

REPRESENTATION UP TO HOMOTOPY OF DOUBLE ALGEBROIDS AND THEIR TRANSGRESSION CLASSES

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ABSTRACT. In this work, we present definition of the representation of double Lie algebroids and find a one-to-one correspondence to introduce representation up to homotopy of double Lie algebroids and gauge equivalence of them. Also dual, tensor product and direct sum representations up to homotopy are made as a double Lie algebroid modules. We conclude this paper by generalization of some Lie algebroid ∞ -representation properties to ∞ -representation of double Lie algebroids and introduce transgression classes for double Lie algebroids.

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1. INTRODUCTION

Double Lie algebroids are introduced as infinitesimal form of double Lie groupoids [6, 7, 8], however the defining an abstract concept of double Lie algebroids is rough. Definition of double Lie algebroids directly arise from double vector bundles and VB-algebroids properties as follow,

A double vector bundle is a commutative square

$$(1.1) \quad \begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ \pi_A \downarrow & & \downarrow q_B \\ A & \xrightarrow{q_A} & M \end{array}$$

satisfying the following four conditions:

- (1) all four sides are vector bundles;
- (2) π_B is a vector bundle morphism over q_A ;
- (3) $+_B : D \times_B D \rightarrow D$ is a vector bundle morphism over $+$: $A \times_M A \rightarrow A$, where $+_B$ is the addition map for the vector bundle $D \rightarrow B$, and
- (4) the scalar multiplication $\mathbb{R} \times D \rightarrow D$ in the bundle $D \rightarrow B$ is a vector bundle morphism over the scalar multiplication $\mathbb{R} \times A \rightarrow A$. [5]

The vector bundles A and B are called the side bundles. The core C of a double vector bundle is the intersection of the kernels of π_A and π_B . It has a natural vector bundle structure over M .

Note that double vector bundles can be dualized in two distinct ways. We denote the dual of D as a vector bundle over A by $D \star A$ and likewise for $D \star B$. The dual $D \star A$ is itself a double vector bundle, with side bundles A and C^* and core B^* .

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ \pi_A \downarrow & & \downarrow q_B \\ A & \xrightarrow{q_A} & M \end{array} \quad \begin{array}{ccc} D \star A & \xrightarrow{\pi_{C^*}} & C^* \\ \pi_A \downarrow & & \downarrow q_{C^*} \\ A & \xrightarrow{q_A} & M \end{array} \quad \begin{array}{ccc} D \star B & \xrightarrow{\pi_B} & B \\ \pi_{C^*} \downarrow & & \downarrow q_B \\ C^* & \xrightarrow{q_{C^*}} & M \end{array}$$

Also by dualizing again $D \star B$ over C^* , we get

$$\begin{array}{ccc} D \star B \star C^* & \xrightarrow{\pi_A} & A \\ \pi_{C^*} \downarrow & & \downarrow q_A \\ C^* & \xrightarrow{q_{C^*}} & M, \end{array}$$

with core B^* . In the same manner, we get a double vector bundle $D \star A \star C^*$ with sides B and C^* and core A^* . [5]

If A is a VB-algebroid with Lie algebroid structures on $A \rightarrow N$ and $M \rightarrow D$, the dual vector bundle $A \star N \rightarrow N$ has a Lie-Poisson structure (a linear Poisson

structure), and the structure on $A \star N$ is also Lie-Poisson with respect to $A \star N \rightarrow C^*$. Dualizing this bundle gives a Lie algebroid structure on $D \star N \star C^* \rightarrow C^*$.

