

RODRIGUES PARAMETERS ON DUAL HYPERBOLIC UNIT SPHERE $\bar{H}_0^2$

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ABSTRACT. Rodrigues parameters depend on the tangent of the half rotation angle in Euclidean space but in Dual space, dual Rodrigues parameters contain both rotation angle and distance corresponding the shortest distance between the straight lines in $\mathbb{R}^3$. In this paper, we give Cayley's formula for the dual hyperbolic spherical motion and explain 3×3 type L-Dual skew symmetric matrices by using properties of this formula. Then, we obtain Rodrigues parameters of dual Hyperbolic unit sphere and show that Rodrigues parameters contain the hyperbolic rotation angle which is being between timelike lines and distance which is the minimal Lorentzian distance between the timelike lines of $\mathbb{R}_1^3$.

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1. INTRODUCTION

Dual numbers were introduced by Clifford in the 19th century. Later on, Study [12], who showed special interest to the representation of directed lines by dual unit

vectors, generalized their first applications to mechanics and defined the mapping that is known by his name. This mapping exists one-to-one correspondence between the vectors of dual unit sphere and the directed line in $\mathbb{R}^3$. In 1996, Uğurlu and Çalışkan [13] defined a Study mapping in Lorentzian space $\mathbb{R}_1^3$ instead of $\mathbb{R}^3$.

Euler-Rodrigues formula was first introduced in Euler's equations in 1775. In 1840, these equations which is known as Rodrigues parameters which depend on tangent of half the rotation angle related to coordinates of the rotation axis were defined independently by Rodrigues [11]. The vector form of this formula was given by Gibbs [4], Bisshopp [1] and Bottema and Roth [2] and they offered Rodrigues formula in planar and spatial motion. Euler-Rodrigues formula has been effectively used in geometric algebras, robotics and theoretical kinematics. Moreover, the Rodrigues formula which is a method of calculating the position of a rotated point, is used in some software applications, such as flight simulators and computer games. For more details, we refer the readers to ([3], [7], [9]). Today, this formula has attracted much attention in kinematics. In 2009, Rodrigues parameters were obtained by Özkaldı and Gündoğan [8] in 3-dimensional Lorentzian space and in 2010, dual Rodrigues parameters were obtained by Karakılıç [6] in unit dual Sphere $\bar{S}^2$. He showed that Rodrigues parameters had relationship with directed lines in $\mathbb{R}^3$. The aim of this study is to give Rodrigues parameters on dual hyperbolic sphere $\bar{H}_0^2$, which allows us to provide the motions of the points on hyperbolic sphere whose rotation axis is a spacelike. Moreover, Rodrigues parameters are used to obtain rotation matrix on dual hyperbolic unit sphere.

Now, we recall a brief summary of the theory of dual numbers, Lorentz space and dual Lorentz space.

A dual number is formal sum $A = a + \varepsilon a^*$ where a and a^* are real numbers and $\varepsilon^2 = 0$. The set of all dual numbers is a commutative ring over the real numbers field and is showed by D . Then

$$D^3 = \{\bar{a} = (A_1, A_2, A_3) \mid A_i \in D, 1 \leq i \leq 3\}$$

is a set that is a module over the ring D and is named D -module or Dual space. The elements of D^3 are called as dual vector. A dual vector $\bar{a}$ may be given in the form $\bar{a} = \vec{a} + \varepsilon \vec{a}^*$ where $\vec{a}$ and $\vec{a}^*$ are the vectors of $\mathbb{R}^3$. For any vectors $\bar{a} = \vec{a} + \varepsilon \vec{a}^*$, $\bar{b} = \vec{b} + \varepsilon \vec{b}^* \in D^3$, the inner product and the vector product of

these vectors are defined by, respectively,

$$\langle \bar{a}, \bar{b} \rangle = \langle \vec{a}, \vec{b} \rangle + \varepsilon \left(\langle \vec{a}, \vec{b}^* \rangle + \langle \vec{a}^*, \vec{b} \rangle \right)$$

and

$$\bar{a} \times \bar{b} = \vec{a} \times \vec{b} + \varepsilon \left(\vec{a} \times \vec{b}^* + \vec{a}^* \times \vec{b} \right).$$

The norm of a dual vector $\bar{a}$ is defined to be

$$\|\bar{a}\| = \|\vec{a}\| + \varepsilon \frac{\langle \vec{a}, \vec{a}^* \rangle}{\|\vec{a}\|}.$$

A dual vector $\bar{a}$ with norm 1 is named dual unit vector. The set of dual unit vectors is

$$\bar{S}^2 = \{ \bar{a} = \vec{a} + \varepsilon \vec{a}^* \in D^3 \mid \|\bar{a}\| = 1, \vec{a}, \vec{a}^* \in \mathbb{R}^3 \}$$

which is called dual unit sphere. The vectors of unit sphere $\bar{S}^2$ are one-to-one correspondence with the directed lines in $\mathbb{R}^3$.

