Journal of Dynamical Systems & Geomatric Theories
ISSN : 1726-037X (Print) 2169-0057 (Online)
Vol. 16(2) November 2018, pp. 139–149
DOI : 10.1080/1726037X.2018.1436274

CHAIN MIXING AND SHADOWING PROPERTY IN ITERATED FUNCTION SYSTEMS

M. FATEHI NIA
DEPARTMENT OF MATHEMATICS
YAZD UNIVERSITY
P.O. BOX 89195-741 YAZD, IRAN
E-MAIL: FATEHINIAM@YAZD.AC.IR

(Received: 14 June 2017, Accepted: 19 November 2017)

Abstract. In this paper we consider the chain mixing and shadowing properties for *iterated function systems*,(**IFS**). Some important results about shadowing property are extended to iterated function systems. We introduce some examples and investigate the relationship between the original shadowing property and shadowing property for**IFS**. This is proved that for every**IFS**with shadowing property chain mixing and topological mixing properties are equivalent.

AMS Classification: 37C50,37C15
Keywords: Chain mixing; topologically mixing; iterated function systems; shadowing property

1. Introduction

The notion of shadowing is an important tool for studying properties of discrete dynamical systems. From numerical point of view, if a dynamical system has the shadowing property, then numerically obtained orbits reflect the real behavior of trajectories of the systems(see [1, 5, 13, 15]).

Iterated function systems(**IFS**), are used for the construction of deterministic fractals and have found numerous applications, in particular to image compression and image processing [2]. Important notions in dynamics like attractors, minimality, transitivity, and shadowing can be extended to IFS (see [3, 4, 7, 8, 9, 10]). The authors defined the shadowing property for a parameterized iterated function system and prove that if a parameterized IFS is uniformly contracting, then it has the shadowing property[11].

The present paper concerns the shadowing and chain mixing properties for **IFS** and some important results are extended to iterated function systems. We show that shadowing property implies that chain mixing and topological mixing properties are equivalent. In [1] the authors have shown that a minimal homeomorphism on a compact metric space does not have the shadowing property. In Example 5 we show that there is a minimal **IFS** which has the shadowing property. In Example 4 we give an **IFS**, $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ such that for every $\lambda \in \Lambda$, $f_\lambda : X \to X$, as a discrete dynamical system, has the shadowing property but $\mathcal{F}$ does not have the shadowing property. Finally, we consider topological mixing and chain mixing properties and show that shadowing property in IFS implies that topological mixing and chain mixing are equivalent.

2. Preliminaries

Let (X, d) be a complete metric space. Let us recall that an *Iterated Function System(IFS)* $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is any family of continuous mappings $f_\lambda : X \to X$, $\lambda \in \Lambda$, where Λ is a finite nonempty set (see[11]).

Let $T = \mathbb{Z}$ or $T = \mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$ and Λ^T denote the set of all infinite sequences $\{\lambda_i\}_{i \in T}$ of symbols belonging to Λ. A typical element of $\Lambda^{\mathbb{Z}_+}$ can be denoted as $\sigma = \{\lambda_0, \lambda_1, ...\}$ and we use the shorted notation

$$\mathcal{F}_{\sigma_n} = f_{\lambda_n} o ... o f_{\lambda_1} o f_{\lambda_0}.$$

A sequence $\{x_n\}_{n \in T}$ in X is called an orbit of the **IFS** $\mathcal{F}$ if there exists $\sigma \in \Lambda^T$ such that $x_{n+1} = f_{\lambda_n}(x_n)$, for each $\lambda_n \in \sigma$.

Given $\delta > 0$, a sequence $\{x_n\}_{n \in T}$ in X is called a $\delta-$pseudo orbit of $\mathcal{F}$ if there exists $\sigma \in \Lambda^T$ such that for every $\lambda_n \in \sigma$, we have $d(x_{n+1}, f_{\lambda_n}(x_n)) < \delta$.