Definition 1.1. [5] *A double Lie algebroid $(A; M; N; D)$ is a double vector bundle $(A; M; N; D)$ with core denoted by C , and with Lie algebroid structures on each of $M \rightarrow D$, $N \rightarrow D$, $A \rightarrow M$ and $A \rightarrow N$ such that each pair of parallel Lie algebroids gives A the structure of a VB-algebroid, and such that $(A \star M \star C^*, A \star N \star C^*)$ with the induced Lie algebroid structures on base C^* as defined above, is a Lie bi-algebroid.*

An important problem in the representation theory of Lie algebroids is the absence of a well-define adjoint representation, by the usual representation definition. Most of quest to resolve this problem has led to a number of generalizations of Lie algebroid representations, which the most popular is the notion representation up to homotopy or ∞ -representation [2, 3, 4, 6].

Let A be a Lie algebroid over M and E be a vector bundle over M . An A -connection on E is a $\mathbb{R}$ -bilinear map $\Delta : \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$, such that

$$\Delta_{f\alpha}s = f\Delta_{\alpha}s$$

and

$$\Delta_{\alpha}fs = f\Delta_{\alpha}s + L_{\rho(\alpha)}(f)(s),$$

where ρ is the anchor of A , $f \in C^{\infty}(M)$, $s \in \Gamma(E)$ and $\alpha \in \Gamma(A)$. Δ is a representation of A on E , where it's curvature is vanish. A representation up to homotopy of Lie algebroid $A \rightarrow M$ is a graded vector bundle ε over M together with a degree 1 operator $\mathcal{D}$ on $\Omega(A, \varepsilon)$ such that $\mathcal{D}^2 = 0$ and such that the Leibniz rule

$$\mathcal{D}(\alpha\omega) = (d_A\alpha)\omega + (-1)^p\alpha(\mathcal{D}\omega)$$

holds for $\alpha \in \Omega^p(A)$ and $\omega \in \Omega(A, \varepsilon)$. [1]

The initial part of this paper is proceed to the notions of representation and representation up to homotopy of double Lie algebroids. Our approach is based on Quillens superconnections [11] and fits into the general theory of structures up to homotopy. Next we study these notions, using the language of supergeometry [12, 13] by Lie algebroid bi-modules as a double Lie algebroid modules and introduce adjoint, tensor, direct sum and dual modules. We conclude this paper by study transgression form of representation up to homotopy of double Lie algebroids and we prove that the transgression classes are well-define at last.

2. ∞ -REPRESENTATION OF DOUBLE LIE ALGEBROID

From now on, by a double Lie algebroid A , we mean a double Lie algebroid as diagram (1.1). A representation of a double Lie algebroid A , is a bi-vector bundle E over M and N together with two flat $(A \rightarrow M)$ -connection Δ_M and $(A \rightarrow N)$ -connection Δ_N .

Proposition 2.1. *There is a 1-1 correspondence between a representation of double Lie algebroid A with a pair of two degree 1 operators δ_M and δ_N on $\Omega_M^\bullet(A, E) = \Gamma_M(\bigwedge^\bullet A^* \otimes E)$ and $\Omega_N^\bullet(A, E) = \Gamma_N(\bigwedge^\bullet A^* \otimes E)$, respectively, which satisfy the Leibniz rule and $\delta_M^2 = \delta_N^2 = 0$.*

Proof. Let (E, Δ_M, Δ_N) be a representation of A . Two degree 1 operators

$$\begin{aligned} \delta_M \omega(\alpha_1, \dots, \alpha_p) &= \sum_{i < j} (-1)^{i+j} \omega([\alpha_i, \alpha_j]_M, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_{p-1}) \\ &\quad + \sum_i (-1)^i \Delta_M(\alpha_i, \omega(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_p)) \end{aligned}$$

and

$$\begin{aligned} \delta_N \omega(\beta_1, \dots, \beta_p) &= \sum_{i < j} (-1)^{i+j} \omega([\beta_i, \beta_j]_N, \dots, \hat{\beta}_i, \dots, \hat{\beta}_j, \dots, \beta_{p-1}) \\ &\quad + \sum_i (-1)^i \Delta_N(\beta_i, \omega(\beta_1, \dots, \hat{\beta}_i, \dots, \beta_p)), \end{aligned}$$

given by the Koszul formula satisfy the Leibniz rule and $\delta_M^2 = \delta_N^2 = 0$. $\square$

By Proposition 2.1 we can generalized the representation notion of double Lie algebroids to the ∞ -representation as follow.