For the vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, let R_1^3 be the 3-dimensional Lorentz space over the field of real numbers R with the Lorentzian inner product $\langle, \rangle$ given by

$$\langle \vec{a}, \vec{b} \rangle_L = -a_1 b_1 + a_2 b_2 + a_3 b_3.$$

A vector $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}_1^3$ is said to be timelike if $\langle \vec{a}, \vec{a} \rangle_L < 0$, spacelike if $\langle \vec{a}, \vec{a} \rangle_L > 0$ and lightlike (null) if $\langle \vec{a}, \vec{a} \rangle_L = 0$ and $a \neq 0$. The norm of $\vec{a}$ is defined $\|\vec{a}\|_L = \sqrt{\langle \vec{a}, \vec{a} \rangle_L}$. The norm $\|\vec{a}\|_L$ is either positive or zero or positive imaginary. If $\|\vec{a}\|_L$ is positive imaginary, then the notation $||\vec{a}||$ is used instead of $\|\vec{a}\|_L$.

The Lorentzian cross product of vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is defined by

$$\vec{a} \times_L \vec{b} = (a_3 b_2 - a_2 b_3, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

Consider that φ is the angle between vectors $\vec{a}$ and $\vec{b}$. There exist the following equations:

1. For the timelike vectors $\vec{a}, \vec{b}$ in $\mathbb{R}_1^3$,

$$\langle \vec{a}, \vec{b} \rangle_L = -||\vec{a}|| \cdot ||\vec{b}|| \cosh \varphi$$

$$||\vec{a} \times_L \vec{b}|| = ||\vec{a}|| \cdot ||\vec{b}|| \sinh \varphi$$

2. For the spacelike vector $\vec{a}$ and positive timelike vector $\vec{b}$ in $\mathbb{R}_1^3$,

$$\begin{aligned} |\langle \vec{a}, \vec{b} \rangle_L| &= \|\vec{a}\|_L \cdot \|\vec{b}\| \sinh \varphi \\ \|\vec{a} \times_L \vec{b}\| &= \|\vec{a}\|_L \cdot \|\vec{b}\| \cosh \varphi. \end{aligned}$$

3. For the spacelike vectors $\vec{a}, \vec{b}$ in $\mathbb{R}_1^3$ that span a timelike vector subspace

$$\begin{aligned} |\langle \vec{a}, \vec{b} \rangle_L| &= \|\vec{a}\|_L \cdot \|\vec{b}\|_L \cosh \varphi \\ \|\vec{a} \times_L \vec{b}\| &= \|\vec{a}\|_L \cdot \|\vec{b}\|_L \sinh \varphi. \end{aligned}$$

Let $\vec{a}$ and $\vec{b}$ be two vectors in $\mathbb{R}_1^n$. Then, the Lorentzian inner product on $\mathbb{R}_1^n$ is given as below

$$\langle \vec{a}, \vec{b} \rangle_L = -a_1 b_1 + \sum_{i=2}^n a_i b_i \quad [10].$$

The set of unit timelike and spacelike vectors are named Hyperbolic and Lorentzian unit sphere and shown by

$$H_0^2 = \{ \vec{a} = (a_1, a_2, a_3) \in \mathbb{R}_1^3 \mid \langle \vec{a}, \vec{a} \rangle_L = -1 \}$$

and

$$S_1^2 = \{ \vec{a} = (a_1, a_2, a_3) \in \mathbb{R}_1^3 \mid \langle \vec{a}, \vec{a} \rangle_L = 1 \}$$

respectively [13].

The Lorentzian inner and cross product of dual vectors $\bar{a} = \vec{a} + \varepsilon \vec{a}^*$, $\bar{b} = \vec{b} + \varepsilon \vec{b}^* \in D^3$ is defined by

$$\langle \bar{a}, \bar{b} \rangle_L = \langle \vec{a}, \vec{b} \rangle_L + \varepsilon \left(\langle \vec{a}, \vec{b}^* \rangle_L + \langle \vec{a}^*, \vec{b} \rangle_L \right)$$

and

$$\bar{a} \times_L \bar{b} = \vec{a} \times_L \vec{b} + \varepsilon \left(\vec{a} \times_L \vec{b}^* + \vec{a}^* \times_L \vec{b} \right)$$

respectively. The Lorentzian causal character of a dual vector $\bar{a} = \vec{a} + \varepsilon \vec{a}^*$ is same with the causal character of $\vec{a}$ [13]. A dual vector $\bar{a}$ is said to be timelike if $\langle \vec{a}, \vec{a} \rangle_L < 0$, spacelike if $\langle \vec{a}, \vec{a} \rangle_L > 0$ and lightlike(or null) if $\langle \vec{a}, \vec{a} \rangle_L = 0$ and $a \neq 0$ which is Lorentzian inner product with signature $(-, +, +)$. The norm of $\bar{a}$ is defined as

$$\|\bar{a}\|_L = \sqrt{\langle \bar{a}, \bar{a} \rangle_L} = \|\vec{a}\|_L + \varepsilon \frac{\vec{a} \cdot_L \vec{a}^*}{\|\vec{a}\|_L}$$

where $\vec{a} \cdot_L \vec{a}^*$ can be used instead of $\langle \vec{a}, \vec{a} \rangle_L$. Then, the set

$$D_1^3 = \{ \bar{a} = \vec{a} + \varepsilon \vec{a}^* \mid \vec{a}, \vec{a}^* \in \mathbb{R}_1^3 \}$$

is called as dual Lorentzian space (L-Dual space).