One says that the **IFS** $\mathcal{F}$ has the *shadowing property* (on T) if, given $\epsilon > 0$, there exists $\delta > 0$ such that for any $\delta-$pseudo orbit $\{x_n\}_{n \in T}$ there is an orbit $\{y_n\}_{n \in T}$,

satisfying the inequality $d(x_n, y_n) \leq \epsilon$ for all $n \in T$. In this case one says that the $\{y_n\}_{n\in T}$ or the point y_0, $\epsilon-$ shadows the $\delta-$pseudo orbit $\{x_n\}_{n\in T}$.
Please note that if Λ is a set with one member then the **IFS** $\mathcal{F}$ is an ordinary discrete dynamical system. In this case the shadowing property for $\mathcal{F}$ is ordinary shadowing property for a discrete dynamical system.
The **IFS** $\mathcal{F} = \{X; f_\lambda|\lambda \in \Lambda\}$ is *uniformly contracting* if there exists

$$\beta = \sup_{\lambda\in\Lambda} \sup_{x\neq y} \frac{d(f_\lambda(x), f_\lambda(y))}{d(x,y)}$$

and this number called also the *contracting ratio*, is less than one[11].
An $\mathcal{F}$ is uniformly expanding if

$$\beta = \inf_{\lambda\in\Lambda} \inf_{x\neq y} \frac{d(f_\lambda(x), f_\lambda(y))}{d(x,y)}$$

and this number, called also the contracting ratio, is greater than one.

Theorem 1. [11] *If a parameterized **IFS** $\mathcal{F} = \{X; f_\lambda|\lambda \in \Lambda\}$ is uniformly contracting, then it has the Shadowing Property on Z_+.*

Theorem 2. [11] *If a parameterized **IFS** $\mathcal{F} = \{X; f_\lambda|\lambda \in \Lambda\}$ is uniformly expanding and if each function $f_\lambda(\lambda \in \Lambda)$ is surjective, then the IFS has the Shadowing Property on Z_+.*

Definition 1. Suppose (X, d) and $(Y, d^{'})$ are compact metric spaces and Λ is a finite set. Let $\mathcal{F} = \{X; f_\lambda|\lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda|\lambda \in \Lambda\}$ are two **IFS** which $f_\lambda : X \to X$ and $g_\lambda : Y \to Y$ are continuous maps for all $\lambda \in \Lambda$. We say that $\mathcal{F}$ is said to be topologically conjugate to $\mathcal{G}$ if there is a homeomorphism $h : X \to Y$ such that $g_\lambda = hof_\lambda oh^{-1}$, for all $\lambda \in \Lambda$.

Definition 2. The iterated function system $\mathcal{F} = \{X; f_\lambda|\lambda \in \Lambda\}$ is minimal if each closed subset $A \subset X$ such that $f_\lambda(A) \subset A$ for all $\lambda \in \Lambda$, is empty or coincides with X.
Equivalently, for a minimal iterated function system $\mathcal{F} = \{X; f_\lambda|\lambda \in \Lambda\}$, for any $x \in X$ the collection of iterates $f_{\lambda_1} o...of_{\lambda_k}(x)$, $k > 0$ and $\lambda_1, ..., \lambda_k \in \Lambda$, is dense in X [14].

3. Shadowing property for iterated function systems

In this section we investigate the structure of **IFS**'s with the shadowing property. It is well known that if $f: X \to X$ and $g: Y \to Y$ are conjugated then f has the shadowing property if and only if so does g. In the next theorem we extend this property for *iterated function systems* .

Theorem 3. *Suppose (X, d) and $(Y, d^{'})$ are compact metric spaces and Λ is a finite set. Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda | \lambda \in \Lambda\}$ be two conjugated **IFS**. Then $\mathcal{F}$ has the shadowing property if and only if so does $\mathcal{G}$.*

Proof. Given $\epsilon > 0$. Suppose $h: X \to Y$ is a homeomorphism that $g_\lambda = hof_\lambda oh^{-1}$, for all $\lambda \in \Lambda$. There is $\gamma > 0$ such that $d(x, y) < \gamma$ implies $(d^{'}(h(x), h(y)) < \epsilon$. If $\mathcal{F}$ has the shadowing property then there is $\beta > 0$ such that each $\beta-$pseudo orbit $\{x_i\}_{i \geq 0}$ for $\mathcal{F}$ is $\gamma-$traced by some point in X.
Let $\delta > 0$ be a number such that $d^{'}(x, y) < \delta$ implies $d(h^{-1}(x), h^{-1}(y) < \beta$. Now we show that each $\delta-$pseudo orbit for $\mathcal{G}$ is $\epsilon-$traced by some point in Y. Suppose $\{y_i\}_{i \geq 0}$ is a $\delta-$pseudo orbit for $\mathcal{G}$, so for each $i \geq 0$ there is $\lambda_i \in \Lambda$ such that $d^{'}(g_{\lambda_i}(y_i), y_{i+1}) < \delta$. Put $x_i = h^{-1}(y_i)$, for all $i \geq 0$. Then $d(f_{\lambda_i}(x_i), x_{i+1}) = d(h^{-1}og_{\lambda_i}(y_i), h^{-1}(y_{i+1})) < \beta$, for all $i \geq 0$. Thus $\{x_i\}_{i \geq 0}$ is a $\beta-$pseudo orbit for $\mathcal{F}$ and so there is an orbit $\{z_i\}_{i \geq 0}$ of $\mathcal{F}$ such that $d(z_i, x_i) < \gamma$ for all $i \geq 0$. Note that for each $i \geq 0$ there is $\mu_i \in \Lambda$ such that $z_{i+1} = f_{\mu_i}(z_i)$. Let $w_i = h(z_i)$, this is clear that

