

## SOME COINCIDENCE POINT RESULTS IN THE PRODUCT SPACE FOR SOLUTIONS OF THE FUZZY OF ORDINARY DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS SYSTEMS

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**ABSTRACT.** In this work, we will introduce the new classes of product spaces and also establish the existence of coincidence points in quasi-ordered fuzzy metric spaces. In order to investigate the integral equations and ordinary differential equations is solutions of the systems. Also study the property of mixed-monotone and comparable property of quasi-ordered fuzzy metric spaces in the product space. Our theorems improve and cover the corresponding recent results announced by Wu (2014) and Roldán et al. (2014).

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## 1. INTRODUCTION

The concept of coincidence points, fixed points and common fixed points has been studied many authors see [1]-[4]. Afterwards, Wu [5] studied the coincidence point and common fixed point in the product space and studied property of comparable or the property of mixed-monotone on feasible product space of quasi-ordered metric spaces. Kramosil and Michàlet [6] were first introduced the concept of fuzzy metric spaces. After that, the concept of fuzzy metric spaces modified by Gorge and Veeramani [7]. Grabiec[9] extended fixed point theorems of Banach and Edelstein are extended to fuzzy metric spaces in the sense of Kramosil and Michàlet. Gregori et al.[10] introduced a notion of precompact fuzzy metric space which provides several satisfactory and presented by Gorge and Veeramani [14]. After that, many authors have studied fuzzy metric spaces (see e.g.[9]-[12]). In 2000, Mihet[13] studied fixed point theorem in fuzzy metric spaces by using E.A. property. Afterwards, in 2014, Roldàn et al.[15] studied and modified multidimensional coincidence point results in fuzzy metric spaces for a fuzzy contractivity condition. In 2016, Khaofong et al.[16] studied and proved coincidence point theorems of multidimensional for  $\varphi$ -weak contraction in fuzzy metric spaces. In particular, Wu[5] presented results on property of mixed-monotone for functions in the product spaces and complete for mixed-monotonically in quasi-ordered metric spaces.

This paper, in Section 2, we are introduce the notions of complete for mixed-monotonically and complete for monotonically in quasi-ordered fuzzy metric spaces. In Section 3, we have obtained the coincidence point results in the product spaces of complete for mixed-monotonically in quasi-ordered fuzzy metric spaces and the coincidence point results in the product spaces of complete for monotonically in quasi-ordered fuzzy metric spaces. Finally, we are presented the application to examine the existence and chain-uniqueness of solutions for the ordinary differential equations system and also the integral equations system are given to support the results.

## 2. PRELIMINARIES

### 2.1. Triangular norms.

**Definition 2.1** ([8]). *Let  $* : [0, 1]^2 \rightarrow [0, 1]$  be a continuous  $t$ -norm if the following conditions are satisfied:*

- (i)  $*$  is associative and commutative,
- (ii)  $*$  is continuous,
- (iii)  $a * 1 = a$  for all  $a \in [0, 1]$ ,
- (iv) for all  $a, b, c, d \in [0, 1]$ ,  $a \leq c$  and  $b \leq d$  such that  $a * b \leq c * d$ .

If  $a_1, a_2, \dots, a_m \in [0, 1]$ , then  $*_{i=1}^m a_i = a_1 * a_2 * \dots * a_m$ .

For each  $a \in [0, 1]$ , the sequence  $\{*^m a\}_{m=1}^\infty$  is defined inductively by  $*^1 a = a$  and  $*^{m+1} a = (*^m a) * a$  for all  $m \geq 1$ .

**Definition 2.2.** A  $t$ -norm  $*$  is said to be of  $H$ -type if the sequence  $\{*^m a\}_{m=1}^\infty$  is equicontinuous at  $a = 1$ , i.e., for all  $\epsilon \in (0, 1)$ , there exists  $\eta \in (0, 1)$  such that if  $a \in (1 - \eta, 1]$ , then  $*^m a > 1 - \epsilon$  for all  $m \in \mathbb{N}$ .

The most important and well known continuous  $t$ -norm of  $H$ -type is  $* := \min$ . Other examples can be found in [13].

**Remark 2.3.** A  $t$ -norm  $*$  is called positive, if  $a * b > 0$  for all  $a, b \in (0, 1)$ .

## 2.2. Fuzzy metric spaces.

We shall study the basic definitions and properties about fuzzy metric spaces. The concept of fuzzy metric spaces is defined by George and Veeramani [7] as follows.

**Definition 2.4** ([7]). Let  $X$  be an arbitrary non-empty set,  $*$  is a continuous  $t$ -norm, and  $M$  is a fuzzy set on  $X \times X \times (0, \infty)$ . The 3-tuple  $(X, M, *)$  is called a fuzzy metric space if satisfying the following conditions, for each  $u, v, w \in X$  and  $t, s > 0$ ,

- (M1)  $M(u, v, t) > 0$ ,
- (M2)  $M(u, v, t) = 1 \iff u = v$ ,
- (M3)  $M(u, v, t) = M(v, u, t)$ ,
- (M4)  $M(u, v, t) * M(v, w, s) \leq M(u, w, t + s)$ ,
- (M5)  $M(u, v, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous.

In this case, we also say that  $(X, M)$  is a fuzzy metric space under  $*$ . In the sequel, we will only consider fuzzy metric spaces satisfying :

- (M6)  $\lim_{t \rightarrow \infty} M(u, v, t) = 1$  for all  $u, v \in X$ .

**Example 2.5** ([7]). Let  $(X, d)$  be a metric space. We define  $a * b = ab$  (or  $a * b = \min\{a, b\}$ ) for all  $a, b \in [0, 1]$ , and define  $M : X \times X \times (0, \infty) \rightarrow [0, 1]$  as

$$M(u, v, t) = \frac{t}{t + d(u, v)},$$

for all  $u, v \in X$  and  $t > 0$ . Then  $(X, M, *)$  is a fuzzy metric space. The fuzzy metric  $M$  induced by the metric  $d$  is said to be the standard fuzzy metric.

**Definition 2.6** ([7]). Let  $(X, M, *)$  be a fuzzy metric space.

- (1) A sequence  $\{u_n\}$  in  $X$  is said to be convergent to a point  $u \in X$  if  $\lim_{n \rightarrow \infty} M(u_n, u, t) = 1$  for all  $t > 0$ .
- (2) A sequence  $\{u_n\}$  in  $X$  is called a Cauchy sequence if, for each  $0 < \epsilon < 1$  and  $t > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $M(u_n, u_m, t) > 1 - \epsilon$  for each  $n, m \geq n_0$ .
- (3)  $(X, M, *)$  is said to be complete if every Cauchy sequence is convergent.
- (4)  $(X, M, *)$  is said to be compact if every sequence has a convergent subsequence.

**Lemma 2.7** ([9]). Let  $(X, M, *)$  be a fuzzy metric space. For all  $u, v \in X$ ,  $M(u, v, \cdot)$  is non-decreasing function.

**Definition 2.8** ([12]). Let  $(X, M, *)$  be a fuzzy metric space. Then the mapping  $M$  is said to be continuous on  $X \times X \times (0, \infty)$  if

$$\lim_{n \rightarrow \infty} M(u_n, v_n, t_n) = M(u, v, t),$$

when  $\{(u_n, v_n, t_n)\}$  is a sequence in  $X^2 \times (0, \infty)$  which converges to a point  $(u, v, t) \in X^2 \times (0, \infty)$ , i.e.,

$$\lim_{n \rightarrow \infty} M(u_n, u, t) = \lim_{n \rightarrow \infty} M(v_n, v, t) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} M(u, v, t_n) = M(u, v, t).$$

**Lemma 2.9** ([11]). If  $(X, M, *)$  be a fuzzy metric space, then  $M$  is a continuous function on  $X \times X \times (0, \infty)$ .

**Definition 2.10** ([15]). Let  $p \in \mathbb{N}$  and let  $(X, M, *)$  be a fuzzy metric space. A mapping  $G : X^p \rightarrow X$  is said to be continuous at a point  $\mathbf{y}_0 \in X^p$  if, for any sequence  $\{\mathbf{y}_m\}_{m \geq 0}$  in  $X^p$  converging to  $\mathbf{y}_0$ , the sequence  $\{G(\mathbf{y}_m)\}_{m \geq 0}$  converges to  $G(\mathbf{y}_0)$ . If  $G$  is continuous at each  $\mathbf{y}_0 \in X^p$ , then  $G$  is said continuous on  $X^p$ .

**Lemma 2.11** ([15]). Let  $(X, M, *)$  be a fuzzy metric space such that  $*$  is a  $t$ -norm of  $H$ -type, let  $k \in (0, 1)$  and let  $\{u_m^1\}_{m \geq 0}, \{u_m^2\}_{m \geq 0}, \dots, \{u_m^n\}_{m \geq 0}$  be  $n$  sequences

in  $X$  such that, for all  $m \geq 1$  and all  $t > 0$ ,

$$*_{i=1}^n M(u_m^i, u_{m+1}^i, kt) \geq *_{i=1}^n M(u_{m-1}^i, u_m^i, t).$$

Then  $\{u_m^1\}_{m \geq 0}, \{u_m^2\}_{m \geq 0}, \dots, \{u_m^n\}_{m \geq 0}$  are Cauchy sequences.

### 2.3. The product spaces and coincidence points.

Let  $X$  be a non-empty set. We consider the product set

$$X^m = \underbrace{X \times X \times \dots \times X}_m \text{ times}.$$

The vector notation  $\mathbf{u} = (u^{(1)}, u^{(2)}, \dots, u^{(m)})$  is the element of  $X^m$ , where  $u^{(i)} \in X$  for  $i = 1, 2, \dots, m$ . In the sequel, let the mapping  $T : X^m \rightarrow X^m$  defined by

$$T(\mathbf{u}) = (T_1(\mathbf{u}), T_2(\mathbf{u}), \dots, T_m(\mathbf{u})),$$

where the function  $T_k : X^m \rightarrow X$ ,  $\forall k = 1, 2, \dots, m$ . The vector element  $\hat{\mathbf{u}} = (\hat{u}^{(1)}, \hat{u}^{(2)}, \dots, \hat{u}^{(m)}) \in X^m$  is called a fixed point of  $T$  iff  $T(\hat{\mathbf{u}}) = \hat{\mathbf{u}}$ , that is,

$$T_k(\hat{u}^{(1)}, \hat{u}^{(2)}, \dots, \hat{u}^{(m)}) = \hat{u}^{(k)},$$

$\forall k = 1, 2, \dots, m$ .

**Definition 2.12** ([5]). Let  $X$  be a non-empty set. We consider the mappings  $T : X^m \rightarrow X^m$  and  $F : X^m \rightarrow X^m$  defined by

$$T(\mathbf{u}) = (T_1(\mathbf{u}), T_2(\mathbf{u}), \dots, T_m(\mathbf{u})),$$

$$F(\mathbf{u}) = (F_1(\mathbf{u}), F_2(\mathbf{u}), \dots, F_m(\mathbf{u})),$$

where the function  $T_k, F_k : X^m \rightarrow X$  for all  $k = 1, 2, \dots, m$ .

- A point  $\hat{\mathbf{u}} \in X^m$  is a coincidence point of  $T$  and  $F$  iff  $T(\hat{\mathbf{u}}) = F(\hat{\mathbf{u}})$ , i.e.,  $T_k(\hat{\mathbf{u}}) = F_k(\hat{\mathbf{u}})$ ,  $\forall k = 1, 2, \dots, m$ .
- A point  $\hat{\mathbf{u}}$  is a common fixed point of  $T$  and  $F$  iff  $T(\hat{\mathbf{u}}) = F(\hat{\mathbf{u}}) = \hat{\mathbf{u}}$ , i.e.,  $T_k(\hat{\mathbf{u}}) = F_k(\hat{\mathbf{u}}) = \hat{u}^{(k)}$ ,  $\forall k = 1, 2, \dots, m$ .

- We say that  $T$  and  $F$  are commutative functions iff  $F(T(\mathbf{u})) = T(F(\mathbf{u}))$  for any  $\mathbf{u} \in X^m$ .

## 2.4. Mixed-monotone mapping and quasi-ordered set.

The set  $(X, \preceq)$  is called a quasi-ordered set iff the binary relation  $\preceq$  defined on  $X$  is reflexive and transitive.

For all  $\mathbf{u}, \mathbf{v} \in X^m$ , the elements  $\mathbf{u}$  and  $\mathbf{v}$  are  $\preceq$ -mixed-comparable iff for any  $k = 1, \dots, m$ ,  $u^{(k)} \preceq v^{(k)}$  or  $v^{(k)} \preceq u^{(k)}$ . Let  $A \subset \{1, 2, \dots, m\}$  and  $B = \{1, 2, \dots, m\} \setminus A$ . We called that  $A$  and  $B$  are disjoint pair of  $\{1, 2, \dots, m\}$ . The binary relation on  $X^m$  define by:

$$\mathbf{u} \preceq_A \mathbf{v} \iff u^{(k)} \preceq v^{(k)} \text{ for } k \in A$$

$$(1) \quad \text{and } v^{(k)} \preceq u^{(k)} \text{ for } k \in B.$$

It is clear that  $(X^m, \preceq_A)$  is a quasi-ordered product set based on  $A$ . Also, we get  $\mathbf{u} \preceq_A \mathbf{v} \iff \mathbf{v} \preceq_B \mathbf{u}$ . We have to say about  $A$  or  $B$  is allowed to be an empty set. (see [5])

**Definition 2.13** ([5]). Suppose that the disjoint pair  $A$  and  $B$  of  $\{1, 2, \dots, m\}$ . Let  $(X, \preceq)$  be a quasi-ordered set, we consider  $(X^m, \preceq_A)$  is a quasi-ordered set, defined by (1).