The dual Hyperbolic and dual Lorentzian unit sphere are denoted by

$$\bar{H}_0^2 = \{\bar{a} = \vec{a} + \varepsilon \vec{a}^* \in D_1^3 \mid \langle \bar{a}, \bar{a} \rangle_L = -1, \vec{a}, \vec{a}^* \in \mathbb{R}_1^3\}$$

and

$$\bar{S}_1^2 = \{\bar{a} = \vec{a} + \varepsilon \vec{a}^* \in D_1^3 \mid \langle \bar{a}, \bar{a} \rangle_L = 1, \vec{a}, \vec{a}^* \in \mathbb{R}_1^3\}$$

respectively. There are two components of dual Hyperbolic unit sphere. The components of $\bar{H}_0^2$ passing through $(1, 0, 0)$ and $(-1, 0, 0)$ are called as a future-pointing dual hyperbolic unit sphere and a past-pointing dual hyperbolic unit sphere and shown by $\bar{H}_0^{2+}$ and $\bar{H}_0^{2-}$, respectively. Thus, with regard to this definition, the following sets can be written

$$\bar{H}_0^{2+} = \{\bar{a} = \vec{a} + \varepsilon \vec{a}^* \in \bar{H}_0^2 \mid a \text{ is a future-pointing timelike unit vector}\}$$

and

$$\bar{H}_0^{2-} = \{\bar{a} = \vec{a} + \varepsilon \vec{a}^* \in \bar{H}_0^2 \mid a \text{ is a past-pointing timelike unit vector}\}.$$

The dual timelike and spacelike vectors of dual Hyperbolic and Lorentzian unit sphere $\bar{H}_0^2$ and $\bar{S}_1^2$ are one-to-one correspondence with the directed timelike and spacelike line of the Lorentzian space $\mathbb{R}_1^3$ [13].

In L-Dual space D_1^3 , the angle between vectors $\bar{a}, \bar{b}$ is defined as follows:

1. Let $\bar{a}$ and $\bar{b}$ be dual timelike vectors in D_1^3 . The dual angle is defined by

$$\langle \bar{a}, \bar{b} \rangle_L = -\|\bar{a}\|_L \cdot \|\bar{b}\|_L \cosh \bar{\phi}.$$

2. Let $\bar{a}$ and $\bar{b}$ be dual timelike vectors in D_1^3 which span a dual timelike vector subspace. The dual angle between these vectors is determined as follows

$$\langle \bar{a}, \bar{b} \rangle_L = \|\bar{a}\|_L \cdot \|\bar{b}\|_L \cosh \bar{\phi}.$$

3. Let $\bar{a}$ be a dual spacelike vector and $\bar{b}$ be a dual timelike vector in D_1^3 . Then, the dual angle is described by

$$\langle \bar{a}, \bar{b} \rangle_L = \|\bar{a}\|_L \cdot \|\bar{b}\|_L \sinh \bar{\phi}.$$

Let us take $\bar{A} = [\bar{a}_{ij}] \in D_n^m$ which is the set of $m \times n$ type dual matrix and $\bar{B} = [\bar{b}_{jk}] \in D_p^n$ where $\bar{a}_{ij} = a_{ij} + \varepsilon a_{ij}^*$ and $\bar{b}_{jk} = b_{jk} + \varepsilon b_{jk}^*$. Lorentzian matrix

multiplication which is called as L-multiplication is defined by

$$\bar{A} \cdot_L \bar{B} = \left[-\bar{a}_{i1} \bar{b}_{1k} + \sum_{j=2}^n \bar{a}_{ij} \bar{b}_{jk} \right],$$

where $\bar{A} \cdot_L \bar{B}$ is matrix of m rows and p columns.

The $n \times n$ identity matrix corresponding to L-multiplication is shown by I_n . I_n is described as follows

$$I_n = \begin{bmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix}_{n \times n}.$$

If $\bar{A} \cdot_L \bar{B} = \bar{B} \cdot_L \bar{A} = I_n$, a $n \times n$ matrix $\bar{A}$ is called as L-invertible where $\bar{B}$ is an $n \times n$ matrix. At that time, $\bar{B}$ is called L-inverse of $\bar{A}$ and is indicated by $\bar{A}^{-1}$. The transpose of a matrix $\bar{A} = [\bar{a}_{ij}] \in D_n^m$ is denoted by $\bar{A}^T$ and is defined as $\bar{A}^T = [\bar{a}_{ji}] \in D_m^n$. A matrix $\bar{A} \in D_n^n$ is called L-dual orthogonal matrix if $\bar{A}^{-1} = \bar{A}^T$ [5].

Let f be a differentiable dual function with dual variable $x + \varepsilon x^*$. Thus, Taylor expansion of dual function f is

$$f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x)$$

where $f'(x)$ is the derivation of f .

2. CAYLEY'S FORMULA FOR THE DUAL HYPERBOLIC SPHERICAL MOTION

$\bar{X} = \bar{A} \cdot_L \bar{x}$ corresponds to a rotation about the origin where $\bar{A}$ is a 3×3 type L-Dual orthogonal matrix and $\bar{x} \in D_1^3$. Suppose that the vector $\bar{x}$ is on dual hyperbolic unit sphere $\bar{H}_0^2$. The length of dual vector is preserved by the dual hyperbolic unit spherical motion, that is

$$\|\bar{X}\|_L = \|\bar{x}\|_L$$

which yields

$$(\bar{X} - \bar{x})^T \cdot_L (\bar{X} + \bar{x}) = 0.$$

Since the equality $\bar{X} = \bar{A} \cdot_L \bar{x}$ expressing a rotation about the origin is existed, the following equations are easily calculated

$$\begin{aligned}\bar{X} + \bar{x} &= \bar{A} \cdot_L \bar{x} + \bar{x} \\ &= (\bar{A} + I) \cdot_L \bar{x}\end{aligned}$$

and

$$\begin{aligned}\bar{X} - \bar{x} &= \bar{A} \cdot_L \bar{x} - \bar{x} \\ &= (\bar{A} - I) \cdot_L \bar{x}.\end{aligned}$$