$$w_{i+1} = h(z_{i+1}) = h(f_{\mu_i}(z_i)) = g_{\mu_i}(h(z_i)) = g_{\mu_i}(w_i),$$

for all $i \geq 0$. Therefor $\{w_i\}_{i \geq 0}$ is an orbit of $\mathcal{G}$ and $d^{'}(w_i, y_i) < \epsilon$ for all $i \geq 0$. □

Theorem 4. *Let X be a compact metric space. If $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is an **IFS** with the shadowing property (on $\mathbb{Z}$), then so is $\mathcal{F}^{-1} = \{X; g_\lambda | \lambda \in \Lambda\}$ where $f_\lambda : X \to X$ is homeomorphism and $g_\lambda = f_\lambda^{-1}$ for all $\lambda \in \Lambda$.*

Proof. Given $\epsilon > 0$. There is $\delta > 0$ such that each $\delta-$pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ is $\epsilon-$traced by a point $y \in X$. Choose $\delta^{'} > 0$ such that $d(x, y) < \delta^{'}$ implies that $d(f_\lambda(x), f_\lambda(y)) < \delta$, for all $\lambda \in \Lambda$. It is enough to see that each $\delta^{'}-$pseudo orbit $\{y_i\}_{i \in \mathbb{Z}}$ for $\mathcal{F}^{-1}$ is $\epsilon-$traced by a point.
Since $\{y_i\}_{i \in \mathbb{Z}}$ is a $\delta^{'}-$pseudo orbit for $\mathcal{F}^{-1}$, for every $i \in \mathbb{Z}$ there is $\lambda_i \in \Lambda$ such that $d(g_{\lambda_i}(y_i), y_{i+1}) < \delta^{'}$. Thus $d(y_i, f_{\lambda_i}(y_{i+1})) < \delta$, for all $i \in \mathbb{Z}$. So $d(f_{\mu_i}(x_i), x_{i+1}) <$

δ, where $x_i = y_{-i}$ and $\mu_i = \lambda_{-i-1}$. Then $\{x_i\}_{i\in\mathbb{Z}}$ is a $\delta-$pseudo orbit for $\mathcal{F}$ and there exists an orbit $\{w_i\}_{i\in\mathbb{Z}}$ for $\mathcal{F}$, satisfying the inequality $d(w_i, x_i) < \epsilon$, for all $i \in \mathbb{Z}$.
Suppose $w_{i+1} = f_{\eta_i}(w_i)$ and put $z_i = w_{-i}$, for all $i \in \mathbb{Z}$.
Since $w_i = f_{\eta_i}^{-1}(w_{i+1}) = g_{\eta_i}(w_{i+1})$, this is clear that $\{z_i\}_{i\in\mathbb{Z}}$ is an orbit of $\mathcal{F}^{-1}$ and $d(z_i, y_i) = d(w_{-i}, x_{-i}) < \epsilon$, for all $i \in \mathbb{Z}$. □

Suppose X is an Euclidean metric space and $B(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$ is the closed $\epsilon-$ball about $x \in X$.