- $\{u_n\}_{n \in \mathbb{N}} \in X$  is a sequence of  $\preceq$ -mixed-monotone iff  $u_n \preceq u_{n+1}$  or  $u_{n+1} \preceq u_n$  for all  $n \in \mathbb{N}$ .
- $\{\mathbf{u}_n\}_{n \in \mathbb{N}} \in X^m$  is a sequence of  $\preceq$ -mixed-monotone iff each sequence  $\{u_n^{(k)}\}_{n \in \mathbb{N}} \in X$  is a sequence of  $\preceq$ -mixed-monotone,  $\forall k = 1, \dots, m$ .
- $\{\mathbf{u}_n\}_{n \in \mathbb{N}} \in X^m$  is a sequence of  $\preceq_A$ -mixed-monotone iff  $\mathbf{u}_n \preceq_A \mathbf{u}_{n+1}$  or  $\mathbf{u}_{n+1} \preceq_A \mathbf{u}_n$  for all  $n \in \mathbb{N}$ .

**Definition 2.14** ([5]). Suppose that the disjoint pair  $A$  and  $B$  of  $\{1, 2, \dots, m\}$ . Let  $(X, \preceq)$  be a quasi-ordered set, consider  $(X^m, \preceq_A)$  is a quasi-ordered set, defined by (1) and the function  $F : (X^m, M) \rightarrow (X^m, M)$ .

- $F$  have the property of sequentially  $\preceq$ -mixed-monotone iff for any sequence  $\{\mathbf{u}_n\}_{n \in \mathbb{N}} \in X^m$  is a  $\preceq$ -mixed-monotone, then  $\{F(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is also a sequence of  $\preceq$ -mixed-monotone.
- $F$  have the property of sequentially  $\preceq_A$ -mixed-monotone iff for any sequence  $\{\mathbf{u}_n\}_{n \in \mathbb{N}} \in X^m$  is a  $\preceq_A$ -mixed-monotone, then  $\{F(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is also a sequence of  $\preceq_A$ -mixed-monotone.

Let  $X$  be a non-empty set. We consider a mappings  $T : X^m \rightarrow X^m$  and  $F : X^m \rightarrow X^m$  satisfying  $T^p(X^m) \subseteq F(X^m)$ ,  $\exists p \in \mathbb{N}$ , where  $T^p(\mathbf{u}) = T(T^{p-1}(\mathbf{u}))$  for any  $\mathbf{u} \in X^m$ . Thus, we have  $T_k^p(\mathbf{u}) = T_k(T^{p-1}(\mathbf{u}))$  for  $k = 1, \dots, m$ . Let an initial element  $\mathbf{u}_0 = (u_0^{(1)}, u_0^{(2)}, \dots, u_0^{(m)}) \in X^m$ , where  $u_0^{(k)} \in X$  for  $k = 1, \dots, m$ , since  $T^p(X^m) \subseteq F(X^m)$ , there exists  $\mathbf{u}_1 \in X^m$  such that  $F(\mathbf{u}_1) = T^p(\mathbf{u}_0)$ . Similarly, there also exists  $\mathbf{u}_2 \in X^m$  such that  $F(\mathbf{u}_2) = T^p(\mathbf{u}_1)$ . By continue this process, we can create a sequence  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  such that

$$(2) \quad F(\mathbf{u}_n) = T^p(\mathbf{u}_{n-1}).$$

for all  $n \in \mathbb{N}$ ; that is,

$$F_k(\mathbf{u}_n) = F_k(u_n^{(1)}, \dots, u_n^{(k)}, \dots, u_n^{(m)}) = T_k^p(u_{n-1}^{(1)}, \dots, u_{n-1}^{(k)}, \dots, u_{n-1}^{(m)}) = T_k^p(\mathbf{u}_{n-1}),$$

$\forall k = 1, \dots, m$ . The concept of seed element for mixed-monotone, see [5].

Let  $(X, \preceq)$  be a quasi-ordered set, and  $A$  and  $B$  be a disjoint pair of  $\{1, 2, \dots, m\}$ . We consider the function  $F_k : X^m \rightarrow X$ .

- The function  $F_k$  is  $(\preceq_A, \preceq)$  – increasing  $\iff \mathbf{u} \preceq_A \mathbf{v}$  implies  $F_k(\mathbf{u}) \preceq F_k(\mathbf{v})$ .
- The function  $F_k$  is  $(\preceq_A, \preceq)$  – decreasing  $\iff \mathbf{u} \preceq_A \mathbf{v}$  implies  $F_k(\mathbf{v}) \preceq F_k(\mathbf{u})$ .

From (1), we have  $F_k$  is  $(\preceq_A, \preceq)$  – decreasing  $\iff F_k$  is  $(\preceq_B, \preceq)$  – increasing, and  $F_k$  is  $(\preceq_A, \preceq)$  – increasing  $\iff F_k$  is  $(\preceq_B, \preceq)$  – decreasing.

**Definition 2.15** ([5]). Let  $(X, \preceq)$  be a quasi-ordered set and a disjoint pair  $A$  and  $B$  of  $\{1, 2, \dots, m\}$ . Consider  $(X^m, \preceq_A)$  is a quasi-ordered set and the mapping  $T : (X^m, \preceq_A) \rightarrow (X^m, \preceq_A)$ .

- The function  $T$  is  $(\preceq_A, \preceq_A)$  – increasing  $\iff \mathbf{u} \preceq_A \mathbf{v}$  implies  $T(\mathbf{u}) \preceq_A T(\mathbf{v})$ .
- The function  $T$  is  $(\preceq_A, \preceq_A)$  – decreasing  $\iff \mathbf{u} \preceq_A \mathbf{v}$  implies  $T(\mathbf{v}) \preceq_A T(\mathbf{u})$ .

**Lemma 2.16** ([5]). *Let  $(X, \preceq)$  be a quasi-ordered set and a disjoint pair  $A$  and  $B$  of  $\{1, 2, \dots, m\}$ . Suppose that the mapping  $T : (X^m, \preceq_A) \rightarrow (X^m, \preceq_A)$  has the property of mixed-monotone on  $X^m$ . Let  $\mathbf{u}_0 \in X^m$  be an initial element, we define the sequence  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  by  $\mathbf{u}_n = T(\mathbf{u}_{n-1})$ . The following statements hold true.*

- (i) *Assume that  $\mathbf{u}_0 \preceq_A \mathbf{u}_1$ , for any  $k \in A$ ,  $\{u_n^{(k)}\}_{n \in \mathbb{N}}$  are  $\preceq$ -increasing and for any  $k \in B$  are  $\preceq$ -decreasing. Otherwise,  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  is a sequence of  $\preceq_A$ -increasing.*
- (ii) *Assume that  $\mathbf{u}_1 \preceq_A \mathbf{u}_0$ , for any  $k \in A$ ,  $\{u_n^{(k)}\}_{n \in \mathbb{N}}$  are  $\preceq$ -decreasing and for any  $k \in B$  are  $\preceq$ -increasing. Otherwise,  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  is a sequence of  $\preceq_A$ -decreasing. In addition, the element  $\mathbf{u}_0$  is an initial element and a seed element for  $\preceq_A$ -monotone in  $X^m$ .*

### 3. MAIN RESULTS

Before proving our main theorem, we introduce the following notion:

**Definition 3.1.** *Let  $(X, M, *, \preceq)$  be a quasi-ordered fuzzy metric spaces.  $(X, M, *, \preceq)$  is complete for mixed-monotonically if and only if for every Cauchy sequence  $\{u_n\}_{n \in \mathbb{N}}$  in  $X$  of  $\preceq$ -mixed-monotone is convergent.*

It's clear that if  $(X, M, *, \preceq)$  is a quasi-ordered fuzzy metric spaces complete, then  $(X, M, *, \preceq)$  is also complete for mixed-monotonically. However, the converse is not necessarily true.

Let  $(X^m, \mathcal{M}, *)$  be a product fuzzy metric space induced by the original fuzzy metric  $M$ . Define by, the following functions for  $\mathbf{u}, \mathbf{v} \in X^m$ ,

$$(4) \quad \mathcal{M}(\mathbf{u}, \mathbf{v}, t) = \min_{k=1, \dots, m} \{M(u^{(k)}, v^{(k)}, t)\}$$

and we have the following

$$(5) \quad \mathcal{M}(\mathbf{u}, \mathbf{v}, t) \geq *_{k=1}^m M(u^{(k)}, v^{(k)}, t).$$

So, we have  $(X^m, \mathcal{M})$  be a product fuzzy metric space. For the product fuzzy metric  $\mathcal{M}$ , we consider the following notions.

- In preserving convergence sense, the fuzzy metrics  $\mathcal{M}$  and  $M$  are compatible iff let  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  be a sequence in  $X^m$ , the following statement holds:  
 $\mathcal{M}(\mathbf{u}_n, \hat{\mathbf{u}}, t) \rightarrow 1 \iff M(u_n^{(k)}, \hat{u}^{(k)}, t) \rightarrow 1, \forall k = 1, \dots, m.$
- In preserving continuity sense, the fuzzy metrics  $\mathcal{M}$  and  $M$  are compatible iff,  $\forall \epsilon > 0, \exists \mathfrak{k} > 0$  is a positive constant, the following statement holds:  
 $\mathcal{M}(\mathbf{u}, \mathbf{v}, t) > 1 - \epsilon \iff M(u^{(k)}, v^{(k)}, t) > \mathfrak{k}(1 - \epsilon), \forall k = 1, \dots, m.$

We can check  $\mathcal{M}$  and  $M$  are compatible with defined in (4) or (5) for the sense of preserving convergence and preserving continuity.

**Proposition 3.2.** *The fuzzy metrics  $\mathcal{M}$  and  $M$  are compatible in preserving continuity sense if and only if  $\mathcal{M}$  and  $M$  are compatible in preserving convergence sense.*

*Proof.* Suppose that  $\mathcal{M}(\mathbf{u}_n, \hat{\mathbf{u}}, t) \rightarrow 1$ . For any  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $\mathcal{M}(\mathbf{u}_n, \hat{\mathbf{u}}, t) > (1 - \epsilon)/\mathfrak{k}$ , for all  $n \geq n_0$ , i.e.,  $M(u_n^{(k)}, \hat{u}^{(k)}, t) > 1 - \epsilon$ ,  $\forall k = 1, 2, \dots, m$  and  $n \geq n_0$ . By definition of converse, for any  $\epsilon > 0$ , there exists  $n_0^{(k)} \in \mathbb{N}$ , thus  $M(u_n^{(k)}, \hat{u}^{(k)}, t) > \mathfrak{k} \cdot (1 - \epsilon)$ ,  $\forall n \geq n_0^{(k)}$  and  $k = 1, \dots, m$ . Let

$$n_0 = \max_{k=1, \dots, m} n_0^{(k)}.$$

Such that  $M(\hat{u}_n^{(k)}, \hat{u}^{(k)}, t) > \mathfrak{k} \cdot (1 - \epsilon)$ ,  $\forall n \geq n_0$  and  $k = 1, \dots, m$ , i.e.,  $\mathcal{M}(\mathbf{u}_n, \hat{\mathbf{u}}, t) > 1 - \epsilon$  for all  $n \geq n_0$ .  $\square$

Let  $(X, M)$  be a fuzzy metric space, and the mapping  $T : (X^m, \mathcal{M}) \rightarrow (X^m, \mathcal{M})$  be a defined on  $(X^m, \mathcal{M})$ . If  $T$  is continuous at a point  $\hat{\mathbf{u}} \in X^m$ , then, for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $\mathbf{u} \in X^m$  with  $\mathcal{M}(\hat{\mathbf{u}}, \mathbf{u}, t) > 1 - \delta$  implies  $\mathcal{M}(T(\hat{\mathbf{u}}), T(\mathbf{u}), t) > 1 - \epsilon$ . Assume that  $\mathcal{M}$  and  $M$  are compatible. Then  $T$  is continuous at a point  $\hat{\mathbf{u}} \in X^m$  if and only if each  $T_k$  is continuous at a point  $\hat{\mathbf{u}}$ ,  $\forall k = 1, \dots, m$ . It's clear that if  $T$  is continuous at a point  $\hat{\mathbf{u}} \in X^m$ , then each  $T_k$  is continuous at a point  $\hat{\mathbf{u}}$  for  $k = 1, 2, \dots, m$ . For any  $\epsilon > 0$ , there exists  $\delta_k > 0$  such that  $\mathcal{M}(\hat{\mathbf{u}}, \mathbf{u}, t) > 1 - \delta_k$  implies  $M(T_k(\hat{\mathbf{u}}), T_k(\mathbf{u}), t) > \mathfrak{k} \cdot (1 - \epsilon)$ ,  $\forall k = 1, \dots, m$ . Let

$$\delta = \min_{k=1, \dots, m} \delta_k.$$

It follows that  $\mathcal{M}(\hat{\mathbf{u}}, \mathbf{u}, t) > 1 - \delta$  implies  $M(T_k(\hat{\mathbf{u}}), T_k(\mathbf{u}), t) > \mathfrak{k} \cdot (1 - \epsilon)$ ,  $\forall k = 1, \dots, m$ , (i.e.,  $\mathcal{M}(T(\hat{\mathbf{u}}), T(\mathbf{u}), t) > 1 - \epsilon$ ). Next, we present another notion of continuity.

**Definition 3.3.** Let  $(X, M)$  be a fuzzy metric space and  $(X^m, \mathcal{M})$  be the corresponding product fuzzy metric spaces. Let  $T : (X^m, \mathcal{M}) \rightarrow (X^m, \mathcal{M})$  and  $F : (X^m, \mathcal{M}) \rightarrow (X^m, \mathcal{M})$  are functions defined on  $(X^m, \mathcal{M})$ . The function  $T$  is continuous at  $\hat{\mathbf{u}} \in X^m$  with respect to  $F$  if and only if,  $\forall \epsilon > 0, \exists \delta > 0$  such that  $\mathbf{u} \in X^m$  with  $\mathcal{M}(\hat{\mathbf{u}}, F(\mathbf{u}), t) > 1 - \delta$  implies  $\mathcal{M}(T(\hat{\mathbf{u}}), T(\mathbf{u}), t) > 1 - \epsilon$ . The function  $T$  is continuous on  $X^m$  with respect to  $F$  if and only if  $T$  is continuous at each  $\hat{\mathbf{u}} \in X^m$ .

It's clear that, for  $F$  is an identity function, if  $T$  is continuous at a point  $\hat{\mathbf{u}}$ , then  $T$  is also continuous at a point  $\hat{\mathbf{u}}$ .