Definition 2.2. *An ∞ -representation of a double Lie algebroid A is a graded bi-vector bundle ε over M and N together with two degree 1 operators D_M and D_N on $\Omega_M(A; \varepsilon)$ and $\Omega_N(A; \varepsilon)$ respectively, such that $D_M^2 = D_N^2 = 0$ and the Leibniz rule*

$$D_i(\alpha\omega) = (d_A^i \alpha)\omega + (-1)^p \alpha D_i \omega$$

holds for $\alpha \in \Omega_i^p(A)$ and $\omega \in \Omega_i(A; \varepsilon)$, where $i \in \{M, N\}$.

We also say that $D_M \times D_N$ is a ∞ -representation of A on ε .

We interest to remark that, In the case where $\varepsilon = E$ is concentrated in degree 0, an ∞ -representation is the same thing as a representation.

Proposition 2.3. *Let $D_M \times D_N$ be a ∞ -representation of double Lie algebroid A on ε . There is a one to one correspondence between (ε, D_M, D_N) and graded bi-vector bundle over M and N together with*

- (1) *two degree 1 operators σ_M and σ_N on ε , $((\varepsilon, \sigma_M, \sigma_N))$ is a two branch complex, i.e. (ε, σ_M) and (ε, σ_N) are two complexes.)*
- (2) *two $(A \rightarrow M)$ -connection Δ_M and $(A \rightarrow N)$ -connections Δ_N on (ε, σ_M) and (ε, σ_N) , respectively,*
- (3) *two $\text{End}_M(\varepsilon)$ -valued 2-form $\omega_2^M \in \Omega_M^2(A; \text{End}^{-1}(\varepsilon))$ and $\text{End}_N(\varepsilon)$ -valued 2-form $\omega_2^N \in \Omega_N^2(A; \text{End}^{-1}(\varepsilon))$ satisfying*

$$\sigma_j(\omega_2^j) + R_{\Delta_j},$$

where $j = M$ or N and R_{Δ_j} is the curvature of Δ_j ,

- (4) *for each $i > 2$ two $\text{End}_M(\varepsilon)$ -valued i -form $\omega_i^M \in \Omega_M^i(A; \text{End}^{1-i}(\varepsilon))$ and $\text{End}_N(\varepsilon)$ -valued i -form $\omega_i^N \in \Omega_N^i(A; \text{End}^{1-i}(\varepsilon))$, satisfying*

$$\sigma_M(\omega_i^j) + d_{\Delta_j}(\omega_{i-1}) + \omega_2^j \circ \omega_{i-2}^j + \omega_3^j \circ \omega_{i-3}^j + \cdots + \omega_{i-2}^j \circ \omega_2^j = 0,$$

for $j = M$ or N .

Moreover, for $j = M$ or N we have

$$D_j = \sigma_j + \Delta_j + \sum_{i>2} \omega_i^j.$$

Proof. It follows immediately from [1, Proposition 3.2]. □

Definition 2.4. *Let ε and ε' be ∞ -representations of A . A morphism from ε to ε' is a $\Omega_M(A) \times \Omega_N(A)$ module morphism*

$$\varphi : \Omega_M(A; \varepsilon) \times \Omega_N(A; \varepsilon) \rightarrow \Omega_M(A; \varepsilon') \times \Omega_N(A; \varepsilon')$$

such that $\varphi \circ (D_M \times D_N) = (D'_M \times D'_N) \circ \varphi$.

As usual, an invertible morphism of ∞ -representations is called an isomorphism.