Theorem 5. [6] *Let ϵ, $\delta > 0$ and Λ be a finite set. Suppose $\mathcal{F} = \{X; f_\lambda|\lambda \in \Lambda\}$ and $\mathcal{G} = \{X; g_\lambda|\lambda \in \Lambda\}$ are two* ***IFS*** *such that $B(f_\lambda(x), \epsilon + \delta) \subseteq g_\lambda(B(x, \epsilon))$, for every $\lambda \in \Lambda$ and $x \in X$. Then every $\delta-$pseudo orbit for $\mathcal{F}$ can be $\epsilon-$shadowed by an orbit in $\mathcal{G}$*

Corollary 1. [6] *The* ***IFS****, $\mathcal{F} = \{X; f_\lambda|\lambda \in \Lambda\}$ has the shadowing property if, for every $\epsilon > 0$, there is a $\delta > 0$ such that $B(f_\lambda(x), \epsilon + \delta) \subseteq f_\lambda(B(x, \epsilon))$ holds for all $x \in X$ and $\lambda \in \Lambda$.*

Theorem 6. *Let Λ be a finite set, $\mathcal{F} = \{X; f_\lambda|\lambda \in \Lambda\}$ is an IFS and let $k > 0$ be an integer. Set $\mathcal{F}^k = \{g_\mu|\mu \in \Pi\} = \{f_{\lambda_k} o...o f_{\lambda_1}|\lambda_1, ..., \lambda_k \in \Lambda\}$.*
a$-$ If $\mathcal{F}$ has the shadowing property (on $\mathbb{Z}_+$) then so does $\mathcal{F}^k$.
b$-$ Suppose that $\mathcal{F}^k$ has the shadowing property (on $\mathbb{Z}_+$), ϵ is an arbitrary positive number and $0 \leq m < k$ is a fixed integer number. There exists $\delta > 0$ such that for every $\delta-$pseudo orbit $\{x_i\}_{i\geq 0}$ in $\mathcal{F}$, $\{x_{ki+m}\}_{i\geq 0}$ is a $\delta-$pseudo orbit in $\mathcal{F}^k$ and can be $\epsilon-$traced by some point.

Proof. a$-$ Suppose $\mathcal{F}$ has the shadowing property and ϵ is an arbitrary positive number. There exists $\delta > 0$ such that if $\{x_i\}_{i\geq 0}$ is a $\delta-$pseudo orbit for $\mathcal{F}$ we can find $x \in X$ and $\sigma \in \Lambda^{\mathbb{Z}_+}$ satisfies, $d(x, x_0) < \epsilon$ and $d(\mathcal{F}_{\sigma_{n-1}}(x), x_n) < \epsilon$ for all $n \geq 1$. Let $\{y_i\}_{i\geq 0}$ be a $\delta-$pseudo orbit for $\mathcal{F}^k$. This means that for any $i \geq 0$ there exists $g_{\mu_i} \in \mathcal{F}^k$ such that $d(g_{\mu_i}(y_{ki}), y_{i+1}) < \delta$. For every $i \geq 0$ write

$$x_{ki} = y_i,\ x_{ki+j} = f_{\lambda^i_{j-1}} o...o f_{\lambda^i_0}(y_i),\ 0 < j < k.$$

Where $g_{\mu_i} = f_{\lambda^i_{k-1}} o...o f_{\lambda^i_0}$. So $\{x_n\}_{n\geq 0}$ is a $\delta-$pseudo orbit for $\mathcal{F}$ and can be $\epsilon-$traced by some point $x \in X$. Then there is $\sigma' \in \Lambda^{\mathbb{Z}_+}$ such that $d(\mathcal{F}_{\sigma'_n}(x), x_n) < \epsilon$

for all $n \geq 0$. This implies that $d(\mathcal{F}_{\sigma'_{ki}}(x), x_{ki}) < \epsilon$ for all $i \geq 0$. So $d(\mathcal{F}_{\sigma'_{ki}}(x), y_i) < \epsilon$ for all $i \geq 0$. Since $\mathcal{F}_{\sigma'_{ki}} \in \mathcal{F}^k$, for all $i \geq 0$ then $\{y_i\}_{i\geq 0}$ is $\epsilon-$shadowed by x in ***IFS*** $\mathcal{F}^k$.

$b-$ Suppose $\mathcal{F}^k$ has the shadowing property. Given $\epsilon > 0$, let δ be an ϵ modulus of shadowing property for $\mathcal{F}^k$. Since for all $\lambda \in \Lambda$, $f_\lambda : X \to X$ is a continuous map of a compact metric space, we can find $0 < \gamma < \delta$ such that each $\gamma-$finite pseudo orbit $\{y_j\}_{0\leq j\leq k}$ satisfies

$$d(f_{\lambda_j} o...o f_{\lambda_0}(y_0), y_{j+1}) < \frac{\delta}{2},\ 0 \leq j < k. \tag{1}$$

Where $d(f_{\lambda_j}(y_j), y_{j+1}) < \gamma$, for all $0 \leq j < k$.