**Proposition 3.4.** Let  $(X, M)$  be a fuzzy metric space, and let  $T : (X^m, \mathcal{M}) \rightarrow (X^m, \mathcal{M})$  and  $F : (X^m, \mathcal{M}) \rightarrow (X^m, \mathcal{M})$  are functions defined on  $(X^m, \mathcal{M})$ . Assume that the fuzzy metric  $\mathcal{M}$  and  $M$  are compatible in preserving continuity sense. Then  $T$  is continuous at  $\hat{\mathbf{u}} \in X^m$  with respect to  $F$  if and only if,  $\forall \epsilon > 0, \exists \delta > 0$  such that  $\mathbf{u} \in X^m$  with  $M(\hat{u}^{(k)}, F_k(\mathbf{u}), t) > 1 - \delta$ , implies that  $M(T_k(\hat{\mathbf{u}}), T_k(\mathbf{u}), t) > 1 - \epsilon, \forall k = 1, \dots, m$ .

*Proof.* Assume that the function  $T$  is continuous a point  $\hat{\mathbf{u}}$ . By definition,  $\forall \epsilon > 0, \exists \hat{\delta} > 0$  such that  $\mathbf{u} \in X^m$  with  $\mathcal{M}(\hat{\mathbf{u}}, F(\mathbf{u}), t) > 1 - \hat{\delta}$  implies  $\mathcal{M}(T(\hat{\mathbf{u}}), T(\mathbf{u}), t) > (1 - \epsilon)/\mathfrak{k}$ . Let  $1 - \delta = \mathfrak{k} \cdot (1 - \hat{\delta})$ . Hereinafter,  $M(\hat{u}^{(k)}, F_k(\mathbf{u}), t) > 1 - \delta \iff \mathcal{M}(\hat{\mathbf{u}}, F(\mathbf{u}), t) > 1 - \hat{\delta}, \forall k = 1, \dots, m$ , which implies  $\mathcal{M}(T(\hat{\mathbf{u}}), T(\mathbf{u}), t) > (1 - \epsilon)/\mathfrak{k}$ , (i.e.,  $M(T_k(\hat{\mathbf{u}}), T_k(\mathbf{u}), t) > 1 - \epsilon, \forall k = 1, \dots, m$ ). For the opposite,  $\forall \epsilon > 0, \exists \delta^\circ > 0$  such that  $M(\hat{u}^{(k)}, F_k(\mathbf{u}), t) > 1 - \delta^\circ, \forall k = 1, \dots, m$  implies  $M(T_k(\hat{\mathbf{u}}), T_k(\mathbf{u}), t) > \mathfrak{k} \cdot (1 - \epsilon), \forall k = 1, \dots, m$ . Let  $1 - \delta = (1 - \delta^\circ)/\mathfrak{k}$ . It follows that  $\mathcal{M}(\hat{\mathbf{u}}, F(\mathbf{u}), t) > 1 - \delta \iff M(\hat{u}^{(k)}, F_k(\mathbf{u}), t) > 1 - \delta^\circ, \forall k = 1, \dots, m$ , which implies that  $M(T_k(\hat{\mathbf{u}}), T_k(\mathbf{u}), t) > \mathfrak{k} \cdot (1 - \epsilon), \forall k = 1, \dots, m$ , (i.e.,  $\mathcal{M}(T(\hat{\mathbf{u}}), T(\mathbf{u}), t) > 1 - \epsilon$ ).  $\square$

**Lemma 3.5.** Let  $(X, M)$  be a fuzzy metric space. If  $u_n \rightarrow u$  as  $n \rightarrow \infty$  with respect to the fuzzy metric  $M$ , then, given any fixed  $y \in X, M(u_n, v, t) \rightarrow M(u, v, t)$  as  $n \rightarrow \infty$ .

We are now in the position to state and prove our main result on fixed point and coincidence point theorems and common fixed point theorems.

### 3.1. Fixed point result.

**Theorem 3.6.** *Let  $X$  be a non-empty set. Assume that the quasi-ordered fuzzy metric spaces  $(X, M, *, \preceq)$  is complete for mixed-monotonically. Suppose that the fuzzy metric  $\mathcal{M}$  and  $M$  compatible in preserving continuity sense. We consider the mappings  $T : (X^m, \mathcal{M}) \rightarrow (X^m, \mathcal{M})$  and  $F : (X^m, \mathcal{M}) \rightarrow (X^m, \mathcal{M})$  satisfying  $T^p(X^m) \subseteq F(X^m)$ ,  $\exists p \in \mathbb{N}$ . Let  $\mathbf{u}_0 \in X^m$  be a seed element of  $\preceq$ -mixed-monotone. Suppose that the functions  $T$  and  $F$  satisfying the following conditions:*

- $F$  has the property of sequentially  $\preceq$ -mixed-monotone;
- $T^p$  is continuous on  $X^m$  with respect to  $F$ ;
- each  $F_k$  is continuous on  $X^m$  for all  $k = 1, \dots, m$ .

Assume that there exists  $\beta \in (0, 1)$  such that, for each  $k = 1, \dots, m$ ,

$$(6) \quad M(T_k^p(\mathbf{u}), T_k^p(\mathbf{v}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}), F_i(\mathbf{v}), t)),$$

for all  $t > 0$  and for any two elements  $\mathbf{u}, \mathbf{v} \in X^m$  be a  $\preceq$ -mixed-comparable, where  $\gamma : [0, 1] \rightarrow [0, 1]$  continuous  $*^m \gamma(a) \geq a$  for each  $a \in [0, 1]$ . Then  $T^p$  has a fixed point  $\hat{\mathbf{u}}$  such that  $\lim_{n \rightarrow \infty} \{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}} = \hat{u}^{(k)}$  for each component  $\hat{u}^{(k)}$  of  $\hat{\mathbf{u}}$  constructed in  $F(\mathbf{u}_n) = T^p(\mathbf{u}_{n-1})$  for all  $k = 1, 2, \dots, m$ .

*Proof.* Consider  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  be a sequence define by constructed (2). Since  $\mathbf{u}_0 \in X^m$  be a seed element of  $\preceq$ -mixed-monotone, i.e.,  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  is a sequence of  $\preceq$ -mixed-monotone, for any  $n \in \mathbb{N}$ ,  $\mathbf{u}_n$  and  $\mathbf{u}_{n+1}$  are  $\preceq$ -mixed-comparable. For any  $t$  and all  $m$ , we define

$$\delta_n(t) := *_{i=1}^m M(F_i(\mathbf{u}_n), F_i(\mathbf{u}_{n-1}), t).$$

We claim that  $\delta_{n+1}(\beta t) \geq \delta_n(t)$  for all  $m$  and all  $t > 0$ .

By the condition (6), we have

$$\begin{aligned} M(F_k(\mathbf{u}_{n+1}), F_k(\mathbf{u}_n), \beta t) &= M(T_k^p(\mathbf{u}_n), T_k^p(\mathbf{u}_{n-1}), \beta t) \\ &\geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}_n), F_i(\mathbf{u}_{n-1}), t)) \end{aligned}$$

$$= \gamma(\delta_n(t)).$$

Since  $*^m\gamma(a) \geq a$  for each  $a \in [0, 1]$ , it follows that

$$\begin{aligned} \delta_{n+1}(\beta t) &= *_{i=1}^m M(F_i(\mathbf{u}_{n+1}), F_i(\mathbf{u}_n), \beta t) \\ &\geq *_{i=1}^m (\gamma(\delta_n(t))) \\ &\geq \delta_n(t) \quad \text{for all } t > 0. \end{aligned}$$

In particular, this means that the sequences  $\{F_1(\mathbf{u}_n)\}_{n \in \mathbb{N}}$ ,  $\{F_2(\mathbf{u}_n)\}_{n \in \mathbb{N}}$ ,  $\dots$ ,  $\{F_m(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  verify

$$*_{i=1}^m M(F_i(\mathbf{u}_{n+1}), F_i(\mathbf{u}_n), \beta t) = \delta_{n+1}(\beta t) \geq \delta_n(t) = *_{i=1}^m M(F_i(\mathbf{u}_n), F_i(\mathbf{u}_{n-1}), t).$$

By Lemma 2.11, we have that  $\{F_1(\mathbf{u}_n)\}_{n \in \mathbb{N}}$ ,  $\{F_2(\mathbf{u}_n)\}_{n \in \mathbb{N}}$ ,  $\dots$ ,  $\{F_m(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  are Cauchy sequences. Since the function  $F$  has the property of sequentially  $\preceq$ -mixed-monotone, i.e.,  $\{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a Cauchy sequence of  $\preceq$ -mixed-monotone for  $k = 1, \dots, m$ , by  $X$  have completeness for  $\preceq$ -mixed-monotone, there exists  $\hat{u}^{(k)} \in X$  such that  $\lim_{n \rightarrow \infty} F_k(\mathbf{u}_n) = \hat{u}^{(k)}$  for  $k = 1, \dots, m$ . Form the fuzzy metrics  $\mathcal{M}$  and  $M$  are compatible in preserving continuity sense, by Proposition 3.2, it follows that  $\lim_{n \rightarrow \infty} F(\mathbf{u}_n) = \hat{\mathbf{u}}$ . Since each  $F_k$  is continuous on  $X^m$ , we also have  $\lim_{n \rightarrow \infty} F_k(F(\mathbf{u}_n)) = F_k(\hat{\mathbf{u}})$ , for all  $k = 1, 2, \dots, m$ .

Since  $T^p$  is continuous on  $X^m$ , by Proposition 3.4,  $\forall \epsilon > 0$ ,  $\exists \delta > 0$  such that  $\mathbf{u} \in X^m$  with  $M(\hat{u}^{(k)}, F_k(\mathbf{u}), t) > 1 - \delta$  for  $k = 1, \dots, m$  implies

$$(7) \quad M(T_k^p(\hat{\mathbf{u}}), T_k^p(\mathbf{u}), t) > 1 - \epsilon.$$

Since  $\lim_{n \rightarrow \infty} F_k(\mathbf{u}_n) = \hat{u}^{(k)}$ ,  $\forall k = 1, \dots, m$ , given  $\alpha := \min\{\epsilon, \delta\}$ , there exists  $n_0 \in \mathbb{N}$  such that

$$(8) \quad M(F_k(\mathbf{u}_n), \hat{u}^{(k)}, t) > 1 - \alpha \geq 1 - \delta,$$

for any  $n \in \mathbb{N}$  with  $n \geq n_0$ ,  $\forall k = 1, \dots, m$ .

For any  $n \geq n_0$  by (7) and (8), it follows that

$$(9) \quad M(T_k^p(\hat{\mathbf{u}}), T_k^p(\mathbf{u}_n), t) > 1 - \delta, \quad \forall k = 1, \dots, m$$

So, we get

$$\begin{aligned} M(T_k^p(\hat{\mathbf{u}}), \hat{u}^{(k)}, t) &\geq M(T_k^p(\hat{\mathbf{u}}), F_k(\mathbf{u}_{n_0+1}), s) * M(F_k(\mathbf{u}_{n_0+1}), \hat{u}^{(k)}, w) \\ &= M(T_k^p(\hat{\mathbf{x}}), T_k^p(\mathbf{x}_{n_0}), s) * M(F_k(\mathbf{x}_{n_0+1}), \hat{x}^{(k)}, w) \\ &> (1 - \delta) * (1 - \delta), \end{aligned}$$

for  $s, w > 0$  with  $t = s + w$ .

Since  $*$  is a  $t$ -norm of  $H$ -type,  $\forall \epsilon > 0, \exists \delta > 0$  such that

$$M(T_k^p(\hat{\mathbf{u}}), \hat{u}^{(k)}, t) > 1 - \epsilon, \quad \forall k = 1, \dots, m.$$

Since  $\epsilon > 0$ , we conclude that  $M(T_k^p(\hat{\mathbf{u}}), \hat{u}^{(k)}, t) = 1$  for  $k = 1, \dots, m$ , which also says that  $T_k^p(\hat{\mathbf{u}}) = \hat{u}^{(k)}$ ,  $\forall k = 1, \dots, m$ , (i.e.  $T^p(\hat{\mathbf{u}}) = \hat{\mathbf{u}}$ ).  $\square$

**Theorem 3.7.** *Let  $X$  be a non-empty set. Assume that the quasi-ordered fuzzy metric spaces  $(X, M, *, \preceq)$  is complete for mixed-monotonically and the fuzzy metric  $\mathcal{M}$  and  $M$  compatible in preserving continuity sense. Let  $A$  and  $B$  be any disjoint pair of  $\{1, 2, \dots, m\}$ . We consider the mappings  $T : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  and  $F : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  satisfying  $T^p(X^m) \subseteq F(X^m)$ ,  $\exists p \in \mathbb{N}$ . Let  $\mathbf{u}_0 \in X^m$  be a seed element of  $\preceq_A$ -mixed-monotone. Suppose that  $T$  and  $F$  satisfying the following conditions:*

- $F$  has the property of sequentially  $\preceq$ -mixed-monotone;
- $T^p$  is continuous on  $X^m$  with respect to  $F$ ;
- each  $F_k$  is continuous on  $X^m$  for  $k = 1, \dots, m$ .

Assume that there exists  $\beta \in (0, 1)$  such that, for each  $k = 1, \dots, m$ ,

$$(10) \quad M(T_k^p(\mathbf{u}), T_k^p(\mathbf{v}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}), F_i(\mathbf{v}), t)),$$

for all  $t > 0$  and for any two elements  $\mathbf{u}, \mathbf{v} \in X^m$  be a  $\preceq$ -mixed-comparable, where  $\gamma : [0, 1] \rightarrow [0, 1]$  continuous  $*^m \gamma(a) \geq a$  for each  $a \in [0, 1]$ . Then  $T^p$  has a fixed point  $\hat{\mathbf{u}}$  such that  $\lim_{n \rightarrow \infty} \{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}} = \hat{u}^{(k)}$  for each component  $\hat{u}^{(k)}$  of  $\hat{\mathbf{u}}$  constructed in  $F(\mathbf{u}_n) = T^p(\mathbf{u}_{n-1})$  for all  $k = 1, 2, \dots, m$ .

