\partial}{\partial \tau}) = -\tau(y_i\frac{\partial}{\partial y_i} + \zeta_\mu\frac{\partial}{\partial\zeta_\mu}), \quad \widetilde{J}(\frac{\partial}{\partial \eta_\mu}) = \frac{\partial}{\partial\zeta_\mu}, \quad \widetilde{J}(\frac{\partial}{\partial\zeta_\mu}) = 0.$$

By using this we can define an operator Π on $Der\mathcal{A}_M$ which is so called " the dynamical connection " and satisfy

$$J \circ \Pi = \widetilde{J} \circ \Pi = \widetilde{J}, \quad \Pi \circ \widetilde{J} = -\widetilde{J}, \quad \Pi \circ J = -J. \tag{23}$$

Clearly Π is an even operator and $|J| = |\tilde{J}|$. By a straightforward computation from (23) we deduce that the local expressions of Π are

$$
\begin{aligned}
&\Pi(\frac{\partial}{\partial t}) = -y_i\frac{\partial}{\partial x_i} + \Pi_i\frac{\partial}{\partial y_i} - \zeta_\mu\frac{\partial}{\partial \eta_\mu} + \Pi_\mu\frac{\partial}{\partial \zeta_\mu},\\
&\Pi(\frac{\partial}{\partial \tau}) = -\tau y_i\frac{\partial}{\partial x_i} + \overline{\Pi}_i\frac{\partial}{\partial y_i} - \tau\zeta_\mu\frac{\partial}{\partial \eta_\mu} + \overline{\Pi}_\mu\frac{\partial}{\partial \zeta_\mu},\\
&\Pi(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i} + \Pi_{ij}\frac{\partial}{\partial y_j} + \Pi_{i\mu}\frac{\partial}{\partial \zeta_\mu},\\
&\Pi(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial y_i},\\
&\Pi(\frac{\partial}{\partial \eta_\mu}) = \frac{\partial}{\partial \eta_\mu} + \overline{\Pi}_{\mu i}\frac{\partial}{\partial y_i} + \overline{\Pi}_{\mu\nu}\frac{\partial}{\partial \zeta_\nu},\\
&\Pi(\frac{\partial}{\partial \zeta_\mu}) = -\frac{\partial}{\partial \zeta_\mu},
\end{aligned}
\tag{24}
$$

where superfunctions $\Pi_\bullet = \Pi_\bullet(t, x_i, y_i; \tau, \eta_\mu, \zeta_\mu), \Pi_{\bullet\bullet} = \Pi_{\bullet\bullet}(t, x_i, y_i; \tau, \eta_\mu, \zeta_\mu)$ and $\overline{\Pi}_{\bullet\bullet} = \Pi_{\bullet\bullet}(t, x_i, y_i; \tau, \eta_\mu, \zeta_\mu)$ are the local coefficients of the operator Π. It is obvious that $|\Pi_i| = |\overline{\Pi}_\mu| = |\Pi_{ij}| = |\overline{\Pi}_{\mu\nu}| = 0$ and $|\Pi_\mu| = |\overline{\Pi}_i| = |\Pi_{i\mu}| = |\overline{\Pi}_{\mu i}| = 1$. From (24) we deduce that $\Pi^3 = \Pi$ and $rank(\Pi) = (2r, 2s)$.

Again define two operators ι and ℓ on $Der\mathcal{A}_M$, given by

$$
\iota = \Pi^2, \quad \ell = Id - \Pi^2. \tag{25}
$$

Then we have

$$
\begin{aligned}
\iota(\frac{\partial}{\partial t}) &= -y_i\frac{\partial}{\partial x_i} - (y_j\Pi_{ji} + \Pi_i + \zeta_\mu\overline{\Pi}_{\mu i})\frac{\partial}{\partial y_i} - \zeta_\mu\frac{\partial}{\partial \eta_\mu}\\
&\quad -(y_i\Pi_{i\mu} + \zeta_\nu\overline{\Pi}_{\nu\mu} + \Pi_\mu)\frac{\partial}{\partial \zeta_\mu},\\
\iota(\frac{\partial}{\partial \tau}) &= -\tau y_i\frac{\partial}{\partial x_i} - (\tau y_j\Pi_{ji} + \overline{\Pi}_i + \tau\zeta_\mu\overline{\Pi}_{\mu i})\frac{\partial}{\partial y_i} - \tau\zeta_\mu\frac{\partial}{\partial \eta_\mu}\\
&\quad -(\tau y_i\Pi_{i\mu} + \tau\zeta_\nu\overline{\Pi}_{\nu\mu} + \overline{\Pi}_\mu)\frac{\partial}{\partial \zeta_\mu},\\
\iota(\frac{\partial}{\partial x_i}) &= \frac{\partial}{\partial x_i}, \quad \iota(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial y_i}, \quad \iota(\frac{\partial}{\partial \eta_\mu}) = \frac{\partial}{\partial \eta_\mu}, \quad \iota(\frac{\partial}{\partial \zeta_\mu}) = \frac{\partial}{\partial \zeta_\mu},
\end{aligned}
$$

and

$$\begin{aligned}\ell(\frac{\partial}{\partial t}) &= \frac{\partial}{\partial t} - \iota(\frac{\partial}{\partial t}), \quad \ell(\frac{\partial}{\partial \tau}) = \frac{\partial}{\partial \tau} - \iota(\frac{\partial}{\partial \tau}) \\ \ell(\frac{\partial}{\partial x_i}) &= 0, \quad \ell(\frac{\partial}{\partial y_i}) = 0, \quad \ell(\frac{\partial}{\partial \eta_\mu}) = 0, \quad \ell(\frac{\partial}{\partial \zeta_\mu}) = 0.\end{aligned}$$