Now, suppose $\{x_n\}_{n\geq 0}$ is a $\gamma-$pseudo orbit for $\mathcal{F}$.

Suppose $d(f_{\lambda_n}(x_n), x_{n+1}) < \gamma$ for all $n \geq 0$. Clearly, for fixed i, $\{x_{ki+m+j}\}_{0\leq j\leq k}$ is a $\gamma-$finite pseudo orbit for $\mathcal{F}$. Then

$$d(f_{\lambda_{ki+m+j}} o...o f_{\lambda_{ki}}(x_{ki+m}), x_{ki+m+j+1}) < \delta,\ 0 \leq j < k.$$

If $j = k-1$ then $d(f_{\lambda_{ki+m+k-1}} o...o f_{\lambda_{ki+m}}(x_{ki+m}), x_{ki+k+m}) < \delta$. Then $\{x_{ki+m}\}_{i\geq 0}$ is a $\delta-$pseudo orbit for $\mathcal{F}^k$ and can be $\epsilon-$traced by some point in ***IFS*** $\mathcal{F}^k$. □

4. Examples

In this section we are going to construct examples of iterated function systems and investigate the shadowing property for them. By using of Corollary 1 we show that the following **IFS** on S^1 has the shadowing property.

Next example shows that the Sierpinski's IFS has the shadowing property.

Example 1. *(Sierpinski Gasket) Consider the solid (filled) equilateral triangle with vertices at* $(0,0)$, $(1,0)$ *and* $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. *Now we define the following iterated function system on* X, *that called the Sierpinski IFS(see Figure* 1*)*[2].

$$f_1(\boldsymbol{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \boldsymbol{x}.$$

$$f_2(\boldsymbol{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

$$f_3(\boldsymbol{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{bmatrix}$$

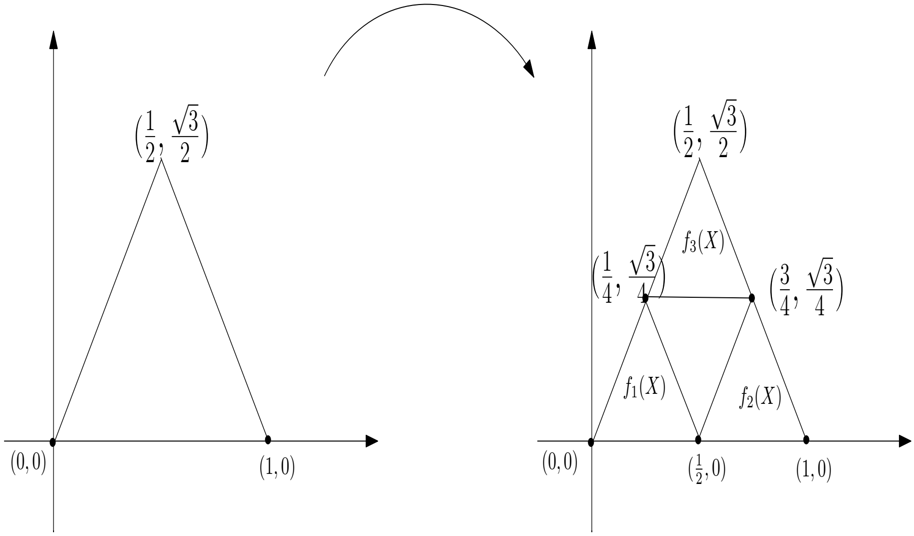

FIGURE 1. First step of Sierpinski's Gasket

This is clear that $\mathcal{F} = \{X; f_1, f_2, f_3\}$ is uniformly contracting and by Theorem 2.1. of [11] *has the shadowing property.*
Let S denote the Sierpinski Gasket generated by $\mathcal{F}$ (see [2]*,for more details), similar argument shows that the IFS $\mathcal{G} = \{S; f_1, f_2, f_3\}$ has the shadowing property.*

Example 2. [6] *Consider the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$. The natural distance on $\mathbb{R}$ induces a distance, d, on S^1. For $m \in \{2, 3\}$ define the map $f_m : S^1 \to S^1$ by*

$$f_m(x) = (mx)\text{mod}1.$$

For given $\epsilon > 0$ we put $\delta = \frac{\epsilon}{5}$. Let $x, y \in [0, 1)$ and $y \in B(f_2(x), \epsilon + \delta)$. There exist $t, s \in [0, 1)$ that $f_2(t) = y = f_2(s)$(if $y = 0$ then $t = s$, otherwise $t \neq s$). Without less of generality, we assume that $\mid 2s - 2x \mid \leq \epsilon + \delta$. This implies that $\mid s - x \mid \leq \frac{\epsilon+\delta}{2} < \epsilon$. So $y = f_2(s) \in f_2(B(x, \epsilon))$.
*Similarly $B(f_3(x), \epsilon + \delta) \subseteq f_3(B(x, \epsilon))$. By Corollary 1, the **IFS** $\mathcal{F} = \{S^1; f_2, f_3\}$ has the shadowing property.*