*Proof.* Consider  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  be a sequence define by constructed (2). Since  $\mathbf{u}_0 \in X^m$  is a seed element of  $\preceq_A$ -mixed-monotone, we have  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  is a sequence of  $\preceq_A$ -mixed-monotone, i.e., for any  $n \in \mathbb{N}$ ,  $\mathbf{u}_{n-1} \preceq_A \mathbf{u}_n$  or  $\mathbf{u}_n \preceq_A \mathbf{u}_{n-1}$ . According to inequalities (10), we have

$$\begin{aligned} M(F_k(\mathbf{u}_{n+1}), F_k(\mathbf{u}_n), \beta t) &= M(T_k^p(\mathbf{u}_n), T_k^p(\mathbf{u}_{n-1}), \beta t) \\ &\geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}_n), F_i(\mathbf{u}_{n-1}), t)) \end{aligned}$$

From the proof of Theorem 3.6, we have  $\{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $X$ , for any fixed  $k$ . Hence, we will consider the following cases.

- We have  $\{F(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a sequence of  $\preceq_A$ -mixed-monotone if  $F$  has the property of sequentially  $\preceq_A$ -mixed-monotone; that is, for any  $n \in \mathbb{N}$ ,  $F(\mathbf{u}_n) \preceq_A F(\mathbf{u}_{n+1})$  or  $F(\mathbf{u}_{n+1}) \preceq_A F(\mathbf{u}_n)$ . Since  $\{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $X$ , for any fixed  $k$ , we also have  $\{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a Cauchy sequence of  $\preceq$ -mixed-monotone,  $\forall k = 1, \dots, m$ .
- Let  $F$  has the property of sequentially  $\preceq$ -mixed-monotone. Form  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  be a sequence of  $\preceq_A$ -mixed-monotone, it follows that  $\{u_n^{(k)}\}_n \in \mathbb{N}$  in  $X$  is a sequence of  $\preceq$ -mixed-monotone,  $\forall k = 1, \dots, m$ . So that, we have  $\{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a Cauchy sequence of  $\preceq$ -mixed-monotone,  $\forall k = 1, \dots, m$ .

Since  $(X, \preceq)$  be a completeness for mixed-monotone, there exists  $\hat{u}^{(k)}$  in  $X$  such that  $\lim_{n \rightarrow \infty} F_k(\mathbf{u}_n) = \hat{u}^{(k)}$ ,  $\forall k = 1, \dots, m$ . The rest of the proof, follows the proof of Theorem 3.6. So, we obtained.  $\square$

**Definition 3.8.** Let  $(X, M, \preceq)$  be a fuzzy metric space endowed with a quasi-order. The fuzzy metric space  $(X, M, \preceq)$  is called complete for monotonically if and only if each Cauchy sequence of  $\preceq$ -monotone  $\{u_n^{(k)}\}_{n \in \mathbb{N}}$  in  $X$  is convergent.

It's clear that if the quasi-ordered fuzzy metric space  $(X, M, \preceq)$  be a complete for mixed-monotonically, then  $(X, M, \preceq)$  is also a complete for monotonically. But, the reverse is not true. Namely, the notion of completeness for mixed-monotone is stronger than completeness for monotone.

**Theorem 3.9.** Let  $X$  be a non-empty set. Assume that the quasi-ordered fuzzy metric spaces  $(X, M, *, \preceq_A)$  is complete for monotonically. Assume that the fuzzy

metric  $\mathcal{M}$  and  $M$  compatible in preserving continuity sense. We consider the mappings  $T : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  and  $F : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  satisfying  $T^p(X^m) \subseteq F(X^m)$ ,  $\exists p \in \mathbb{N}$ . Let  $\mathbf{u}_0 \in X^m$  be a seed element of  $\preceq_A$ -monotone. Suppose that the functions  $T$  and  $F$  satisfying the following conditions:

- $F$  has  $\preceq_A$ -monotone ;
- $T^p$  is continuous on  $X^m$  with respect to  $F$ ;
- each  $F_k$  is continuous on  $X^m$  for all  $k = 1, \dots, m$ .

Assume that there exists  $\beta \in (0, 1)$  such that, for each  $k = 1, \dots, m$ ,

$$(11) \quad M(T_k^p(\mathbf{u}), T_k^p(\mathbf{v}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}), F_i(\mathbf{v}), t)),$$

for all  $t > 0$  and for any  $\mathbf{u}, \mathbf{v} \in X^m$  with  $\mathbf{u} \preceq_A \mathbf{v}$  or  $\mathbf{v} \preceq_A \mathbf{u}$ , where  $\gamma : [0, 1] \rightarrow [0, 1]$  continuous  $*^m \gamma(a) \geq a$  for each  $a \in [0, 1]$ . Then  $T^p$  has a fixed point  $\hat{\mathbf{u}}$  such that  $\lim_{n \rightarrow \infty} \{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}} = \hat{u}^{(k)}$  for each component  $\hat{u}^{(k)}$  of  $\hat{\mathbf{u}}$  constructed in  $F(\mathbf{u}_n) = T^p(\mathbf{u}_{n-1})$  for all  $k = 1, 2, \dots, m$ .

*Proof.* We consider  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  be the sequence define by constructed (2). Since  $\mathbf{u}_0 \in X^m$  is a seed element of  $\preceq_A$ -monotone, i.e.,  $\mathbf{u}_n \preceq_A \mathbf{u}_{n+1}$ ,  $\forall n \in \mathbb{N}$  or  $\mathbf{u}_{n+1} \preceq_A \mathbf{u}_n$ ,  $\forall n \in \mathbb{N}$ , according to inequalities (11),

$$\begin{aligned} M(F_k(\mathbf{u}_{n+1}), F_k(\mathbf{u}_n), \beta t) &= M(T_k^p(\mathbf{u}_n), T_k^p(\mathbf{u}_{n-1}), \beta t) \\ &\geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}_n), F_i(\mathbf{u}_{n-1}), t)) \end{aligned}$$

Since  $F$  has  $\preceq_A$ -monotone, then  $F(\mathbf{u}_n) \preceq_A F(\mathbf{u}_{n+1})$  for any  $n \in \mathbb{N}$  or  $F(\mathbf{u}_{n+1}) \preceq_A F(\mathbf{u}_n)$  for any  $n \in \mathbb{N}$ . We have

$$\mathcal{M}(F(\mathbf{u}_{n+1}), F(\mathbf{u}_n), t) \geq *_{i=1}^m M(F_i(\mathbf{u}_n), F_i(\mathbf{u}_{n-1}), t).$$

By the proof of Theorem 3.6, we can prove that  $\{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}} \in X$  is a Cauchy sequence, for any fixed  $k = 1, \dots, m$ . Since  $F$  has  $\preceq_A$ -monotone and  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  be a sequence of  $\preceq_A$ -monotone, we have  $\{F(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a sequence of  $\preceq_A$ -monotone.

- We say that  $\{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a Cauchy sequence of  $\preceq$ -increasing, for  $k \in A$  and is a Cauchy sequence of  $\preceq$ -decreasing, for  $k \in B$  if  $\{F(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a sequence of  $\preceq_A$ -increasing.
- We say that  $\{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a Cauchy sequence of  $\preceq$ -decreasing, for  $k \in A$  and is a Cauchy sequence of  $\preceq$ -increasing, for  $k \in B$  if  $\{F(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a sequence of  $\preceq_A$ -decreasing.

Since  $X$  has completeness for monotone,  $\exists \hat{x}^{(k)} \in X$  such that  $\lim_{n \rightarrow \infty} F_k(\mathbf{u}_n) = \hat{u}^{(k)}, \forall k = 1, \dots, m$ . The remaining proof follows from the proof Theorem 3.6.  $\square$

### 3.2. Coincidence point result.

**Definition 3.10.** Let  $(X, M, *, \preceq)$  be a fuzzy metric space endowed with a quasi-order. The fuzzy metric space  $(X, M, *, \preceq)$  preserves of the mixed-monotone convergence if and only if for each sequence of  $\preceq$ -mixed-monotone  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  that converges to a point  $\hat{\mathbf{u}}$ , we have  $\mathbf{u}_n \preceq \hat{\mathbf{u}}$  or  $\hat{\mathbf{u}} \preceq \mathbf{u}_n$  for any  $n \in \mathbb{N}$ .

**Remark 3.11.** Let  $(X, M, *, \preceq)$  be a fuzzy metric space and preserve of the mixed-monotone convergence. Assume that  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  is a sequence in  $X^m$  such that each sequence  $\{u_n^{(k)}\}_{n \in \mathbb{N}}$  is a convergence sequence of  $\preceq$ -mixed-monotone with a limit point  $\hat{u}^{(k)}$  for any  $k = 1, \dots, m$ . We have the observations as follows.

- (a)  $\mathbf{u}_n$  and  $\hat{\mathbf{u}}$  are  $\preceq$ -mixed-comparable, for any  $n \in \mathbb{N}$ .
- (b) For any  $n \in \mathbb{N}$ , there exists a disjoint pair  $A_n$  and  $B_n$  (that depend on  $n$ ) of  $\{1, 2, \dots, m\}$  such that  $\mathbf{u}_n \preceq_{A_n} \hat{\mathbf{u}}$  or  $\hat{\mathbf{u}} \preceq_{B_n} \mathbf{u}_n$ , where  $A_n$  or  $B_n$  is an empty set.

**Theorem 3.12.** Let  $X$  be a non-empty set. Suppose that the quasi-ordered fuzzy metric spaces  $(X, M, *, \preceq)$  is complete for mixed-monotonically,  $(X, M, *, \preceq)$  preserves of the mixed-monotone convergence. Suppose that the fuzzy metric  $\mathcal{M}$  and  $M$  compatible in preserving convergence sense. We consider the mappings  $T : (X^m, \mathcal{M}) \rightarrow (X^m, \mathcal{M})$  and  $F : (X^m, \mathcal{M}) \rightarrow (X^m, \mathcal{M})$  satisfying  $T^p(X^m) \subseteq F(X^m)$ ,  $\exists p \in \mathbb{N}$ . Let  $\mathbf{u}_0 \in X^m$  be a seed element of  $\preceq$ -mixed-monotone. Suppose that the functions  $T$  and  $F$  satisfying the following conditions:

- $T$  and  $F$  are commutative function;
- $F$  has the property of sequentially  $\preceq$ -mixed-monotone ;
- each  $F_k$  is continuous on  $X^m$  for all  $k = 1, \dots, m$ .

Assume that there exists  $\beta \in (0, 1)$  such that, for each  $k = 1, \dots, m$ ,

$$(12) \quad M(T_k^p(\mathbf{u}), T_k^p(\mathbf{v}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}), F_i(\mathbf{v}), t)),$$

for all  $t > 0$  and for any two elements  $\mathbf{u}, \mathbf{v} \in X^m$  be a  $\preceq$ -mixed-comparable, where  $\gamma : [0, 1] \rightarrow [0, 1]$  continuous  $*^m \gamma(a) \geq a$  for each  $a \in [0, 1]$ . Then there exists  $\hat{\mathbf{u}} \in X^m$  of  $T$  such that  $T^p(\hat{\mathbf{u}}) = F(\hat{\mathbf{u}})$ , we say that  $\hat{\mathbf{u}}$  is a coincidence point of  $T^p$  and  $F$ . such that  $\lim_{n \rightarrow \infty} \{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}} = \hat{u}^{(k)}$  for each component  $\hat{u}^{(k)}$  of  $\hat{\mathbf{u}}$  constructed in  $F(\mathbf{u}_n) = T^p(\mathbf{u}_{n-1})$  for all  $k = 1, 2, \dots, m$ .

*Proof.* From Theorem 3.6, we create a sequence  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  in  $X^m$  such that  $\lim_{n \rightarrow \infty} F(\mathbf{u}_n) = \hat{\mathbf{u}}$ . Since each  $F_k$  is continuous on  $X^m$ , we also have  $\lim_{n \rightarrow \infty} F_k(F(\mathbf{u}_n)) = F_k(\hat{\mathbf{u}})$ , for all  $k = 1, \dots, m$ . Given any  $\delta > 0$ , exists  $n_0 \in \mathbb{N}$  such that

$$M(F_k(F(\mathbf{u}_n)), F_k(\hat{\mathbf{u}}), t) > 1 - \delta$$

for all  $n \in \mathbb{N}$  with  $n \geq n_0$  and  $\forall k = 1, \dots, m$ . From Lemma 2.9, we obtain that

$$(13) \quad \lim_{n \rightarrow \infty} M(F_k(F(\mathbf{u}_n)), F_k(\hat{\mathbf{u}}), t) = 1$$

$\forall t > 0$  and  $k = 1, \dots, m$ .

Hence,  $\exists p \in \mathbb{N}$ , we have

$$M(T_k^p(\hat{\mathbf{u}}), T_k^p(F(\mathbf{u}_n)), t) \geq \gamma(*_{i=1}^m M(F_i(\hat{\mathbf{u}}), F_i(F(\mathbf{u}_n)), t)) \quad (\text{by (12)})$$

$$= \gamma(*_{i=1}^m M(F_i(F(\mathbf{u}_n)), F_i(\hat{\mathbf{u}}), t))$$

$$> \gamma(*_{i=1}^m (1 - \delta))$$

$$> \gamma(1 - \epsilon)$$

$$\geq 1 - \epsilon.$$

Since for any  $\epsilon > 0$ , such that  $M(T_k^p(\hat{\mathbf{u}}), T_k^p(F(\mathbf{u}_n)), t) = 1, \forall k = 1, \dots, m$ , then, we have

$$(14) \quad \lim_{n \rightarrow \infty} M(T_k^p(\hat{\mathbf{u}}), T_k^p(F(\mathbf{u}_n)), t) = 1$$

$\forall t > 0$  and  $k = 1, \dots, m$ .