It is clear that $\mathcal{C} = Im\ \iota$, then if we put $\mathsf{L} = Im\ \ell$, we have

$$Der\mathcal{A}_M = \mathcal{C} \oplus \mathsf{L}.$$

From (5) we concolude that

$$Der\mathcal{A}_M = \mathcal{D} \oplus \mathcal{H} \oplus \mathsf{L}.$$

Furthermore, Π has eigenvalues $0, -1$, and 1. From (24) the eigenspaces corresponding to the eigenvalue -1 are $\langle \frac{\partial}{\partial y_i} \rangle \oplus \langle \frac{\partial}{\partial \zeta_\mu} \rangle$, the eigenspaces corresponding to the eigenvalue 1 are

$$\langle H_i := \frac{\partial}{\partial x_i} + \frac{1}{2}\Pi_{ij}\frac{\partial}{\partial y_j} + \frac{1}{2}\Pi_{i\mu}\frac{\partial}{\partial \zeta_\mu} \rangle \oplus \langle H_\mu := \frac{\partial}{\partial \eta_\mu} + \frac{1}{2}\overline{\Pi}_{\mu i}\frac{\partial}{\partial y_i} + \frac{1}{2}\overline{\Pi}_{\mu\nu}\frac{\partial}{\partial \eta_\nu} \rangle$$

and the eigenspaces corresponding to the eigenvalue 0 are

$$\begin{aligned}\langle T_e := \frac{\partial}{\partial t} &+ y_i\frac{\partial}{\partial x_i} + (y_j\Pi_{ji} + \Pi_i + \zeta_\mu\overline{\Pi}_{\mu i})\frac{\partial}{\partial y_i} \\ &+ \zeta_\mu\frac{\partial}{\partial \eta_\mu} + (y_i\Pi_{i\mu} + \zeta_\nu\overline{\Pi}_{\nu\mu} + \Pi_\mu)\frac{\partial}{\partial \zeta_\mu} \rangle\end{aligned}$$

and

$$\begin{aligned}\langle T_o := \frac{\partial}{\partial \tau} &+ \tau y_i\frac{\partial}{\partial x_i} + (\tau y_j\Pi_{ji} + \overline{\Pi}_i + \tau\zeta_\mu\overline{\Pi}_{\mu i})\frac{\partial}{\partial y_i} \\ &+ \tau\zeta_\mu\frac{\partial}{\partial \eta_\mu} + (\tau y_i\Pi_{i\mu} + \tau\zeta_\nu\overline{\Pi}_{\nu\mu} + \overline{\Pi}_\mu)\frac{\partial}{\partial \zeta_\mu} \rangle.\end{aligned}$$

In the light of these computations, the following decomposition of $Der\mathcal{A}_M$ may be considered for coordinate system $(t, x_i, y_i; \tau, \eta_\mu, \zeta_\mu)$

$$Der\mathcal{A}_M = \langle \frac{\partial}{\partial y_i} \rangle \oplus \langle \frac{\partial}{\partial \zeta_\mu} \rangle \oplus \langle H_i \rangle \oplus \langle H_\mu \rangle \oplus \langle T_e \rangle \oplus \langle T_o \rangle.$$

A simple computation shows that $\widetilde{J}(\Gamma_{even} + \Gamma_{odd}) = 0$ and $J(\Gamma_{even} + \Gamma_{odd}) = -(y_i\frac{\partial}{\partial y_i} + \zeta_\mu\frac{\partial}{\partial \zeta_\mu})(1 + \tau)$.

Theorem 5.1. *If the hypothesis of Theorem 4.4 is satisfied for* Γ*, then the operator* $\widetilde{\Pi} := -\mathcal{L}_\Gamma \widetilde{J}$ *is a dynamical connection with the following coefficients:*

$$\begin{aligned}
\widetilde{\Pi}_i &= \Gamma_i - y_j \frac{\partial \Gamma_i}{\partial y_j} - \zeta_\mu \frac{\partial \Gamma_i}{\partial \zeta_\mu}, \qquad \widetilde{\Pi}_\mu = \Gamma_\mu - y_j \frac{\partial \Gamma_\mu}{\partial y_j} - \zeta_\nu \frac{\partial \Gamma_\mu}{\partial \zeta_\nu}, \\
\widetilde{\widetilde{\Pi}}_i &= y_i + \Gamma_i \tau - (-1)^{|\Gamma_i|} \tau y_i \frac{\partial \Gamma_i}{\partial y_j} - (-1)^{|\Gamma_i|} \tau \zeta_\nu \frac{\partial \Gamma_i}{\partial \zeta_\nu}, \\
\widetilde{\widetilde{\Pi}}_\mu &= \zeta_\mu - \Gamma_\mu \tau - (-1)^{|\Gamma_\mu|} \tau y_j \frac{\partial \Gamma_\mu}{\partial y_j} - (-1)^{|\Gamma_\mu|} \tau \zeta_\nu \frac{\partial \Gamma_\mu}{\partial \zeta_\nu}, \\
\widetilde{\Pi}_{ij} &= \frac{\partial \Gamma_j}{\partial y_i}, \quad \widetilde{\Pi}_{i\mu} = \frac{\partial \Gamma_\mu}{\partial y_i}, \\
\widetilde{\widetilde{\Pi}}_{\mu i} &= (-1)^{|\Gamma_i|} \frac{\partial \Gamma_i}{\partial \zeta_\nu}, \quad \widetilde{\widetilde{\Pi}}_{\nu\mu} = (-1)^{|\Gamma_\nu|} \frac{\partial \Gamma_\nu}{\partial \zeta_\mu}.
\end{aligned}$$

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