Except for $\bar{A} \cdot_L \bar{x} = -\bar{x}$, it is clear that $\bar{A} + I$ is regular and when the above mentioned equalities are considered,

$$(\bar{A} + I)^{-1} \cdot_L (\bar{X} + \bar{x}) = \bar{x}.$$

can be briefly written. Thus, if $\bar{x}$ selected on $\bar{H}_0^2$ is taken into account,

$$\bar{X} - \bar{x} = (\bar{A} - I) \cdot_L (\bar{A} + I)^{-1} \cdot_L (\bar{X} + \bar{x})$$

is obtained. Suppose that the matrix $(\bar{A} - I) \cdot_L (\bar{A} + I)^{-1}$ is shortly symbolized as $\bar{Z}$

$$\bar{Z} = (\bar{A} - I) \cdot_L (\bar{A} + I)^{-1},$$

it is possible to write the below expression

$$\bar{X} - \bar{x} = \bar{Z} \cdot_L (\bar{X} + \bar{x}).$$

Due to the fact that $\bar{Z}^T = -\bar{Z}$, $\bar{Z}$ is a skew-symmetric matrix. After solving the equation $\bar{Z} = (\bar{A} - I) \cdot_L (\bar{A} + I)^{-1}$ for $\bar{A}$, the formula which is called as L-Dual Cayley's formula is attained as below:

$$\bar{A} = (I - \bar{Z})^{-1} \cdot_L (I + \bar{Z}),$$

or its equivalent form;

$$\bar{A} = (I + \bar{Z}) \cdot_L (I - \bar{Z})^{-1}.$$

Since $\bar{Z}$ is skew-symmetric, it is allowed to compute the matrix $\bar{A}^T$ as follows;

$$\begin{aligned}\bar{A}^T &= \left(I + \bar{Z} \right)^T \cdot_L \left(\left(I - \bar{Z} \right)^{-1} \right)^T \\ &= \left(I + \bar{Z}^T \right) \cdot_L \left(I - \bar{Z}^T \right)^{-1} \\ &= \left(I - \bar{Z} \right) \cdot_L \left(I + \bar{Z} \right)^{-1}.\end{aligned}$$

Considering the above mentioned matrices $\bar{A}$ and $\bar{A}^T$, it is clear that

$$\begin{aligned}\bar{A}^T \cdot_L \bar{A} &= \left(I - \bar{Z} \right) \cdot_L \left(I + \bar{Z} \right)^{-1} \cdot_L \left(I + \bar{Z} \right) \cdot_L \left(I - \bar{Z} \right)^{-1} \\ &= I_n.\end{aligned}$$

Then, $\bar{A}$ is a L-Dual Orthogonal Matrix.

Corollary 1. *An orthogonal L-Dual matrix $\bar{A}$ is generated by every skew-symmetric L-Dual matrix $\bar{Z}$.*

3. 3×3 TYPE L-DUAL SKEW-SYMMETRIC MATRICES

By using the equality $\bar{Z} \cdot_L \bar{y} = \bar{z} \times_L \bar{y}$, let us determine the matrix $\bar{Z}$.

$$\text{Let us take } \bar{Z} = \begin{bmatrix} \bar{z}_{11} & \bar{z}_{12} & \bar{z}_{13} \\ \bar{z}_{21} & \bar{z}_{22} & \bar{z}_{23} \\ \bar{z}_{31} & \bar{z}_{32} & \bar{z}_{33} \end{bmatrix}, \quad \bar{y} = (\bar{y}_1, \bar{y}_2, \bar{y}_3), \quad \bar{z} = (\bar{z}_1, \bar{z}_2, \bar{z}_3),$$

where $\bar{y}$ is a dual timelike vector which is centered at the $\bar{H}_0^2$. If we consider the equality $\bar{Z} \cdot_L \bar{y} = \bar{z} \times_L \bar{y}$, the following equalities are easily calculated

$$\begin{aligned}z_{11} &= 0, \quad z_{12} = z_3, \quad z_{13} = -z_2 \\ z_{21} &= -z_3, \quad z_{22} = 0, \quad z_{23} = -z_1 \\ z_{31} &= z_2, \quad z_{32} = z_1, \quad z_{33} = 0\end{aligned}$$

and

$$\begin{aligned}z_{11}^* &= 0, \quad z_{12}^* = z_3^*, \quad z_{13}^* = -z_2^* \\ z_{21}^* &= -z_3^*, \quad z_{22}^* = 0, \quad z_{23}^* = -z_1^* \\ z_{31}^* &= z_2^*, \quad z_{32}^* = z_1^*, \quad z_{33}^* = 0.\end{aligned}$$