Since  $T$  and  $F$  are commutative, we get  $F(T^p(\mathbf{u})) = T^p(F(\mathbf{u}))$ ,  $\forall \mathbf{u} \in X^m$ , which also implies

$$F_k(F(\mathbf{u}_n)) = F_k(T^p(\mathbf{u}_{n-1})) = T_k^p(F(\mathbf{u}_{n-1})).$$

Now, we have

$$\begin{aligned} M(T_k^p(\hat{\mathbf{u}}), F_k(\hat{\mathbf{u}}), t) &\geq M(T_k^p(\hat{\mathbf{u}}), T_k^p(F(\mathbf{u}_n)), s) * M(T_k^p(F(\mathbf{u}_n)), F_k(\hat{\mathbf{u}}), w) \\ &= M(T_k^p(\hat{\mathbf{u}}), T_k^p(F(\mathbf{u}_n)), s) * M(F_k(F(\mathbf{u}_{n+1})), F_k(\hat{\mathbf{u}}), w) \end{aligned}$$

we can take limit, such that

$$\begin{aligned} \lim_{n \rightarrow \infty} M(T_k^p(\hat{\mathbf{u}}), F_k(\hat{\mathbf{u}}), t) &\geq \lim_{n \rightarrow \infty} \left[ M(T_k^p(\hat{\mathbf{u}}), T_k^p(F(\mathbf{u}_n)), s) \right. \\ &\quad \left. * M(F_k(F(\mathbf{u}_{n+1})), F_k(\hat{\mathbf{u}}), w) \right] \\ &= \lim_{n \rightarrow \infty} M(T_k^p(\hat{\mathbf{u}}), T_k^p(F(\mathbf{u}_n)), s) \\ &\quad * \lim_{n \rightarrow \infty} M(F_k(F(\mathbf{u}_{n+1})), F_k(\hat{\mathbf{u}}), w) \\ &= 1 * 1 \quad \text{by (13) and (14)} \\ &= 1 \end{aligned}$$

$\forall k = 1, \dots, m$  and  $s, w > 0$  with  $t = s + w$ .

We conclude that  $M(T_k^p(\hat{\mathbf{u}}), F_k(\hat{\mathbf{u}}), t) = 1$ , which says that  $T_k^p(\hat{\mathbf{u}}) = F_k(\hat{\mathbf{u}})$ ,  $\forall k = 1, 2, \dots, m$ , (i.e.,  $T^p(\hat{\mathbf{u}}) = F(\hat{\mathbf{u}})$ ).  $\square$

If  $p = 1$ , we obtain the following corollary.

**Corollary 3.13.** *Let  $X$  be a non-empty set. Suppose that the quasi-ordered fuzzy metric spaces  $(X, M, *, \preceq)$  is complete for mixed-monotonically,  $(X, M, *, \preceq)$  preserves of the mixed-monotone convergence. Assume that the fuzzy metric  $\mathcal{M}$  and  $M$  compatible in preserving convergence sense. We consider the mappings  $T : (X^m, \mathcal{M}) \rightarrow (X^m, \mathcal{M})$  and  $F : (X^m, \mathcal{M}) \rightarrow (X^m, \mathcal{M})$  satisfying  $T(X^m) \subseteq F(X^m)$ . Let  $\mathbf{u}_0 \in X^m$  be a seed element of  $\preceq$ -mixed-monotone. Suppose that the functions  $T$  and  $F$  satisfying the following conditions:*

- $T$  and  $F$  are commutative function;
- $F$  has the property of sequentially  $\preceq$ -mixed-monotone;
- each  $F_k$  is continuous on  $X^m$  for  $k = 1, \dots, m$ .

Assume that there exists  $\beta \in (0, 1)$  such that, for each  $k = 1, \dots, m$ ,

$$(15) \quad M(T_k(\mathbf{u}), T_k(\mathbf{v}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}), F_i(\mathbf{v}), t)).$$

for all  $t > 0$  and for any two elements  $\mathbf{u}, \mathbf{v} \in X^m$  be a  $\preceq$ -mixed-comparable, where  $\gamma : [0, 1] \rightarrow [0, 1]$  continuous  $*^m \gamma(a) \geq a$  for each  $a \in [0, 1]$ . Then there exists  $\hat{\mathbf{u}} \in X^m$  of  $T$  such that  $T(\hat{\mathbf{u}}) = F(\hat{\mathbf{u}})$ , we say that  $\hat{\mathbf{u}}$  is a coincidence point of  $T$  and  $F$ . Such that  $\lim_{n \rightarrow \infty} \{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}} = \hat{\mathbf{u}}^{(k)}$  for each component  $\hat{\mathbf{u}}^{(k)}$  of  $\hat{\mathbf{u}}$  constructed in  $F(\mathbf{u}_n) = T(\mathbf{u}_{n-1})$ , for all  $k = 1, 2, \dots, m$ .

**Theorem 3.14.** *Let  $X$  be a non-empty set. Assume that the quasi-ordered fuzzy metric spaces  $(X, M, *, \preceq)$  is complete for mixed-monotonically,  $(X, M, *, \preceq)$  preserves the mixed-monotone convergence. Suppose that the fuzzy metric  $\mathcal{M}$  and  $M$  compatible in preserving convergence sense. Let  $A$  and  $B$  be a disjoint pair of  $\{1, 2, \dots, m\}$ . We consider the mappings  $T : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  and  $F : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  satisfying  $T^p(X^m) \subseteq F(X^m)$ ,  $\exists p \in \mathbb{N}$ . Let  $\mathbf{u}_0 \in X^m$  be a seed element of  $\preceq$ -mixed-monotone. Suppose that the functions  $T$  and  $F$  satisfying the following conditions:*

- $T$  and  $F$  are commutative function;
- $F$  has the property of sequentially  $\preceq_A$ -mixed-monotone for each disjoint pair  $A^\circ$  and  $B^\circ$  of  $\{1, \dots, m\}$ ;
- each  $F_k$  is continuous on  $X^m$  for  $k = 1, \dots, m$ .

Assume that there exists  $\beta \in (0, 1)$  such that, for each  $k = 1, \dots, m$ ,

$$(16) \quad M(T_k^p(\mathbf{u}), T_k^p(\mathbf{v}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}), F_i(\mathbf{v}), t)),$$

for all  $t > 0$  and for any two elements  $\mathbf{u}, \mathbf{v} \in X^m$  be a  $\preceq$ -mixed-comparable and each disjoint pair  $A^\circ$  and  $B^\circ$  of  $\{1, \dots, m\}$  with  $\mathbf{v} \preceq_{A^\circ} \mathbf{u}$  or  $\mathbf{u} \preceq_{A^\circ} \mathbf{v}$ , where  $\gamma : [0, 1] \rightarrow [0, 1]$  continuous  $*^m \gamma(a) \geq a$  for each  $a \in [0, 1]$ . Then there exists  $\hat{\mathbf{u}} \in X^m$  of  $T$  such that  $T^p(\hat{\mathbf{u}}) = F(\hat{\mathbf{u}})$ , we say that  $\hat{\mathbf{u}}$  is a coincidence point of  $T^p$  and  $F$ . Such that  $\lim_{n \rightarrow \infty} \{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}} = \hat{u}^{(k)}$  for each component  $\hat{u}^{(k)}$  of  $\hat{\mathbf{u}}$  constructed in  $F(\mathbf{u}_n) = T^p(\mathbf{u}_{n-1})$  for all  $k = 1, 2, \dots, m$ .

*Proof.* The same argument of Theorem 3.12, we create a sequence  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow \infty} F_k(\mathbf{u}_n) = \hat{u}^{(k)}$  and  $\lim_{n \rightarrow \infty} F_k(F(\mathbf{u}_n)) = F_k(\hat{\mathbf{u}})$ , where  $\{F(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a sequence of  $\preceq$ -mixed-monotone,  $\forall k = 1, 2, \dots, m$ . Form  $\lim_{n \rightarrow \infty} F_k(F(\mathbf{u}_n)) = F_k(\hat{\mathbf{u}})$ , given any  $\delta > 0$ ,  $\exists n_0 \in \mathbb{N}$  for all  $n \in \mathbb{N}$  with  $n \geq n_0$  such that

$$(17) \quad M(F_k(F(\mathbf{u}_n)), F_k(\hat{\mathbf{u}}), t) > 1 - \delta, \quad \forall k = 1, \dots, m.$$

Form  $\{F(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a convergent sequence of  $\preceq$ -mixed-monotone, by (b) of Remark 3.11, we get for any  $n \in \mathbb{N}$ , there exists a subset  $A_n$  of  $\{1, \dots, m\}$  such that

$$(18) \quad F(\mathbf{u}_n) \preceq_{A_n} \hat{\mathbf{u}} \quad \text{or} \quad \hat{\mathbf{u}} \preceq_{A_n} F(\mathbf{u}_n).$$

Since  $F$  has the  $\preceq_{A^\circ}$ -comparable property for each subset  $A^\circ$  of  $\{1, \dots, m\}$ , such that

$$(19) \quad F(F(\mathbf{u}_n)) \preceq_{A^\circ} F(\hat{\mathbf{u}}) \quad \text{or} \quad F(\hat{\mathbf{u}}) \preceq_{A^\circ} F(F(\mathbf{u}_n)).$$

For each  $n \geq n_0$ , we obtain

$$(20) \quad M(T_k^p(\hat{\mathbf{u}}), T_k^p(F(\mathbf{u})), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\hat{\mathbf{u}}), F_i(F(\mathbf{u})), t)).$$

The remaining proof follows from Theorem 3.7. The proof is complete.  $\square$

Next, we will prove the coincidence point for regardless the continuity of  $T^p$ . Howsoever, we introduce the notion of monotone convergence.

**Definition 3.15.** Let  $(X, M, \preceq)$  be a fuzzy metric space endowed with a quasi-order. For each sequence of  $\preceq$ -monotone  $\{u_n\}_{n \in \mathbb{N}}$  that converges to  $\hat{u}$  if and only if  $(X, M, \preceq)$  preserves the monotone convergence, satisfied either one of the following conditions:

- For each  $n \in \mathbb{N}$ ,  $u_n \preceq \hat{x}$  if  $\{u_n\}_{n \in \mathbb{N}}$  is a sequence of  $\preceq$ -increasing;
- For each  $n \in \mathbb{N}$ ,  $\hat{u} \preceq u_n$  if  $\{u_n\}_{n \in \mathbb{N}}$  is a sequence of  $\preceq$ -decreasing.

**Theorem 3.16.** *Let  $X$  be a non-empty set. Assume that the quasi-ordered fuzzy metric spaces  $(X, M, *, \preceq)$  is complete for monotonically,  $(X, M, *, \preceq)$  preserves of the monotone convergence. Suppose that the fuzzy metric  $\mathcal{M}$  and  $M$  compatible in preserving convergence sense. We consider the mappings  $T : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  and  $F : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  satisfying  $T^p(X^m) \subseteq F(X^m)$ ,  $\exists p \in \mathbb{N}$ . Let  $\mathbf{u}_0 \in X^m$  be a seed element of  $\preceq_A$ -monotone. Suppose that the functions  $T$  and  $F$  satisfying the following conditions:*

- $T$  and  $F$  are commutative function;
- $F$  is  $\preceq_A$ -monotone;
- each  $F_k$  is continuous on  $X^m$  for all  $k = 1, 2, \dots, m$ .

Assume that there exists  $\beta \in (0, 1)$  such that, for each  $k = 1, \dots, m$ ,

$$(21) \quad M(T_k^p(\mathbf{u}), T_k^p(\mathbf{v}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}), F_i(\mathbf{v}), t)),$$

for all  $t > 0$  and for any  $\mathbf{u}, \mathbf{v} \in X^m$ , with  $\mathbf{u} \preceq_A \mathbf{v}$  or  $\mathbf{v} \preceq_A \mathbf{u}$ , where  $\gamma : [0, 1] \rightarrow [0, 1]$  continuous  $*^m \gamma(a) \geq a$  for each  $a \in [0, 1]$ . Then there exists  $\hat{\mathbf{u}} \in X^m$  of  $T$  such that  $T^p(\hat{\mathbf{u}}) = F(\hat{\mathbf{u}})$ , which says that  $\hat{\mathbf{u}}$  is a coincidence point of  $T^p$  and  $F$ . Such that  $\lim_{n \rightarrow \infty} \{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}} = \hat{u}^{(k)}$  for each component  $\hat{u}^{(k)}$  of  $\hat{\mathbf{u}}$  constructed in  $F(\mathbf{u}_n) = T^p(\mathbf{u}_{n-1})$  for all  $k = 1, 2, \dots, m$ .

*Proof.* By the proof of Theorem 3.12, we create a sequence  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow \infty} F_k(\mathbf{u}_n) = \hat{u}^{(k)}$  and  $\lim_{n \rightarrow \infty} F_k(F(\mathbf{u}_n)) = F_k(\hat{\mathbf{u}})$ , since  $\{F(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a sequence of  $\preceq_A$ -monotone. By  $A$  and  $B$  is a disjoint pair, we have, for each  $n \in \mathbb{N}$ ,  $F(\mathbf{u}_n) \preceq_A \hat{\mathbf{u}}$  or  $\hat{\mathbf{u}} \preceq_A F(\mathbf{u}_n)$ . Since  $\lim_{n \rightarrow \infty} F_k(F(\mathbf{u}_n)) = F_k(\hat{\mathbf{u}})$ , given  $\forall \delta > 0$ , there exists  $n_0 \in \mathbb{N}$  such that

$$(22) \quad M(F_k(F(\mathbf{u}_n)), F_k(\hat{\mathbf{u}}), t) > 1 - \delta$$

for any  $n \in \mathbb{N}$  with  $n \geq n_0$ ,  $\forall k = 1, 2, \dots, m$ . Since  $\{F(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  is a  $\preceq_A$ -monotone, we have  $F(F(\mathbf{u}_n)) \preceq_A F(\hat{\mathbf{u}})$  or  $F(\hat{\mathbf{u}}) \preceq_A F(F(\mathbf{u}_n))$ . For each  $n_0 \in \mathbb{N}$ , such that  $M(T_k^p(\hat{\mathbf{u}}), T_k^p(F(\mathbf{u}_n)), t) \geq \gamma(*_{i=1}^m M(F_i(\hat{\mathbf{u}}), F_i(F(\mathbf{u}_n)), t))$  (by (21))

$$\begin{aligned}
&= \gamma(*_{i=1}^m M(F_i(F(\mathbf{u}_n)), F_i(\hat{\mathbf{u}}), t)) \\
&> \gamma(*_{i=1}^m (1 - \delta)) \\
&> \gamma(1 - \epsilon) \\
&\geq 1 - \epsilon.
\end{aligned}$$

For each  $n \geq n_0$ , we obtain

$$(23) \quad M(T_k^p(\hat{\mathbf{u}}), T_k^p(F(\mathbf{u})), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\hat{\mathbf{u}}), F_i(F(\mathbf{u})), t)).$$

The remaining proof follows from the proof of Theorem 3.7. The proof is complete.  $\square$

### 3.3. The common fixed point result.

Next, we will show the uniqueness of the common fixed point on the  $\preceq$ -mixed-comparable sense.