Example 3. *The sequence space on two symbols is the set*

$$\Sigma = \{(x_0, x_1, x_2, \cdots) \mid x_i = 0 \text{ or } 1\}.$$

Let $x = (x_0, x_1, x_2, \cdots)$ and $y = (y_0, y_1, y_2, \cdots)$ be two elements in Σ. We define the distance between them to be $\rho(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^{|i|}}$. This is well known that (Σ, ρ) is a compact metric space. Let $\sigma : \Sigma \longrightarrow \Sigma$ be the corresponding shift map, i.e. $(\sigma(x))_i = x_{i+1}$ and $\alpha_2 : \Sigma \to \Sigma$ be a map which is defined by

$$\alpha_2(x_0x_1x_2\cdots) = (x_1x_3x_4x_5\cdots).$$

*Take $\mathcal{F} = \{\Sigma; \sigma, \alpha_2\}$. Since σ and α_2 aren't injective, the **IFS** $\mathcal{F}$ isn't uniforming expanding. We show that $\mathcal{F}$ has the shadowing property.*
*Let $\epsilon > 0$. There exists an integer number, $N(\epsilon)$, that $\frac{1}{2^{N(\epsilon)}} \le \epsilon < \frac{1}{2^{N(\epsilon)-1}}$. Choose δ such that $0 < \delta < \frac{1}{2^{N(\epsilon)-1}} - \epsilon$. Let $x = (x_0x_1x_2x_3\cdots)$ be an arbitrary point in Σ. In [6], this is proved $B(\alpha_2(x), \epsilon + \delta) \subseteq \alpha_2(B(x, \epsilon))$ and $B(\sigma(x), \epsilon + \delta) \subseteq \sigma(B(x, \epsilon))$. By Corollary 1, the **IFS** $\mathcal{F} = \{\Sigma; \sigma, \alpha_2\}$ has the shadowing property.*

The following example shows that if for every $\lambda \in \Lambda$, $f_\lambda : X \to X$, as a discrete dynamical system, has the shadowing property then
$\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is not necessary an **IFS** with the shadowing property.

Example 4. *Let $X = [0, 1]$. This is clear that the constant function $f_1(x) = 1$ has the shadowing property and by [5] the tent map*

$$f_2(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2}, \\ -2x + 2 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

has the shadowing property. But we show that $\mathcal{F} = \{X; f_1, f_2\}$ does not have the shadowing property.
Let $\epsilon = \frac{1}{4}$ and δ be an arbitrary positive number. Then we can choose $0 < x_0 < \min\{\delta, \frac{1}{16}\}$ and natural number $n \ge 3$ such that $\frac{1}{8} \le f_2^{n-2}(x_0) < \frac{1}{4} \le f_2^{n-1}(x_0)$. So $d(f_2(1), x_0) < \delta$, $d(f_1(f_2^{n-2}(x_0)), 1) < \delta$ and $d(f_2(f_2^{j-1}(x_0), f_2^{j}(x_0)) < \delta$ for each $1 < j \le n-2$.
Thus we can construct a $\delta-$pseudo orbit by putting $\{x_i\}_{i\ge 0} = \{x_0, f_2(x_0), ..., f_2^{n-2}(x_0), 1, x_0, f_2(x_0), ..., f_2^{n-2}(x_0), 1, x_0, ...\}$. Consider a chain $\{y_i\}_{i\ge 0}$ and a sequence $\{\lambda_0, \lambda_1, ...\} \in \{1, 2\}^{\mathbb{Z}_+}$ such that $d(y_i, x_i) < \epsilon$ for all $0 \le i \le n-1$ and $y_{i+1} = f_{\lambda_i}(y_i)$ for all $i \ge 0$. Since $x_n = 1$, if $d(x_n, y_n) < \epsilon$ then $y_n = 1$ and this implies that $y_i = 0$ or 1, for all $i \ge n$. Therefore $d(y_{nk-1}, x_{nk-1}) \ge \epsilon$, for all $k > 1$. Thus $\{x_i\}_{i\ge 0}$ cannot be $\epsilon-$traced by $\{y_i\}_{i\ge 0}$. This finishes the proof.