Consequently, if we take account of the above expressions, we can give

$$\begin{aligned}\bar{z}_{11} &= 0 & \bar{z}_{12} &= \bar{z}_3 & \bar{z}_{13} &= -\bar{z}_2 \\ \bar{z}_{21} &= -\bar{z}_3 & \bar{z}_{22} &= 0 & \bar{z}_{23} &= -\bar{z}_1 \\ \bar{z}_{31} &= \bar{z}_2 & \bar{z}_{32} &= \bar{z}_1 & \bar{z}_{33} &= 0\end{aligned}$$

and

$$\bar{Z} = \begin{bmatrix} 0 & \bar{z}_3 & -\bar{z}_2 \\ -\bar{z}_3 & 0 & -\bar{z}_1 \\ \bar{z}_2 & \bar{z}_1 & 0 \end{bmatrix}.$$

4. L-DUAL RODRIGUES EQUATION FOR ROTATIONS

Let $\bar{A}$ be a L-Dual orthogonal matrix. We think the equation of a rotation about the origin, i.e., $\bar{A} \cdot_L \bar{x} = \bar{X}$. By using L-Dual Cayley's formula mentioned section 2, we obtain the following equality

$$\bar{X} - \bar{x} = \bar{Z} \cdot_L (\bar{X} + \bar{x}).$$

If we consider the relation between the Lorentzian cross product and skew-symmetric matrix, we can write

$$\bar{X} - \bar{x} = \bar{z} \times_L (\bar{X} + \bar{x}).$$

The above mentioned equality is called as the L-Dual Rodrigues equation and $\bar{z}$ referred in this equation is called as the L-Dual Rodrigues vector.

Theorem 1. *In D_1^3 , the rotation matrix $\bar{A}$ transforms the timelike vectors to time-like vectors, the spacelike vectors to spacelike vectors and the null vectors to null vectors.*

Proof. Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) = (x_1 + \varepsilon x_1^*, x_2 + \varepsilon x_2^*, x_3 + \varepsilon x_3^*) \in D_1^3$. From L-Dual Cayley's formula, it is easily seen that $\bar{A} = (I - \bar{Z})^{-1} \cdot_L (I + \bar{Z})$. In this case, if required procedures are made, the matrix $\bar{A} \cdot_L \bar{x}$ is as follows

$$\bar{A} \cdot_L \bar{x} = \frac{1}{\bar{z}_1^2 - \bar{z}_2^2 - \bar{z}_3^2 + 1} \times \begin{bmatrix} -(\bar{z}_1^2 + \bar{z}_2^2 + \bar{z}_3^2 + 1) & 2(\bar{z}_3 - \bar{z}_1 \bar{z}_2) & -2(\bar{z}_2 + \bar{z}_1 \bar{z}_3) \\ -2(\bar{z}_3 + \bar{z}_1 \bar{z}_2) & -(\bar{z}_1^2 + \bar{z}_2^2 - \bar{z}_3^2 - 1) & -2(\bar{z}_1 + \bar{z}_2 \bar{z}_3) \\ 2(\bar{z}_2 - \bar{z}_1 \bar{z}_3) & 2(\bar{z}_1 - \bar{z}_2 \bar{z}_3) & -(\bar{z}_1^2 - \bar{z}_2^2 + \bar{z}_3^2 - 1) \end{bmatrix} \cdot_L \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}$$

$$= \frac{1}{\bar{z}_1^2 - \bar{z}_2^2 - \bar{z}_3^2 + 1} \times \begin{bmatrix} \bar{x}_1 \left(\bar{z}_1^2 + \bar{z}_2^2 + \bar{z}_3^2 + 1 \right) + 2\bar{x}_2 (\bar{z}_3 - \bar{z}_1 \bar{z}_2) - 2\bar{x}_3 (\bar{z}_2 + \bar{z}_1 \bar{z}_3) \\ 2\bar{x}_1 (\bar{z}_3 + \bar{z}_1 \bar{z}_2) - \bar{x}_2 \left(\bar{z}_1^2 + \bar{z}_2^2 - \bar{z}_3^2 - 1 \right) - 2\bar{x}_3 (\bar{z}_1 + \bar{z}_2 \bar{z}_3) \\ -2\bar{x}_1 (\bar{z}_2 - \bar{z}_1 \bar{z}_3) + 2\bar{x}_2 (\bar{z}_1 - \bar{z}_2 \bar{z}_3) - \bar{x}_3 \left(\bar{z}_1^2 - \bar{z}_2^2 + \bar{z}_3^2 - 1 \right) \end{bmatrix}.$$

Here,

$$\begin{aligned} \langle \bar{z}, \bar{z} \rangle_L &= \langle (z_1 + \varepsilon z_1^*, z_2 + \varepsilon z_2^*, z_3 + \varepsilon z_3^*), (z_1 + \varepsilon z_1^*, z_2 + \varepsilon z_2^*, z_3 + \varepsilon z_3^*) \rangle_L \\ &= -z_1^2 + z_2^2 + z_3^2 + 2\varepsilon (-z_1 z_1^* + z_2 z_2^* + z_3 z_3^*) \end{aligned}$$

is simply computed where $-z_1^2 + z_2^2 + z_3^2 \neq 1$. So, we have the equality

$$\left\langle \bar{A} \cdot_L \bar{x}, \bar{A} \cdot_L \bar{x} \right\rangle_L = -\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 = \langle \bar{x}, \bar{x} \rangle_L.$$