**Definition 3.17.** *Let  $(X, \preceq)$  be a quasi-ordered set. We consider the mappings  $T : X^m \rightarrow X^m$  and  $F : X^m \rightarrow X^m$  defined on  $X^m$ . In the  $\preceq$ -mixed-comparable sense, the common fixed point  $\hat{\mathbf{u}} \in X^m$  of  $T^p$  and  $F$  is a unique if and only if for any other common fixed point  $\hat{\mathbf{v}}$  of  $T$  and  $F$ , if  $\hat{\mathbf{v}}$  and  $\hat{\mathbf{u}}$  are  $\preceq$ -mixed-comparable, then  $\hat{\mathbf{v}} = \hat{\mathbf{u}}$ .*

**Theorem 3.18.** *Let  $X$  be a non-empty set. Suppose that the quasi-ordered fuzzy metric spaces  $(X, M, *, \preceq)$  is complete for mixed-monotonically,  $(X, M, *, \preceq)$  preserves of the mixed-monotone convergence. Assume that the fuzzy metric  $\mathcal{M}$  and  $M$  are compatible in preserving convergence sense. We consider the mappings  $T : (X^m, \mathcal{M}) \rightarrow (X^m, \mathcal{M})$  and  $F : (X^m, \mathcal{M}) \rightarrow (X^m, \mathcal{M})$  satisfying  $T^p(X^m) \subseteq F(X^m)$ ,  $\exists p \in \mathbb{N}$ . Let  $\mathbf{u}_0 \in X^m$  be a seed element of  $\preceq$ -mixed-monotone. Suppose that the functions  $T$  and  $F$  satisfying the following conditions:*

- $T$  and  $F$  are commutative function;
- $F$  has the  $\preceq$ -comparable and property of sequentially  $\preceq$ -mixed-monotone;
- $T^p$  is continuous on  $X^m$  with respect to  $F$ ;
- each  $F_k$  is continuous on  $X^m$  for all  $k = 1, 2, \dots, m$ .

Assume that there exists  $\beta \in (0, 1)$  such that, for each  $k = 1, \dots, m$ ,

$$(24) \quad M(T_k^p(\mathbf{u}), T_k^p(\mathbf{v}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}), F_i(\mathbf{v}), t))$$

for all  $t > 0$  and for any two elements  $\mathbf{u}, \mathbf{v} \in X^m$  be a  $\preceq$ -mixed-comparable, where  $\gamma : [0, 1] \rightarrow [0, 1]$  continuous  $*^m \gamma(a) \geq a$  for each  $a \in [0, 1]$ . Then  $T^p$  and  $F$  have a unique common fixed point  $\hat{\mathbf{u}}$  in the  $\preceq$ -mixed-comparable sense. Such that  $\lim_{n \rightarrow \infty} \{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}} = \hat{u}^{(k)}$  for each component  $\hat{u}^{(k)}$  of  $\hat{\mathbf{u}}$  constructed in  $F(\mathbf{u}_n) = T^p(\mathbf{u}_{n-1})$  for all  $k = 1, 2, \dots, m$ .

*Proof.* From Proposition 3.2 and Theorem 3.7, we obtain  $F(\hat{\mathbf{u}}) = T^p(\hat{\mathbf{u}})$ . By Theorem 3.6, we also obtain  $T^p(\hat{\mathbf{u}}) = \hat{\mathbf{u}}$ . So, we obtain

$$\hat{\mathbf{u}} = F(\hat{\mathbf{u}}) = T^p(\hat{\mathbf{u}})$$

such that  $\hat{\mathbf{u}}$  is a common fixed point of  $T^p$  and  $F$ . Next we shows the uniqueness. Given a point  $\hat{\mathbf{v}}$  be another common fixed point of  $T^p$  and  $F$ . Thus,  $\hat{\mathbf{v}}$  and  $\hat{\mathbf{u}}$  are  $\preceq$ -mixed-comparable, form  $F$  has the  $\preceq$ -comparable property such that  $F(\hat{\mathbf{u}})$  and  $F(\hat{\mathbf{v}})$  are  $\preceq$ -mixed-comparable. If  $F_k(\hat{\mathbf{u}}) \neq F_k(\hat{\mathbf{v}})$  i.e.,  $M(F_k(\hat{\mathbf{u}}), F_k(\hat{\mathbf{v}}), t) \neq 1$ ,  $\exists \delta > 0$ , such that  $M(F_k(\hat{\mathbf{u}}), F_k(\hat{\mathbf{v}}), t) = 1 - \delta$  for some  $k$ , then we obtain

$$\begin{aligned} 1 &\neq M(F_k(\hat{\mathbf{u}}), F_k(\hat{\mathbf{v}}), \beta t) = M(T_k^p(\hat{\mathbf{u}}), T_k^p(\hat{\mathbf{v}}), \beta t) \\ &\geq \gamma(*_{i=1}^m M(F_i(\hat{\mathbf{u}}), F_i(\hat{\mathbf{v}}), t)) \\ &\geq \gamma(*_{i=1}^m (1 - \delta)) \\ &> \gamma(1 - \epsilon) \\ &\geq 1 - \epsilon. \end{aligned}$$

For any  $\epsilon > 0$ , we summarize that  $M(F_k(\hat{\mathbf{u}}), F_k(\hat{\mathbf{v}}), \beta t) = 1$ ,  $\forall k = 1, \dots, m$ . This contradiction says that  $F_k(\hat{\mathbf{u}}) = F_k(\hat{\mathbf{v}})$ ,  $\forall k = 1, \dots, m$ , (i.e.,  $F(\hat{\mathbf{u}}) = F(\hat{\mathbf{v}})$ ). Therefore, by inequality, we have

$$M(\hat{\mathbf{u}}, \hat{\mathbf{v}}, t) \geq M(\hat{\mathbf{u}}, F(\hat{\mathbf{u}}), s) * M(F(\hat{\mathbf{u}}), F(\hat{\mathbf{v}}), w) * M(F(\hat{\mathbf{v}}), \hat{\mathbf{v}}, z)$$

$$= 1 \quad \text{for } s, w, z > 0 \text{ with } t = s + w + z,$$

which says that  $\hat{\mathbf{u}} = \hat{\mathbf{v}}$ . □

Next, we shall study a chain-uniqueness of common fixed point. Consider the fuzzy metric space  $(X, M, *, \preceq)$ , for a disjoint pair  $A$  and  $B$  of  $\{1, \dots, m\}$ , form (1), we define a quasi-order  $\preceq_A$  on  $X^m$ . This time, given any  $\mathbf{u} \in X^m$ , we can define the chain containing  $\mathbf{u}$  denote by  $\mathfrak{C}(\preceq_A, \mathbf{u})$  as follows:

$$\begin{aligned} \mathfrak{C}(\preceq_A, \mathbf{u}) &= \{\mathbf{v} \in X^m : \mathbf{v} \preceq_A \mathbf{u} \text{ or } \mathbf{u} \preceq_A \mathbf{v}\} \\ &= \{\mathbf{v} \in X^m : \mathbf{u} \text{ and } \mathbf{v} \text{ are comparable with respect to } '\preceq_A'\}. \end{aligned}$$

**Definition 3.19** ([5]). *Let  $(X, \preceq)$  be a quasi-ordered set. We consider the mappings  $T : X^m \rightarrow X^m$  and  $F : X^m \rightarrow X^m$  defined on  $X^m$ . A point  $\hat{\mathbf{u}} \in X^m$  is called that chain-unique common fixed point of  $T$  and  $F$  if and only if for a point  $\mathbf{u}$  is other common fixed point of  $T$  and  $F$ , if  $\mathbf{u} \in \mathfrak{C}(\preceq_{A^\circ}, \mathbf{u})$ , for some disjoint pair  $A^\circ$  and  $B^\circ$  of  $\{1, 2, \dots, m\}$ , then  $\mathbf{u} = \hat{\mathbf{u}}$ .*

**Theorem 3.20.** *Let  $X$  be a non-empty set. Suppose that the quasi-ordered fuzzy metric spaces  $(X, M, *, \preceq)$  is complete for mixed-monotonically,  $(X, M, *, \preceq)$  preserves of the mixed-monotone convergence. Assume that the fuzzy metric  $\mathcal{M}$  and  $M$  are compatible in preserving continuity sense. We consider the mappings  $T : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  and  $F : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  satisfying  $T^p(X^m) \subseteq F(X^m)$ ,  $\exists p \in \mathbb{N}$ . Let  $\mathbf{u}_0 \in X^m$  be a seed element of  $\preceq_A$ -mixed-monotone. Suppose that the functions  $T$  and  $F$  satisfying the following conditions:*

- $T$  and  $F$  are commutative function;
- $F$  has the property of sequentially  $\preceq_A$ -mixed-monotone or the property of sequentially  $\preceq$ -mixed-monotone;
- $T^p$  is continuous on  $X^m$  with respect to  $F$ ;
- each  $F_k$  is continuous on  $X^m$  for all  $k = 1, \dots, m$ .

Assume that there exists  $\beta \in (0, 1)$  such that, for each  $k = 1, \dots, m$ ,

$$(25) \quad M(T_k^p(\mathbf{u}), T_k^p(\mathbf{v}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}), F_i(\mathbf{v}), t)).$$

for all  $t > 0$  and for any two  $\preceq$ -mixed-comparable elements  $\mathbf{u}, \mathbf{v} \in X^m$ , where  $\gamma : [0, 1] \rightarrow [0, 1]$  continuous  $*^m \gamma(a) \geq a$  for each  $a \in [0, 1]$ . Then  $T^p$  and  $F$  have a chain-unique common fixed point  $\hat{\mathbf{u}}$ .  $\hat{\mathbf{u}}^{(k)}$  of  $\hat{\mathbf{u}}$  is a limit point of the sequence  $\{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}}$  constructed in  $F(\mathbf{u}_n) = T^p(\mathbf{u}_{n-1})$  for all  $k = 1, 2, \dots, m$ .

*Proof.* From Proposition 3.2 and Theorem 3.14, we have  $\hat{\mathbf{u}}$  is a common fixed point of  $T^p$  and  $F$ . We can prove that the chain-uniqueness, given a point  $\hat{\mathbf{v}}$  be another common fixed point of  $T^p$  and  $F$  with  $\hat{\mathbf{v}} \preceq_{A^\circ} \hat{\mathbf{u}}$  or  $\hat{\mathbf{u}} \preceq_{A^\circ} \hat{\mathbf{v}}$ , for some disjoint pair  $A^\circ$  and  $B^\circ$  of  $\{1, \dots, m\}$ , i.e.,  $\hat{\mathbf{v}} = F(\hat{\mathbf{v}}) = T^p(\hat{\mathbf{v}})$ . Since Theorem 3.12, we obtain  $F(\hat{\mathbf{u}}) = F(\hat{\mathbf{v}})$ . Thus, we have  $\hat{\mathbf{u}} = \hat{\mathbf{v}}$ .  $\square$

**Theorem 3.21.** Let  $X$  be a non-empty set. Assume that the quasi-ordered fuzzy metric spaces  $(X, M, *, \preceq)$  is complete for monotonically,  $(X, M, *, \preceq)$  preserves of the mixed-monotone convergence. Suppose that the fuzzy metric  $\mathcal{M}$  and  $M$  are compatible in preserving continuity sense. We consider the mappings  $T : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  and  $F : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  satisfying  $T^p(X^m) \subseteq F(X^m)$ ,  $\exists p \in \mathbb{N}$ . Let  $\mathbf{u}_0 \in X^m$  be a seed element of  $\preceq_A$ -monotone. Suppose that the functions  $T$  and  $F$  satisfying the following conditions:

- $T$  and  $F$  are commutative function;
- $F$  has  $\preceq_A$ -monotone;
- $T^p$  is continuous on  $X^m$  with respect to  $F$ ;
- each  $F_k$  is continuous on  $X^m$  for all  $k = 1, \dots, m$ .

Assume that there exists  $\beta \in (0, 1)$  such that, for each  $k = 1, \dots, m$ ,

$$(26) \quad M(T_k^p(\mathbf{u}), T_k^p(\mathbf{v}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}), F_i(\mathbf{v}), t)).$$

for all  $t > 0$  and for any  $\mathbf{u}, \mathbf{v} \in X^m$ , with  $\mathbf{u} \preceq_A \mathbf{v}$  or  $\mathbf{v} \preceq_A \mathbf{u}$ , where  $\gamma : [0, 1] \rightarrow [0, 1]$  continuous  $*^m \gamma(a) \geq a$  for each  $a \in [0, 1]$ . Then  $T^p$  and  $F$  have a point  $\hat{\mathbf{u}}$  is chain-unique common fixed point. Such such that  $\lim_{n \rightarrow \infty} \{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}} = \hat{\mathbf{u}}^{(k)}$  for each component  $\hat{\mathbf{u}}^{(k)}$  of  $\hat{\mathbf{u}}$  constructed in  $F(\mathbf{u}_n) = T^p(\mathbf{u}_{n-1})$  for all  $k = 1, 2, \dots, m$ .