We know that there are two natural projection map

$$\mu_M : \Omega_M(A; \varepsilon) \rightarrow \Gamma_M(\varepsilon) \quad \text{and} \quad \mu_N : \Omega_N(A; \varepsilon) \rightarrow \Gamma_N(\varepsilon),$$

which

$$\ker \mu_M = \bigoplus_{p>0} \Omega_M^p(A) \otimes \Gamma_M(A)$$

and

$$\ker \mu_N = \bigoplus_{p>0} \Omega_N^p(A) \otimes \Gamma_N(A).$$

Then we have:

Definition 2.5. A gauge transformation of $\Omega_M(A; \varepsilon) \times \Omega_N(A; \varepsilon)$ is a degree-preserving $\Omega_M(A) \times \Omega_N(A)$ module automorphism u such that the following diagram commutes:

$$\begin{array}{ccc} \Omega_M(A; \varepsilon) \times \Omega_N(A; \varepsilon) & \xrightarrow{u} & \Omega_M(A; \varepsilon) \times \Omega_N(A; \varepsilon) \\ \mu_M \times \mu_N \downarrow & & \downarrow \mu_M \times \mu_N \\ \Gamma_M(A) \times \Gamma_N(A) & \xrightarrow{id} & \Gamma_M(A) \times \Gamma_N(A) \end{array}$$

Two ∞ -representations $D_M \times D_N$ and $D'_M \times D'_N$, that $u(D_M \times D_N) = (D_M \times D_N)u$ for a gauge transformation u , are said to be gauge-equivalent.

3. DOUBLE LIE ALGEBRA MODULES

Let A be a Lie algebroid, an A -module, is a vector bundle $\mathcal{B} \rightarrow A[1]$ equipped with a degree 1 operator Q on $\Gamma(\mathcal{B})$ such that $Q^2 = 0$ and such that the Leibniz rule

$$Q(\alpha\beta) = (d_A\alpha)\beta + (-1)^p\alpha(Q\beta)$$

holds for $\alpha \in C_p^\infty(A[1]) = \Omega^p(A)$ and $\beta \in \Gamma(\mathcal{B})$.

Definition 3.1. A double Lie algebroid module over double Lie algebroid A is a $(A \rightarrow M)$ -($A \rightarrow N$) bi-module.

A morphism of double Lie algebroid modules over A , from $(\mathcal{B}, Q_M, Q_N)$ to $(\mathcal{B}', Q'_M, Q'_N)$ is a linear map $\psi : \mathcal{B} \rightarrow \mathcal{B}'$ such that the following diagram commutes:

$$\begin{array}{ccc} A_M[1] & \xrightarrow{id} & A_M[1] \\ \uparrow & & \uparrow \\ \mathcal{B} & \xrightarrow{\psi} & \mathcal{B}' \\ \downarrow & & \downarrow \\ A_N[1] & \xrightarrow{id} & A_N[1], \end{array}$$

$$\psi Q_M = Q'_M \psi \text{ and } \psi Q_N = Q'_N \psi.$$

Lemma 3.2. *There is a one-to-one correspondence between ∞ -representation ε of double Lie algebroid A and double Lie algebroid module $(\pi_M \times \pi_N)^*\varepsilon$.*

Proof. Let $\varepsilon = \bigoplus E_i[-i]$ be a ∞ -representation of A and $j \in \{M, N\}$. We can consider $(A \rightarrow j)$ -module structure of the form $(\pi_M \times \pi_N)^*\varepsilon \rightarrow A_j[1]$ where π_j is the projection map from $A_j[1]$ to j [10]. Thus $(\pi_M \times \pi_N)^*\varepsilon$ is a double Lie algebroid module structure over A . Conversely, $(A \rightarrow j)$ -module of sections $\Gamma_j((\pi_M \times \pi_N)^*\varepsilon)$ is canonically isomorphic to $\Omega_j(A) \otimes_{C^\infty(M)} \Gamma_j(\varepsilon) = \Omega_j(A; \varepsilon)$. $\square$