Thus, the proof is completed. $\square$

In L-Dual Rodrigues equation $\bar{X} - \bar{x} = \bar{z} \times_L (\bar{X} + \bar{x})$, let us take account $\bar{z}$ spacelike vector and $\bar{z}$ be perpendicular to $\bar{x}$. Because of $\bar{x} \in \bar{H}_0^2$ and from Theorem 1, it is clear that $\bar{X}$ is a timelike vector and $\bar{X} \in \bar{H}_0^2$. Due to

$$\left\| \vec{X} + \vec{x} \right\|_L^2 = \left\langle \vec{X}, \vec{X} \right\rangle_L - 2 \left\| \vec{x} \right\| \cdot \left\| \vec{X} \right\| \cosh \phi + \langle \vec{x}, \vec{x} \rangle_L < 0,$$

then $\bar{X} + \bar{x} = \left(\vec{X} + \vec{x} \right) + \varepsilon \left(\vec{X}^* + \vec{x}^* \right)$ is a timelike vector. When we consider the above mentioned geometric comments about $\bar{z}$ and $\bar{X} + \bar{x}$, the fact that $\bar{z}$ is perpendicular to $\bar{X} + \bar{x}$ is easily obtained. Let $\bar{\phi}$ be the dual angle between $\bar{z}$ and $\bar{X} + \bar{x}$. Thinking the norm of the expression

$$\bar{X} - \bar{x} = \bar{z} \times_L (\bar{X} + \bar{x}),$$

it is possible to write the following equality

$$\left\| \bar{X} - \bar{x} \right\|_L = \left\| \bar{z} \right\|_L \left\| \bar{X} + \bar{x} \right\|_L \cosh \bar{\phi}.$$

Since $\bar{X} + \bar{x}$ is perpendicular to $\bar{z}$, then

$$\left\langle \bar{z}, \bar{X} + \bar{x} \right\rangle_L = \left\| \bar{z} \right\|_L \left\| \bar{X} + \bar{x} \right\|_L \sinh \bar{\phi} = 0.$$

On the other hand, because of $\sinh \bar{\phi} = 0$, $\bar{\phi}$ is zero. Thus, $\cosh \bar{\phi} = 1$. Then, we can obtain the below statement

$$\left\| \bar{X} - \bar{x} \right\|_L = \left\| \bar{z} \right\|_L \left\| \bar{X} + \bar{x} \right\|_L.$$

Moreover, $\vec{X} = \vec{X} + \varepsilon \vec{X}^*$ and

$$\begin{aligned} \langle \vec{X}, \vec{X} \rangle_L &= \langle \vec{X} + \varepsilon \vec{X}^*, \vec{X} + \varepsilon \vec{X}^* \rangle_L \\ &= \langle \vec{X}, \vec{X} \rangle_L + 2\varepsilon \langle \vec{X}, \vec{X}^* \rangle_L. \end{aligned}$$

is computed. Since $\vec{X}$ is perpendicular to $\vec{X}^*$, $\langle \vec{X}, \vec{X} \rangle_L = \langle \vec{X}, \vec{X} \rangle_L$ is attained. Furthermore, by getting

$$\|\vec{X} - \vec{x}\|_L = \sqrt{\langle \vec{X}, \vec{X} \rangle_L - 2\langle \vec{X}, \vec{x} \rangle_L + \langle \vec{x}, \vec{x} \rangle_L}$$

and letting $\langle \vec{X}, \vec{X} \rangle_L = \langle \vec{x}, \vec{x} \rangle_L = -1$, we have

$$\|\vec{X} - \vec{x}\|_L = 2 \sinh \frac{\bar{\phi}}{2}.$$

Similarly,

$$\|\vec{X} + \vec{x}\|_L = 2 \cosh \frac{\bar{\phi}}{2}.$$

Because of the following equality

$$\|\vec{X} - \vec{x}\|_L = \|\vec{z}\|_L \|\vec{X} + \vec{x}\|_L,$$

it is briefly seen that

$$(1) \quad \|\vec{z}\|_L = \tanh \frac{\bar{\phi}}{2}.$$

Using the properties of dual numbers and Lorentzian product, we get

$$(2) \quad \|\vec{z}\|_L = \|\vec{z}\|_L + \varepsilon \frac{\vec{z} \cdot_L \vec{z}^*}{\|\vec{z}\|_L}$$

and

$$(3) \quad \tanh \frac{\bar{\phi}}{2} = \tanh \frac{\phi}{2} + \varepsilon \frac{\phi^*}{2} \left(1 - \tanh^2 \frac{\phi}{2} \right)$$

where ϕ is the hyperbolic rotation angle and ϕ^* is the minimal Lorentzian distance.

The equations (2) and (3) imply

$$(4) \quad \|\vec{z}\|_L + \varepsilon \frac{\vec{z} \cdot_L \vec{z}^*}{\|\vec{z}\|_L} = \tanh \frac{\phi}{2} + \varepsilon \frac{\phi^*}{2} \left(1 - \tanh^2 \frac{\phi}{2} \right).$$

From (4), it is clear that

$$\|\vec{z}\|_L = \tanh \frac{\phi}{2}$$

and

$$(5) \quad \frac{\vec{z} \cdot_L \vec{z}^*}{\|\vec{z}\|_L} = \frac{\phi^*}{2} \left(1 - \tanh^2 \frac{\phi}{2} \right).$$