*Proof.* From Proposition 3.2 and Theorem 3.16, we obtain  $F(\hat{\mathbf{u}}) = T^p(\hat{\mathbf{u}})$ . By Theorem 3.9, we also obtain  $T^p(\hat{\mathbf{u}}) = \hat{\mathbf{u}}$ . Thus, we have  $\hat{\mathbf{u}} = F(\hat{\mathbf{u}}) = T^p(\hat{\mathbf{u}})$ . We shows that  $\hat{\mathbf{u}}$  is a common fixed point of  $T^p$  and  $F$ . We can prove that the  $\preceq_A$ -chain-uniqueness, given a point  $\hat{\mathbf{v}}$  is another common fixed point of  $T^p$  and  $F$ , thus

$\hat{\mathbf{v}}$  and  $\hat{\mathbf{u}}$  are comparable, i.e.,  $\hat{\mathbf{v}} = F(\hat{\mathbf{v}}) = T^p(\hat{\mathbf{v}})$ . By Theorem 3.16, we have  $F(\hat{\mathbf{u}}) = F(\hat{\mathbf{v}})$ . Consequently, by the triangle inequality, we get

$$M(\hat{\mathbf{u}}, \hat{\mathbf{v}}, t) \geq M(\hat{\mathbf{u}}, F(\hat{\mathbf{u}}), s) * M(F(\hat{\mathbf{u}}), F(\hat{\mathbf{v}}), w) * M(F(\hat{\mathbf{v}}), \hat{\mathbf{v}}, z) = 1$$

for  $s, w, z > 0$  with  $t = s + w + z$ .

Which says that  $\hat{\mathbf{u}} = \hat{\mathbf{v}}$ . This completes the proof.  $\square$

#### 4. SOME APPLICATIONS

##### 4.1. Applications to the integral equations system.

Let the space of all continuous functions  $C([0, 1], \mathbb{R})$  from  $[0, 1]$  into  $\mathbb{R}$ . We define the product space of  $C([0, 1], \mathbb{R})$  for  $m$  times denote by  $C^m([0, 1], \mathbb{R})$ . It follows that, we consider  $(C([0, 1], \mathbb{R}), M, \preceq)$  is complete for mixed-monotonically or complete for monotonically,  $(C([0, 1], \mathbb{R}), M, \preceq)$  preserves the monotone convergence.

Let  $G : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^+$  be a continuous functions and  $f^{(k)} : [0, 1] \times \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $\forall k = 1, 2, \dots, m$ , we consider the integral equations system as follows:

$$(27) \quad \int_0^1 G(s, t)[f^{(k)}(s, \mathbf{z}(s)) + \beta z^{(k)}(s)]ds = z^{(k)}(t).$$

for all  $k = 1, \dots, m$ , where  $\beta \geq 0$ . We will find  $\mathbf{z}^* \in C^m([0, 1], \mathbb{R})$  such that the integral equations system (27) are all satisfied, where  $z^{(k*)} \in C([0, 1], \mathbb{R})$  is the  $k$ th component of  $\mathbf{z}^*$ ,  $\forall k = 1, \dots, m$ . Then the solution  $\mathbf{z}^*$  will be in chain-uniqueness sense.

Let  $h : [0, 1] \rightarrow \mathbb{R}^m$  be a vector-valued function defined on  $[0, 1]$ , let  $h^{(k)}$  be a  $k$ th component function of  $\mathbf{h}$ ,  $\forall k = 1, 2, \dots, m$ . The integral of  $\mathbf{h}$  on  $[0, 1]$  is defined on  $\mathbb{R}^m$  as the following:

$$\int_0^1 \mathbf{h}(t) = \left( \int_0^1 h^{(1)}(t)dt, \int_0^1 h^{(2)}(t)dt, \dots, \int_0^1 h^{(m)}(t)dt \right) \in \mathbb{R}^m.$$

We define  $\mathbf{f} : [0, 1] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  is a vector-valued function by  $\mathbf{f} = (f^{(1)}, f^{(2)}, \dots, f^{(m)})$ . Then the integral equations system as seen in (27) can be re-written as the following

vector form of integral equation:

$$(28) \quad \int_0^1 G(s, t)[\mathbf{f}(s, \mathbf{z}(s)) + \beta \mathbf{z}(s)]ds = \mathbf{z}(t),$$

where  $\beta \geq 0$ . In the same way, we say that  $\mathbf{z}^* \in C^m([0, 1], \mathbb{R})$  such that equation (28) is satisfied, we also says that  $\mathbf{z}^*$  is a solution of equation (28).

**Definition 4.1.** Let  $(C([0, 1], \mathbb{R}), M, \preceq)$  be a quasi-ordered fuzzy metric space.

(a) We have unique solution  $\mathbf{z}^*$  of the integral equations system (28) in the  $\preceq$ -mixed-comparable sense if and only if it satisfies the following conditions:

- $\mathbf{z}^*$  is a solution of equation (28);
- if  $\bar{\mathbf{z}}$  is another solution of equation (28),  $\mathbf{z}^*$  and  $\bar{\mathbf{z}}$  are  $\preceq$ -mixed-comparable, then  $\mathbf{z}^* = \bar{\mathbf{z}}$ .

Let  $A$  and  $B$  be a disjoint pair of  $\{1, \dots, m\}$ , we consider the product space  $(C^m([0, 1], \mathbb{R}), M, \preceq_A)$ .

(b) We have a  $\preceq_A$ -chain-unique solution  $\mathbf{z}^*$  of the integral equations system (28) iff it satisfies the following conditions:

- $\mathbf{z}^*$  is a solution of equation (28);
- if  $\bar{\mathbf{z}}$  is another solution of equation (28) satisfying  $\mathbf{z}^* \preceq_A \bar{\mathbf{z}}$  or  $\bar{\mathbf{z}} \preceq_A \mathbf{z}^*$ , then  $\mathbf{z}^* = \bar{\mathbf{z}}$ .

**Lemma 4.2.** Let  $(X, M, *, \preceq)$  be a quasi-ordered fuzzy metric spaces and it's complete for monotonically. Assume that the fuzzy metric  $\mathcal{M}$  and  $M$  compatible in preserving continuity sense. Let  $A$  and  $B$  be any disjoint pair of  $\{1, 2, \dots, m\}$ . We consider the functions  $T : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  is continuous on  $X^m$  and satisfying any one of the following conditions:

- (a)  $T$  has  $(\preceq_A, \preceq_A)$ -increasing;
- (b)  $p$  is an even integer and  $T$  is  $(\preceq_A, \preceq_A)$ -decreasing.

Assume that there exists  $\beta \in (0, 1)$  such that, for any  $\mathbf{u} \preceq_A \mathbf{v}$  or  $\mathbf{v} \preceq_A \mathbf{u}$ , the following inequalities

$$M(T_k^p(\mathbf{u}), T_k^p(\mathbf{v}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}), F_i(\mathbf{v}), t)),$$

for all  $t > 0$ ,  $\forall k = 1, \dots, m$  and  $\exists p \in \mathbb{N}$ , where  $\gamma : [0, 1] \rightarrow [0, 1]$  continuous  $*^m \gamma(a) \geq a$  for each  $a \in [0, 1]$ . If there exists  $\mathbf{u}_0 \in X^m$  such that  $\mathbf{u}_0 \preceq_A T^p(\mathbf{u}_0)$  or

$T^p(\mathbf{u}_0) \preceq_A \mathbf{u}_0$ , then  $T^p$  has a fixed point  $\hat{\mathbf{u}} \in X^m$ , such that  $\lim_{n \rightarrow \infty} \{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}} = \hat{u}^{(k)}$  for each component  $\hat{u}^{(k)}$  of  $\hat{\mathbf{u}}$  constructed in  $F(\mathbf{u}_n) = T^p(\mathbf{u}_{n-1})$  for all  $k = 1, 2, \dots, m$ .

*Proof.* We consider the following cases.

- If  $F$  is  $(\preceq_A, \preceq_A)$ -increasing, then  $T^p$  is  $(\preceq_A, \preceq_A)$ -increasing.
- If  $F$  is  $(\preceq_A, \preceq_A)$ -decreasing, then  $T^p$  is also  $(\preceq_A, \preceq_A)$ -increasing, for  $p$  is an even integer.

By Lemma 2.16, according to  $\mathbf{u}_n = T^p(\mathbf{u}_{n-1})$ , we have  $\mathbf{u}_0 \preceq_A \mathbf{u}_1$  or  $\mathbf{u}_1 \preceq_A \mathbf{u}_0$ . Since  $\mathbf{u}_1 = T^p(\mathbf{u}_0)$  and  $\mathbf{u}_2 = T^p(\mathbf{u}_1)$ , it follows that  $\mathbf{u}_0 \preceq_A \mathbf{u}_1$  implies  $\mathbf{u}_1 \preceq_A \mathbf{u}_2$  and  $\mathbf{u}_1 \preceq_A \mathbf{u}_0$  implies  $\mathbf{u}_2 \preceq_A \mathbf{u}_1$ . Therefore, if  $\mathbf{u}_0 \preceq_A \mathbf{u}_1$ , then we can constructed a sequence of  $\preceq_A$ -increasing  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$ , and if  $\mathbf{u}_1 \preceq_A \mathbf{u}_0$ , then we can constructed  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  is a sequence of  $\preceq_A$ -decreasing, we says that the initial element  $\mathbf{u}_0$  is a seed element of  $\preceq_A$ -monotone in  $X^m$ . Thus, immediate results from Theorem 3.9 by let  $F$  is the identity function.  $\square$

**Lemma 4.3.** *Let  $(X, M, *, \preceq)$  be a quasi-ordered fuzzy metric spaces.  $(X, M, *, \preceq)$  is complete for monotonically and preserves of the monotone convergence. Suppose that the fuzzy metric  $\mathcal{M}$  and  $M$  are compatible in preserving continuity sense. Given any disjoint pair  $A$  and  $B$  of  $\{1, 2, \dots, m\}$ . A mapping  $T : (X^m, \mathcal{M}, \preceq_A) \rightarrow (X^m, \mathcal{M}, \preceq_A)$  is continuous on  $X^m$  and satisfying any one of the following conditions:*

- (a)  $T$  has  $(\preceq_A, \preceq_A)$ -increasing;
- (b)  $p$  is an even integer and  $T$  is  $(\preceq_A, \preceq_A)$ -decreasing.

Assume that there exists  $\beta \in (0, 1)$  such that, for any  $\mathbf{u} \preceq_A \mathbf{v}$  or  $\mathbf{v} \preceq_A \mathbf{u}$ , the following inequalities

$$M(T_k^p(\mathbf{u}), T_k^p(\mathbf{v}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{u}), F_i(\mathbf{v}), t)),$$

for all  $t > 0$ ,  $\forall k = 1, \dots, m$  and  $\exists p \in \mathbb{N}$ , where  $\gamma : [0, 1] \rightarrow [0, 1]$  continuous  $*^m \gamma(a) \geq a$  for each  $a \in [0, 1]$ . If there exists  $\mathbf{u}_0 \in X^m$  such that  $\mathbf{u}_0 \preceq_A T^p(\mathbf{u}_0)$  or  $T^p(\mathbf{u}_0) \preceq_A \mathbf{u}_0$ , then  $T^p$  has a  $\preceq_A$ -chain-unique fixed point  $\hat{\mathbf{u}} \in X^m$ , such that  $\lim_{n \rightarrow \infty} \{F_k(\mathbf{u}_n)\}_{n \in \mathbb{N}} = \hat{u}^{(k)}$  for each component  $\hat{u}^{(k)}$  of  $\hat{\mathbf{u}}$  constructed in  $F(\mathbf{u}_n) = T^p(\mathbf{u}_{n-1})$  for all  $k = 1, 2, \dots, m$ .

*Proof.* By the proof of Lemma 4.2, we have  $\mathbf{u}_0$  is a seed element of  $\preceq_A$ -monotone in  $X^m$ . Thus, immediate results by Theorem 3.16, if  $F$  is the identity function.  $\square$

In the applications to the integral equations system and the differential equations system we using  $p = 1$  only.

**Theorem 4.4.** *Let  $(C([0, 1], \mathbb{R}), M, \preceq)$  be a quasi-ordered fuzzy metric spaces.  $(C([0, 1], \mathbb{R}), M, \preceq)$  is complete for monotonically and preserves of the monotone convergence. Let  $A$  and  $B$  be a disjoint pair of  $\{1, 2, \dots, m\}$ . Define the mappings  $T : (C^m([0, 1], \mathbb{R}), \mathcal{M}, \preceq_A) \rightarrow (C^m([0, 1], \mathbb{R}), \mathcal{M}, \preceq_A)$  by*

$$T(\mathbf{z})(t) = \int_0^1 G(s, t)[\mathbf{f}(s, \mathbf{z}(s)) + \beta \mathbf{z}(s)]ds$$

where  $\mathcal{M}$  is defined in (4) or (5). Suppose that it satisfies the following conditions:

- $T$  is  $(\preceq_A, \preceq_A)$ -increasing;

Assume that there exists  $\beta \in (0, 1)$  such that, for any  $\mathbf{z}, \bar{\mathbf{z}} \in C^m([0, 1], \mathbb{R})$  with  $\mathbf{z} \preceq_A \bar{\mathbf{z}}$  or  $\bar{\mathbf{z}} \preceq_A \mathbf{z}$ , the following inequalities

$$(29) \quad M(T_k(\mathbf{z}), T_k(\bar{\mathbf{z}}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{z}), F_i(\bar{\mathbf{z}}), t)),$$

are satisfied for all  $k = 1, \dots, m$ ;

- there exists  $\mathbf{z}_0 \in C^m([0, 1], \mathbb{R})$  such that  $\mathbf{z}_0 \preceq_A T(\mathbf{z}_0)$  or  $\mathbf{z}_0 \succcurlyeq_I T(\mathbf{z}_0)$ .

Then there exists a  $\preceq_A$ -chain-unique solution of the integral equations system (28).