Theorem 3.3. *Let A be a double Lie algebroid:*

- (1) *There is a one-to-one correspondence between isomorphism classes of double Lie algebroid modules over A , and isomorphism classes (semi) bounded ∞ -representations of A .*
- (2) *For any (semi) bounded bi-vector bundle ε over M and N , there is one-to-one correspondence between statomorphism classes of double Lie algebroid modules over A with standard graded bi-vector bundle ε and gauge equivalence classes of ∞ -representations of A on ε .*

The cohomology of double algebroid A with values in bi-vector bundle $\mathcal{B}$ over M and N , denoted $H^\bullet(A; \mathcal{B})$, is a two branch complex, which is structured by the cohomologies of two complex $(\Gamma_M(\mathcal{B}), Q_M)$ and $(\Gamma_N(\mathcal{B}), Q_N)$.

The results of previous theorem imply that $H^\bullet(A; \mathcal{B})$ is isomorphic to the cohomology of A with values in any ∞ -representation arising from $\mathcal{B}$.

Example 3.4. *Let A be a double Lie algebroid and $j \in \{M, N\}$. The adjoint module of Lie algebroid $A \rightarrow j$ is the tangent bundle $TA_j[1]$, with the operator $Q_j := [d_j^A, \cdot]$. The low-degree cohomology $H^\bullet(A_j, TA_j[1])$ of $A \rightarrow j$ with values in the adjoint module is described in [14]. The adjoint module of double Lie algebroid A is $(A \rightarrow M)$ -($A \rightarrow N$) bi-module $(TA_M[1] \times TA_N[1], Q_M, Q_N)$.*

Let A be a double Lie algebroid and let $(\mathcal{B}_1, Q_1, Q'_1)$ and $(\mathcal{B}_2, Q_2, Q'_2)$ be double Lie algebroid modules over A . Then $(\mathcal{B}_1 \otimes \mathcal{B}_2, Q, Q')$ is a double Lie algebroid module over A , where

$$Q(\beta_1 \otimes \beta_2) = Q_1\beta_1 \otimes \beta_2 + (-1)^{|\beta_1|}\beta_1(Q_2\beta_2)$$

and

$$Q'(\alpha_1 \otimes \alpha_2) = Q'_1\alpha_1 \otimes \alpha_2 + (-1)^{|\alpha_1|}\alpha_1(Q'_2\alpha_2)$$

for $\beta_i \in \Gamma_M(\mathcal{B}_i)$ and $\alpha_i \in \Gamma_N(\mathcal{B}_i)$.

Similarly, the direct sum $\mathcal{B}_1 \oplus \mathcal{B}_2$ inherits an A bi-module structure given by

$$Q(\beta_1 + \beta_2) = Q_1\beta_1 + Q_2\beta_2$$

and

$$Q'(\beta_1 + \beta_2) = Q'_1\beta_1 + Q'_2\beta_2.$$

Let $(\mathcal{B}, Q, Q')$ be a double Lie algebroid module over A , and let $\mathcal{B}_j^* \rightarrow A_j[1]$ be the bi-vector bundle dual to $\mathcal{B} \rightarrow A_j$. We denote by $\langle \cdot, \cdot \rangle_j$ the pairing taking $\Gamma(\mathcal{B}_j^*) \otimes \Gamma_j(\mathcal{B})$ to $C^\infty(A_j[1]) = \Omega_j(A)$, where $j = M$ or N . The induced A bi-module structure (Q^*, Q'^*) on $B_M^* \times B_N^*$ is uniquely determined by the equation

$$d_A^M \langle b, \beta \rangle_M = \langle Q^*b, \beta \rangle_M + (-1)^{|b|} \langle b, Q\beta \rangle_M,$$

for $b \in \Gamma(\mathcal{B}_M^*)$ and $\beta \in \Gamma_M(\mathcal{B})$, and

$$d_A^N \langle a, \alpha \rangle_N = \langle Q'^*a, \alpha \rangle_N + (-1)^{|a|} \langle a, Q'\alpha \rangle_N,$$

for $a \in \Gamma(\mathcal{B}_N^*)$ and $\alpha \in \Gamma_N(\mathcal{B})$.