Let us denote the unit vector which is the same direction of $\vec{z}$ by $\vec{s}$, that is $\vec{s} = \frac{\vec{z}}{\|\vec{z}\|_L}$ and $\vec{s} = (s_1, s_2, s_3)$ is called the Unit Rodrigues vector. So (5) yields,

$$(6) \quad \vec{s} \cdot_L \vec{z}^* = \frac{\phi^*}{2} \left(1 - \tanh^2 \frac{\phi}{2} \right).$$

Moreover, let us define the L -Dual Rodrigues vector, by $\bar{s}$, $\bar{s} = \frac{\bar{z}}{\|\bar{z}\|_L}$. Taking advantage of the properties of the dual numbers and Lorentzian product, we obtain

$$(7) \quad \bar{s} = \vec{s} + \varepsilon \vec{s}^* = \frac{\bar{z}}{\|\bar{z}\|_L} = \frac{\vec{z}}{\|\vec{z}\|_L} + \varepsilon \left(\frac{\vec{z}^*}{\|\vec{z}\|_L} - \frac{\vec{z} \cdot (\vec{z} \cdot_L \vec{z}^*)}{\|\vec{z}\|_L^3} \right)$$

which is $\vec{s} = (s_1, s_2, s_3)$ and $\vec{s}^* = (s_1^*, s_2^*, s_3^*)$.

Thinking the equality (7), the below equations can be easily attained

$$(8) \quad s_1^* = \frac{z_1^* \|\vec{z}\|_L^2 - z_1 (-z_1 z_1^* + z_2 z_2^* + z_3 z_3^*)}{\|\vec{z}\|_L^3}$$

$$(9) \quad s_2^* = \frac{z_2^* \|\vec{z}\|_L^2 - z_2 (-z_1 z_1^* + z_2 z_2^* + z_3 z_3^*)}{\|\vec{z}\|_L^3}$$

$$(10) \quad s_3^* = \frac{z_3^* \|\vec{z}\|_L^2 - z_3 (-z_1 z_1^* + z_2 z_2^* + z_3 z_3^*)}{\|\vec{z}\|_L^3}.$$

Having regarded the expressions (8), (9) and (10) together, it is seemed that the results z_1^*, z_2^*, z_3^* are found

$$(11) \quad z_1^* = \frac{s_1^* \|\vec{z}\|_L^3 + z_1 (z_2 z_2^* + z_3 z_3^*)}{\|\vec{z}\|_L^2 + z_1^2}$$

$$(12) \quad z_2^* = \frac{s_2^* \|\vec{z}\|_L^3 - z_1 z_2 z_1^* + z_2 z_3 z_3^*}{\|\vec{z}\|_L^2 - z_2^2}$$

$$(13) \quad z_3^* = \frac{s_3^* \|\vec{z}\|_L^3 - z_1 z_3 z_1^* + z_2 z_3 z_2^*}{\|\vec{z}\|_L^2 - z_3^2}.$$

Because the real Rodrigues parameters are

$$z_1 = s_1 \tanh \frac{\phi}{2}$$

$$z_2 = s_2 \tanh \frac{\phi}{2}$$

$$z_3 = s_3 \tanh \frac{\phi}{2}$$

and

$$\|\vec{z}\|_L = \tanh \frac{\phi}{2},$$

we can rewrite the equation (11), (12) and (13); for example,

$$z_1^* = \frac{s_1^* \tanh^3 \frac{\phi}{2} + s_1 s_2 \tanh^2 \frac{\phi}{2} z_2^* + s_1 s_3 \tanh^2 \frac{\phi}{2} z_3^*}{\tanh^2 \frac{\phi}{2} (1 + s_1^2)}.$$

Simplifying the above equation, we find as below:

$$(14) \quad z_1^* = \frac{s_1^* \tanh \frac{\phi}{2} + s_1 s_2 z_2^* + s_1 s_3 z_3^*}{1 + s_1^2}.$$

Similarly,

$$(15) \quad z_2^* = \frac{s_2^* \tanh \frac{\phi}{2} - s_1 s_2 z_1^* + s_2 s_3 z_3^*}{1 - s_2^2}$$

$$(16) \quad z_3^* = \frac{s_3^* \tanh \frac{\phi}{2} - s_1 s_3 z_1^* + s_2 s_3 z_2^*}{1 - s_3^2}$$

are obtained.

Note that $\bar{s} = \vec{s} + \varepsilon \vec{s}^*$ is unit dual vector if and only if $\|\vec{s}\|_L = 1$ and $\langle \vec{s}, \vec{s}^* \rangle_L = 0$. Due to this fact, it is possible to attain

$$(17) \quad \begin{aligned} \sqrt{-s_1^2 + s_2^2 + s_3^2} &= 1 \\ -s_1 s_1^* + s_2 s_2^* + s_3 s_3^* &= 0. \end{aligned}$$

So, $\vec{s}$ is spacelike, i.e. $-s_1^2 + s_2^2 + s_3^2 = 1$.