*Proof.* Form  $\mathcal{M}$  is defined in (4) or (5), we get the fuzzy metrics  $\mathcal{M}$  and  $M$  are compatible in preserving convergence sense. By condition (a) in Lemma 4.3 such that  $T$  has a  $\preceq_A$ -chain-unique fixed point  $\mathbf{z}^*$  in  $C^m([0, 1], \mathbb{R})$ . That is, we get

$$\int_0^1 G(s, t)[\mathbf{f}(s, F(\mathbf{z}^*(s))) + \beta F(\mathbf{z}^*(s))]ds = T(\mathbf{z}^*) = \mathbf{z}^*,$$

we says that  $\mathbf{z}^*$  is a  $\preceq_A$ -chain-unique solution of the vector form equation (29).  $\square$

#### 4.2. Applications to the differential equations system.

Let  $(C([0, 1], \mathbb{R}), M, \preceq)$  be a quasi-ordered fuzzy metric space and following the ordinary differential equations system:

$$(30) \quad \begin{cases} z'(t) = f^{(k)}(t, \mathbf{z}(t)) & \text{for } t \in [0, 1] \\ z_k(0) = z_k(1) \end{cases} \quad \text{for } k = 1, \dots, m$$

where  $z_k \in C([0, 1], \mathbb{R})$  and for  $k = 1, \dots, m$ , the function  $f^{(k)} : [0, 1] \times \mathbb{R}^m \rightarrow \mathbb{R}$  are continuous and  $\mathbf{z}(t) = (z_1(t), z_2(t), \dots, z_m(t))$ .

**Definition 4.5.** Let  $(C([0, 1], \mathbb{R}), M, \preceq)$  be a quasi-ordered fuzzy metric space and  $C^m([0, 1], \mathbb{R})$  is a product space.

(a) We have a unique solution  $\mathbf{z}^*$  of the boundary value problem system (30) in the  $\preceq$ -mixed-comparable sense if and only if it satisfies the following conditions:

- $\mathbf{z}^*$  is a solution of equation (30);
- if  $\bar{\mathbf{z}}$  is another solution of equation (30) such that  $\mathbf{z}^*$  and  $\bar{\mathbf{z}}$  are  $\preceq$ -mixed-comparable, then  $\mathbf{z}^* = \bar{\mathbf{z}}$ .

Let  $A$  and  $B$  be a disjoint pair of  $\{1, \dots, m\}$ .

(b) We have a  $\preceq_A$ -chain-unique solution  $\mathbf{z}^*$  of the boundary value problem system (30) if and only if it satisfies the following conditions:

- $\mathbf{z}^*$  is a solution of equation (30);
- if  $\bar{\mathbf{z}}$  is another solution of equation (30) satisfying  $\mathbf{z}^* \preceq_A \bar{\mathbf{z}}$  or  $\bar{\mathbf{z}} \preceq_A \mathbf{z}^*$ , then  $\mathbf{z}^* = \bar{\mathbf{z}}$ .

**Definition 4.6.** Let  $(C([0, 1], \mathbb{R}), \preceq)$  be a quasi-ordered set.

- The quasi-order  $\preceq$  is compatible with the addition iff for all  $a, b, c \in C([0, 1], \mathbb{R})$ , where  $a \preceq b$ , we get  $a + c \preceq b + c$ .
- The quasi-order  $\preceq$  is compatible with the non-negative multiplication iff for all  $a, b \in C([0, 1], \mathbb{R})$  with  $a \preceq b$ , we have  $ac \preceq bc$  for any non-negative function  $c$  in  $C([0, 1], \mathbb{R})$ .

**Remark 4.7.** Suppose that the quasi-order  $\preceq$  is compatible with the addition. If  $a_1 \preceq b_1$  and  $a_2 \preceq b_2$ , we want to claim  $a_1 + a_2 \preceq b_1 + b_2$ . By definition, we immediately have

$$a_1 + a_2 \preceq b_1 + a_2 \quad \text{and} \quad b_1 + a_2 \preceq b_1 + b_2.$$

The transitivity proves the desired claim.

**Remark 4.8.** Let  $\preceq$  be a quasi-order in  $C([0, 1], \mathbb{R})$  and it is compatible with the addition and non-negative multiplication. Then we get the observations as following.

- For  $\mathbf{z}_1, \mathbf{z}_2, \bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2 \in C^m([0, 1], \mathbb{R})$ . If  $\mathbf{z}_1 \preceq_A \mathbf{z}_2$ , and  $\bar{\mathbf{z}}_1 \preceq_A \bar{\mathbf{z}}_2$ , then, from Remark 4.7, we have  $\mathbf{z}_1 + \bar{\mathbf{z}}_1 \preceq_A \mathbf{z}_2 + \bar{\mathbf{z}}_2$ .

- If  $\mathbf{z}_1, \mathbf{z}_2 \in C^m([0, 1], \mathbb{R})$  with  $\mathbf{z}_1 \preccurlyeq_A \mathbf{z}_2$ , then  $c \cdot \mathbf{z}_1 \preccurlyeq_A c \cdot \mathbf{z}_2$  for any non-negative function  $c$  in  $C([0, 1], \mathbb{R})$  by the compatibility for non-negative function.

We consider a disjoint pair  $A$  and  $B$  of  $\{1, 2, \dots, m\}$ . Suppose that the function  $\mathbf{f}: [0, 1] \times R^m \rightarrow R^m$  defined by

$$\mathbf{f} = (f^{(1)}, f^{(2)}, \dots, f^{(m)})$$

is  $(\preccurlyeq_A, \preccurlyeq_A)$ -increasing iff  $\mathbf{z}_1, \mathbf{z}_2 \in C^m([0, 1], \mathbb{R})$  with  $\mathbf{z}_1 \preccurlyeq_A \mathbf{z}_2$  implies  $\mathbf{f}(\cdot, \mathbf{z}_1) \preccurlyeq_A \mathbf{f}(\cdot, \mathbf{z}_2)$ . Similarly, by Definition 4.6 we can define the concept of monotone.

**Definition 4.9.** Let  $(C([0, 1], \mathbb{R}), \preceq)$  be a quasi-ordered set and it is compatible. Consider mapping  $\mathcal{G}: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ , we have  $\mathcal{G}(s, \cdot)$  in  $C([0, 1], \mathbb{R})$  for fixed  $s \in [0, 1]$  and  $\mathcal{G}(\cdot, t)$  in  $C([0, 1], \mathbb{R})$  for fixed  $t \in [0, 1]$ . We called that the integral of function  $\mathcal{G}$  is compatible if and only if for each fixed  $t \in [0, 1]$ ,  $\mathcal{G}_1(\cdot, t) \preceq \mathcal{G}_2(\cdot, t)$  implies

$$\int_0^1 \mathcal{G}_1(s, t) ds \preceq \int_0^1 \mathcal{G}_2(s, t) ds.$$

**Theorem 4.10.** Let  $(C([0, 1], \mathbb{R}), M, \preceq)$  be a quasi-ordered fuzzy metric space. It's complete for monotonically and preserves of the monotone convergence. Given a disjoint pair  $A$  and  $B$  of  $\{1, 2, \dots, m\}$ , and for  $\beta > 0$ . The function  $G: [0, 1] \times [0, 1] \rightarrow \mathbb{R}^+$  define by

$$(31) \quad G(s, t) = \begin{cases} \frac{e^{\beta(1+s-t)}}{e^\beta - 1} & \text{if } 0 \leq s < t \leq 1, \\ \frac{e^{\beta(s-t)}}{e^\beta - 1} & \text{if } 0 \leq t < s \leq 1, \end{cases}$$

and the mapping  $T: (C^m([0, 1], \mathbb{R}), \mathcal{M}, \preccurlyeq_A) \rightarrow (C^m([0, 1], \mathbb{R}), \mathcal{M}, \preccurlyeq_A)$  by

$$T(\mathbf{z})(t) = \int_0^1 G(s, t)[\mathbf{f}(s, \mathbf{z}(s)) + \beta \mathbf{z}(s)] ds,$$

where  $\mathcal{M}$  is defined in (4) or (5). Suppose that it satisfied the conditions as following:

- the quasi-order  $\preceq$  in  $C([0, 1], \mathbb{R})$  is compatible with the addition and non-negative multiplication;
- $\mathbf{f}$  is  $(\preccurlyeq_A, \preccurlyeq_A)$ -increasing;

- for any  $\mathbf{z} \in C^m([0, 1], \mathbb{R})$ , the integrals of functions

$$(32) \quad \mathcal{G}_k(s, t) = G(s, t)[f^{(k)}(s, \mathbf{z}(s)) + \beta z_k(s)]$$

are compatible for  $k = 1, 2, \dots, m$ .

Assume that there exists  $\beta \in (0, 1)$  such that, for any  $\mathbf{z}, \bar{\mathbf{z}} \in C^m([0, 1], \mathbb{R})$  with  $\mathbf{z} \preceq_A \bar{\mathbf{z}}$  or  $\bar{\mathbf{z}} \preceq_A \mathbf{z}$ , the following inequalities

$$(33) \quad M(T_k(\mathbf{z}), T_k(\bar{\mathbf{z}}), \beta t) \geq \gamma(*_{i=1}^m M(F_i(\mathbf{z}), F_i(\bar{\mathbf{z}}), t)),$$

are satisfied for all  $k = 1, \dots, m$ ;

- there exists  $\mathbf{z}_0 \in C^m([0, 1], \mathbb{R})$  such that  $\mathbf{z}_0 \preceq_A F(\mathbf{z}_0)$  or  $\mathbf{z}_0 \succ_I F(\mathbf{z}_0)$ .

Then the boundary value problem system (30) has a  $\preceq_A$ -chain-unique solution.

*Proof.* Consider problem (30), we have

$$(34) \quad \begin{cases} z'_k(t) + \beta z_k(t) = f^{(k)}(t, \mathbf{z}(t)) + \beta z_k(t) & \text{for } t \in [0, 1], \\ z_k(0) = z_k(1), \end{cases} \quad \forall k = 1, \dots, m$$

We can rewrite problem (34) as a integral equations system. Multiplying  $e^{\beta t}$  on both sides, we get

$$e^{\beta t} \cdot z'_k(t) + e^{\beta t} \cdot \beta z_k(t) = [f^{(k)}(t, \mathbf{z}(t)) + \beta z_k(t)] \cdot e^{\beta t},$$

which is equivalent to

$$(e^{\beta t} \cdot z_k(t))' = [f^{(k)}(t, \mathbf{z}(t)) + \beta z_k(t)] \cdot e^{\beta t} \quad \text{for } t \in [0, 1]$$

Taking integration on both sides of equation, we get

$$(35) \quad e^{\beta t} \cdot z_k(t) = z_k(0) + \int_0^t [f^{(k)}(s, \mathbf{z}(s)) + \beta z_k(s)] \cdot e^{\beta s} ds, \quad \text{for } t \in [0, 1]$$

Since  $z_k(0) = z_k(1)$ , we obtain

$$e^{\beta} \cdot z_k(0) = e^{\beta} \cdot z_k(1) = z_k(0) + \int_0^1 [f^{(k)}(s, \mathbf{z}(s)) + \beta z_k(s)] \cdot e^{\beta s} ds,$$

which implies

$$(36) \quad z_k(0) = \int_0^1 \frac{e^{\beta s}}{e^{\beta} - 1} \cdot [f^{(k)}(s, \mathbf{z}(s)) + \beta z_k(s)] ds.$$

From (35) and (36), for  $t \in [0, 1]$ , we have

$$\begin{aligned}
e^{\beta t} \cdot z_k(t) &= \int_0^1 \frac{e^{\beta s}}{e^\beta - 1} \cdot [f^{(k)}(s, \mathbf{z}(s)) + \beta z_k(s)] ds \\
&\quad + \int_0^t [f^{(k)}(s, \mathbf{z}(s)) + \beta z_k(s)] \cdot e^{\beta s} ds \\
&= \int_0^t \frac{e^{\beta s + \beta}}{e^\beta - 1} \cdot [f^{(k)}(s, \mathbf{z}(s)) + \beta z_k(s)] ds \\
&\quad + \int_t^1 \frac{e^{\beta s}}{e^\beta - 1} \cdot [f^{(k)}(s, \mathbf{z}(s)) + \beta z_k(s)] ds
\end{aligned}$$

Thus, we obtain the following integral equations:

$$(37) \quad z_k(t) = \int_0^1 G(s, t) [f^{(k)}(s, \mathbf{z}(s)) + \beta z_k(s)] ds, \quad \forall k = 1, \dots, m$$

where  $G(s, t)$  is given in equation (31). If  $\mathbf{z}^*$  is a solution of equation (37), then  $\mathbf{z}^*$  is a solution of equation (34), also  $\mathbf{z}^*$  is a solution of equation (30).

Form  $G(s, t) \geq 0$ ,  $\preceq$  is compatible with the addition and non-negative multiplication, and  $f$  is  $(\preceq_A, \preceq_A)$ -increasing, for fixed  $t \in [0, 1]$ , by Remark 4.8, we get  $G(s, t) \cdot [f(s, \mathbf{z}(s)) + \beta \mathbf{z}(s)]$  is  $(\preceq_A, \preceq_A)$ -increasing. For  $k = 1, 2, \dots, m$ , by (32) the integrals of the functions  $G_k$  are compatible with the quasi-order  $(\preceq)$ , we have  $T$  is  $(\preceq_A, \preceq_A)$ -increasing. Theorem 3.14 says that the integral equations system (37) has  $\preceq_A$ -chain-unique solution  $\mathbf{z}^*$ , also says that  $\mathbf{z}^*$  is a solution of problem (30). Anywise, this  $\preceq_A$ -chain-uniqueness is about the integral equations system (37), it is not the  $\preceq_A$ -chain-uniqueness about the problem (30). Now, given  $\hat{\mathbf{w}}$  be another solution of the problem system (30) such that  $\hat{\mathbf{z}}$  and  $\mathbf{z}^*$  are comparable. By the derivative of (37), we have  $\hat{\mathbf{z}}$  is also a solution of the integral equations system (37). Form the  $\preceq_A$ -chain-uniqueness in Theorem 4.4, such that  $\mathbf{z}^* = \hat{\mathbf{z}}$ ; that is,  $\mathbf{z}^*$  is a  $\preceq_A$ -chain-unique solution of the problem system (30).  $\square$

## 5. CONCLUSION

In this paper we have proved coincidence point results in the product spaces of complete for mixed-monotone quasi-ordered fuzzy metric spaces. Our results extend and improve the corresponding recent results announced by Hsien-Chung Wu (2014) and Roldán et.al. (2014).

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