4. TRANSGRESSION CLASSES OF DOUBLE LIE ALGEBROID

Let A be a double algebroid, and ε be a bounded in degree graded bi-vector bundle over M and N .

Definition 4.1. A double Lie algebroid superconnection of A on ε is operator $\mathcal{D}_M \times \mathcal{D}_N$ on $\Omega_M(A; \varepsilon) \times \Omega_N(A; \varepsilon)$, which $\mathcal{D}_M$ and $\mathcal{D}_N$ satisfying in the Leibniz rule. A double Lie algebroid superconnection of A is called flat, if $\mathcal{D}_M^2 = \mathcal{D}_N^2 = 0$.

Let $\varepsilon = \bigoplus E_i[-i]$ and choose two metrics on E_i for each i . The adjoint operator $\mathcal{D}_M^\dagger \times \mathcal{D}_N^\dagger$ given by the equation

$$d_j^A \langle \omega_1, \omega_2 \rangle_j = \langle \mathcal{D}_j \omega_1, \omega_2 \rangle_j + (-1)^{|\omega_1|} \langle \omega_1, \mathcal{D}^\dagger \omega_2 \rangle_j,$$

where $j = M$ or N and $\omega_1, \omega_2 \in \Omega_j(A; \varepsilon)$, satisfies the Leibniz rule.

Lemma 4.2. Adjoint superconnection $\mathcal{D}_M^\dagger \times \mathcal{D}_N^\dagger$ is flat, for a flat double Lie algebroid superconnection $\mathcal{D}_M \times \mathcal{D}_N$.

Remark 4.3. Adjoint superconnection $\mathcal{D}_M^\dagger \times \mathcal{D}_N^\dagger$ is generally not homogeneous of degree 1, although $\mathcal{D}_M \times \mathcal{D}_N$ is a ∞ -representation.

For any double Lie algebroid superconnection $\mathcal{D}_M \times \mathcal{D}_N$ of A , we have two Chern-Weil forms

$$ch_k(\mathcal{D}_M) := str(\mathcal{D}_M^{2k}) \in \Omega_M^{2k}(A)$$

and

$$ch_k(\mathcal{D}_N) := str(\mathcal{D}_N^{2k}) \in \Omega_N^{2k}(A),$$

where str denotes the supertrace. Note that if $\mathcal{D}_M \times \mathcal{D}_N$ is an ∞ -representation, then two Chern-Weil forms $ch_k(\mathcal{D}_M)$ and $ch_k(\mathcal{D}_N)$ obviously vanish.

Let I be the unit interval, and consider the product double Lie algebroid $A \times TTI$. Let $t, \dot{t}$ be the canonical coordinate on $T[1]TI$ and $j = M$ or N .

Any p -form $\xi \in \Omega_{j \times TTI}^p(A \times TTI)$ can be uniquely written as $\xi_0(t) + \dot{t}\xi_1(t)$ where ξ_0 and ξ_1 are t dependent elements of $\Omega_j^p(A)$ and $\Omega_j^{p-1}(A)$, respectively. We can define a degree -1 map from $\Omega_{j \times TTI}(A \times TTI)$ to $\Omega_j(A)$ by the Berezian integral, as follow

$$\int \xi := \int_{T[1]TI} dt d\dot{t} \xi = \int_0^1 dt \xi_1. \quad [10]$$

Let π_j be the projection map from $i \times TTI$ to i and $i = M$ or N . For two double Lie algebroid superconnection $\mathcal{D}_M \times \mathcal{D}_N$ and $\mathcal{D}'_M \times \mathcal{D}'_N$ of A on ε , $\mathcal{T}_{\mathcal{D}_M, \mathcal{D}'_M} \times \mathcal{T}_{\mathcal{D}_N, \mathcal{D}'_N}$ forms a double Lie algebroid superconnection of A on $(\pi_M \times \pi_N)^*\varepsilon$, where