Thus, let us suppose that $s_i \neq 0$ for all $i = 1, 2, 3$, $s_2 \neq \pm 1$ and $s_3 \neq \pm 1$. If one or two of them are equal to zero, then one can obtain the solutions of z_i^* 's from (14),(15),(16). We take $s_1 \neq 0$, $s_2 \neq 0$, $s_3 \neq 0$. So the denominators, $1 - s_i^2$, are also different from zero by (17). If we use (6), it is allowed to compose the following equality

$$\vec{s} \cdot_L \vec{z}^* = -s_1 z_1^* + s_2 z_2^* + s_3 z_3^* = \frac{\phi^*}{2} \left(1 - \tanh^2 \frac{\phi}{2} \right).$$

Since $s_1 \neq 0$,

$$(18) \quad z_1^* = \frac{\frac{\phi^*}{2} \left(\tanh^2 \frac{\phi}{2} - 1 \right) + s_2 z_2^* + s_3 z_3^*}{s_1}.$$

The equation (14) and (18) yields,

$$(19) \quad s_2 z_2^* + s_3 z_3^* = s_1 s_1^* \tanh \frac{\phi}{2} + (1 + s_1^2) \left(1 - \tanh^2 \frac{\phi}{2} \right).$$

If we substitute (19) into (18), we have

$$z_1^* = s_1 \left(1 - \tanh^2 \frac{\phi}{2} \right) \frac{\phi^*}{2} + s_1^* \tanh \frac{\phi}{2}.$$

Similarly,

$$z_2^* = s_2 \left(1 - \tanh^2 \frac{\phi}{2} \right) \frac{\phi^*}{2} + s_2^* \tanh \frac{\phi}{2}$$

$$z_3^* = s_3 \left(1 - \tanh^2 \frac{\phi}{2} \right) \frac{\phi^*}{2} + s_3^* \tanh \frac{\phi}{2}$$

are obtained.

Corollary 2. *Taking into consideration the calculations and geometric comments which are done through the article, it is no doubt that the undermentioned results constitute*

$$(20) \quad \begin{aligned} z_1 &= s_1 \tanh \frac{\phi}{2} \\ z_2 &= s_2 \tanh \frac{\phi}{2} \\ z_3 &= s_3 \tanh \frac{\phi}{2} \end{aligned}$$

and

$$(21) \quad \begin{aligned} z_1^* &= s_1 \left(1 - \tanh^2 \frac{\phi}{2} \right) \frac{\phi^*}{2} + s_1^* \tanh \frac{\phi}{2} \\ z_2^* &= s_2 \left(1 - \tanh^2 \frac{\phi}{2} \right) \frac{\phi^*}{2} + s_2^* \tanh \frac{\phi}{2} \\ z_3^* &= s_3 \left(1 - \tanh^2 \frac{\phi}{2} \right) \frac{\phi^*}{2} + s_3^* \tanh \frac{\phi}{2}. \end{aligned}$$

If we consider (20) and (21), $\tilde{z} = (z_1, z_2, z_3) + \varepsilon (z_1^*, z_2^*, z_3^*)$ depends on the hyperbolic rotation angle ϕ being between timelike lines and distance ϕ^* which is being the minimal Lorentzian distance between the timelike lines of $\mathbb{R}_1^3$.

The relationship between the axis of rotation and timelike lines of $\mathbb{R}_1^3$ which correspond to points on dual hyperbolic unit sphere is given in the following diagram:

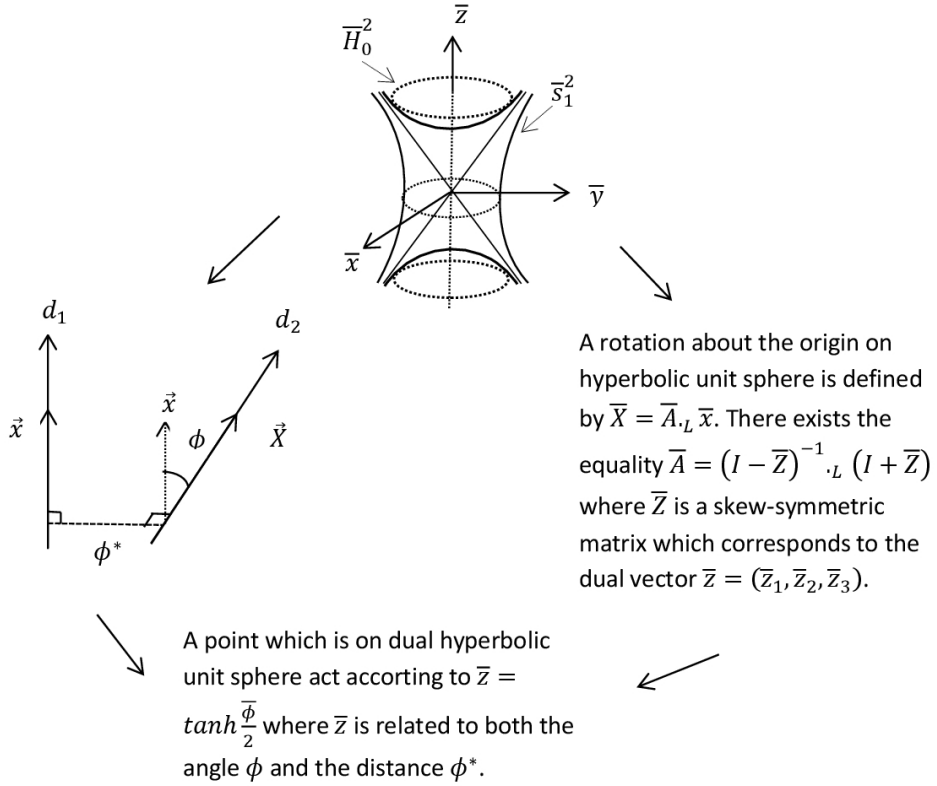


Figure 1. Rodrigues Parameters Diagram

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