In the next example we introduce a minimal **IFS** that has the shadowing property.

Example 5. *Take the following maps $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ given by*

$$f_1(x) = (\frac{1}{2} + 2\alpha)x - \alpha;\ f_2(x) = (\frac{1}{2} + 2\alpha)x + \frac{1}{2} - \alpha$$

*where $0 < \alpha < \frac{1}{4}$. This is clear that $\mathcal{G} = \{\mathbb{R}; f_1, f_2\}$ is an uniformly contracting **IFS**. By Theorem 7.1. of* [2] *and Lemma 4.1. of* [16]*, there exists a compact subset A of $\mathbb{R}$ with nonempty interior such that $f_1(A) \subset A$, $f_2(A) \subset A$ and $\mathcal{F} = \{A; f_1, f_2\}$ is minimal. But $\mathcal{F}$ is an uniformly contracting **IFS** and by Theorem 2.1. of* [11] *has the shadowing property.*

5. Chain mixing and topological mixing

In this section we will recall and extend some well-known general results in discrete dynamical systems for **IFS**s. Also, we give two non trivial examples of **IFS**s that are chain mixing.

Definition 3. The IFS $\mathcal{F}$ is topologically mixing if for every nonempty sets $U, V \subset X$, there is $N > 0$ such that for all $n > N$ we can find $\sigma \in \Lambda^{\mathbb{Z}_+}$ which $\mathcal{F}_{\sigma_n}(U) \cap V \neq \emptyset$.

Definition 4. We say that $\mathcal{F}$ is $\epsilon-$chain mixing, if there is an $N > 0$ such that for every $x, y \in X$ and $n > N$ there is an $\epsilon-$chain from x to y of length exactly n.

The IFS $\mathcal{F}$ is chain mixing if it is $\epsilon-$chain mixing, for every $\epsilon > 0$. In the next example we give an **IFS** consists two irrational rotation that is topological mixing.

Example 6. *Consider a circle S^1 with coordinate $x \in [0; 1)$ and we denote by d the metric on S^1 induced by the usual distance on the real line. Let $\pi(x) : \mathbb{R} \to S^1$ be the covering projection defined by the relations*

$$\pi(x) \in [0; 1) and\ \pi(x) = x(x\ mod\ 1)$$

with respect to the considered coordinates on S^1. Let $\pi : [0, 1] \to S^1$ be a map defined by $\pi(t) = (\cos(2\pi t), \sin(2\pi t))$ Let $F_1 : [0, 1] \to [0, 1]$ be a homeomorphism defined by $F_1(t) = t + \alpha$ and $F_2 : [0, 1] \to [0, 1]$ be a homeomorphism defined by $F_2(t) = t + \beta$, where $\alpha < \beta$ are two distinct positive numbers that $\beta - \alpha$ is irrational.
Assume that f_i is a homeomorphism on S^1 defined by
$f_i(\cos(2\pi t), \sin(2\pi t)) = (\cos(2\pi F_i(t)), \sin(2\pi F_i(t)))$, for $i \in \{0, 1\}$.

*Consider the **IFS*** $\mathcal{F} = \{S^1, f_\lambda | \lambda \in \{1,2\}\}$. *We show that* $\mathcal{F}$ *is chain mixing. Fix* $\epsilon > 0$. *For each sequence* $\sigma \in \Lambda^{\mathbb{Z}_+}$ *and* $n > 1$, $\mathcal{F}_{\sigma_n}(x) = x + k\alpha + (n-k)\beta$, *when* k *is the cardinality of the sets* $\{\lambda_i \mid \lambda_i = 1, 1 \le i \le n\}$. *Since* $\beta - \alpha$ *is irrational, similar argument to Example* 2 *in* [12] *shows that the set* $\{k\alpha + (n-k)\beta \text{ mod } 1 \mid n > 1,\ k \le n\}$ *is dense in* $[0,1]$. *So, for every* x, y *in* S^1 *we can find* $N_0 > 1$ *such that there exists an* $\epsilon-$*chain from* x *to* y *of length* n, *for all* $n > N_0$.