$$\mathcal{T}_{\mathcal{D}_i, \mathcal{D}'_i}(x) = t\mathcal{D}_i(x) + (1-t)\mathcal{D}'_i(x),$$

for any t -independent section $x \in \Gamma_i(\varepsilon)$ of $(\pi_M \times \pi_N)^*\varepsilon$.

Let A be a Lie algebroid. For two A -superconnection $\mathcal{D}_1$ and $\mathcal{D}_2$ the transgression forms $cs_k(\mathcal{D}_1, \mathcal{D}_2) \in \Omega^{2k-1}(A)$ are defined as follow

$$cs_k(\mathcal{D}_1, \mathcal{D}_2) := \int ch_k(\mathcal{T}_{\mathcal{D}_1, \mathcal{D}_2}). \quad [10]$$

Theorem 4.4. *Given an ∞ -representation $\mathcal{D} = \mathcal{D}_M \times \mathcal{D}_N$, the cohomology classes of $cs_k(\mathcal{D}) := cs_k(\mathcal{D}_M, \mathcal{D}_M^\dagger) \times cs_k(\mathcal{D}_N, \mathcal{D}_N^\dagger)$ are independent of the choice of metric and invariant with respect to gauge transformations.*

Proof. Let $i = M$ or N and two metrics $\langle \cdot, \cdot \rangle_1^i$ and $\langle \cdot, \cdot \rangle_2^i$ on ε . The space of metrics is convex, therefore there exists a path ϑ^i of metrics such that $\vartheta^i(0)(x, y) = \langle x, y \rangle_1^i$ and $\vartheta^i(1)(x, y) = \langle x, y \rangle_2^i$, and $u_t^i \in \text{End}_i(\varepsilon)$, such that $\vartheta^i(t)(x, y) = \vartheta^i(0)(u_t^i(x), y)$. The adjoint operators $\mathcal{D}_i^{\dagger t}$ corresponds to $\vartheta^i(t)$ satisfy the equation

$$\mathcal{D}_i^{\dagger t} = (u_t^i)^{-1} \mathcal{D}_i^{\dagger 0} u_t^i.$$

By [10, Proposition 7.5] we have

$$[cs_k(\mathcal{D}_i, \mathcal{D}_i^{\dagger 0})] = [cs_k(\mathcal{D}_i, \mathcal{D}_i^{\dagger 1})] + [cs_k(\mathcal{D}_i^{\dagger 0}, \mathcal{D}_i^{\dagger 1})],$$

and $[cs_k(\mathcal{D}_i^{\dagger 0}, \mathcal{D}_i^{\dagger 1})] = 0$, [10, Proposition 7.6]. Therefore the cohomology classes $[cs_k(\mathcal{D})]$ are metric invariance.

The space of gauge transformations is

$$\bigoplus_{p,k,q,l} (\Omega_M^p \times \Omega_N^q) \otimes (Hom_M(E_k, E_{k-p}) \times Hom_N(E_l, E_{l-q})),$$

which is path connected. Hence the gauge-invariance follows as above. $\square$

CONCLUSION

We observed that A gauge transformation of $\Omega_M(A; \varepsilon) \times \Omega_N(A; \varepsilon)$ is precisely the same thing as an automorphism of $(\pi_M \times \pi_N)^*\varepsilon$ that preserves the image of both zero sections of A over M and N . In other words, changes of decomposition correspond to gauge transformations of $\mathcal{D} = \mathcal{D}_M \times \mathcal{D}_N$. Moreover the cohomology classes of $cs_k(\mathcal{D})$ are well-defined invariants of double Lie algebroid modules.

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