Theorem 7. *Suppose that the IFS* $\mathcal{F}$ *has the shadowing property then* $\mathcal{F}$ *is topologically mixing if and only if it is chain mixing.*

Proof. Let $\mathcal{F}$ be topologically mixing. Given $\epsilon > 0$ and $a,\ b \in X$. So, there exists $N > 0$ such that for every $n > N$ we can find $\sigma \in \Lambda^{\mathbb{Z}_+}$, which $\mathcal{F}_{\sigma_n}(B_\epsilon(a)) \cap B_\epsilon(b) \neq \emptyset$. Then, there is a chain of length n from a point in $B_\epsilon(a)$ to a point in $B_\epsilon(b)$. Thus, there is an $\epsilon-$chain from a to b of length $n+2$.
Conversely, suppose that $\mathcal{F}$ is chain mixing and has the shadowing property. Assume that U and V are two arbitrary non-empty sets. Take $a \in U$ and $b \in V$ and $\epsilon > 0$ such that $B_\epsilon(a) \subset U$ and $B_\epsilon(b) \subset V$. Consider $\delta > 0$ as ϵ modulus of shadowing. Since $\mathcal{F}$ is chain mixing, there exists $N > 0$ such that for every $n > N$, we can find $\sigma \in \Lambda^{\mathbb{Z}_+}$ which $x, \mathcal{F}_{\sigma_1}(x), ..., \mathcal{F}_{\sigma_n}(x), y$ is a $\delta-$chain from x to y. This implies that, for every $n > N$, we can find $\sigma \in \Lambda^{\mathbb{Z}_+}$ which $\mathcal{F}_{\sigma_n}(U) \cap V \neq \emptyset$.

□

Theorems 7 and 1 imply the following result.

Corollary 2. *If a parameterized **IFS*** $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ *is uniformly contracting, then chain mixing and topological mixing properties are equivalent.*

Theorems 7 and 2 imply the following result.

Corollary 3. *If a parameterized **IFS*** $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ *is uniformly expanding and if each function* $f_\lambda(\lambda \in \Lambda)$ *is surjective, then chain mixing and topological mixing properties are equivalent.*

References

[1] N. Aoki, K. Hiraide. Toplogical theory of dynamical systems. North-Holland Mathematical Library, 1994.

[2] M.F. Barnsley. Fractals everywhere, Academic Press, Boston 1988.

[3] M.F. Barnsley, A. Vince. The Conley attractors of an iterated function system. Bull. Aust. Math. Soc., Vol. 88, Number 2, 267-279 (2013).

[4] M.F. Barnsley, A. Vince. Fractal continuation of analytic (fractal) functions, ArXiv:1209.6100v1

[5] E.M. Coven, I. Kan, J.A. Yorke. Pseudo-orbit shadowing in the family of tent maps. Trans. Amer. Math. Soc., Vol. 308, 308: 227-241 (1988).

[6] M. Fatehi Nia, Weak Shadowing as a Generic Property for IFS's, Journal of Mathematical Extension, 2018.

[7] M. Fatehi Nia, Iterated function systems with the average shadowing property, Topology Proc.,Vol. 48, 261-275 (2016).

[8] M. Fatehi Nia, Asymptotic Average shadowing property and chain transitivity for multiple flow systems, Differential Equations and Dynamical Systems, (2016), doi:10.1007/s12591-016-0326-6.

[9] M. Fatehi Nia, Parameterized IFS with the asymptotic average shadowing property, Qual. Theory Dyn. Syst., Vol. 15, 367-381 (2016).

[10] M. Fatehi Nia, Adding machine maps and minimal sets for iterated function systems, J. Dyn. Syst. Geom. Theor.,Vol. 15, Number 1, 71-83 (2017).

[11] V. Glavan, V. Gutu. Shadowing in parameterized IFS, Fixed Point Theory. Vol. 7, Number 2, 263-274 (2006).

[12] J. Palis and W. de Melo, Geometric theory of dynamical systems, Springer, New York, 1982.

[13] K. Palmer. Shadowing in dynamical systems. Theory and applications, Kluwer Acad. Publ., Dordrecht, 2000.

[14] A. Raibekas. C^2-Iterated function systems on the circle and symbolic blender-like, PhD thesis, IMPA, Rio de Janeiro, Brazil 2011.

[15] K. Sakai. Various shadowing properties for positively expansive maps, Topology. Appl., Vol. 131, Number 10, 15-31 (2003).

[16] D.F. Younis. Robustly minimal iterated function systems on compact manifolds, Master thesis, University of Amsterdam, 2011.