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MEROMORPHIC SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS OF [P,Q]-ORDER

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Abstract. In this paper, we investigate the growth and value distribution of meromorphic solutions to higher order homogeneous and nonhomogeneous linear differential equations in which the coefficients are entire functions of finite $[p, q]$-order. We get the results about $[p, q]$-order and the $[p, q]$-convergence exponent of solutions for such equations.

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1. Introduction and main results

Throughout this article, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [8], [12], [24], [25]). For the definition of iterated order of meromorphic function, we use the same definition as in [15], [16]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right), p \in \mathbb{N}$. We also define for all $r \in (0, +\infty)$ sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right), p \in \mathbb{N}$.

Definition 1.1 ([15], [16]) Let f be a meromorphic function. Then the iterated p-order $\rho_p(f)$ of f is defined by

$$\rho_p(f) := \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r}, \ (p \geq 1 \text{ is an integer}).$$

For $p=1$, this notation is called order and for $p=2$ hyper-order. If f is an entire function, then the iterated p-order $\rho_p(f)$ of f is defined by

$$\rho_p(f) := \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log r}, \ (p \geq 1 \text{ is an integer}),$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

The Lebesgue linear measure of a set $E \subset [0,\infty)$ is $m(E) = \int\limits_E dt$, and the logarithmic measure of a set $F \subset (1,+\infty)$ is $m_l(F) = \int\limits_F \frac{dt}{t}$. The upper density of $E \subset [0,\infty)$ is given by

$$\overline{dens}(E) = \limsup_{r\to+\infty} \frac{m(E\cap[0,r])}{r}$$

and the upper logarithmic density of the set $F \subset (1,+\infty)$ is defined by

$$\overline{\log dens}(F) = \limsup_{r\longrightarrow+\infty} \frac{m_l(F\cap[1,r])}{\log r}.$$

Proposition 1.1 ([4]) For all $H \subset (1,+\infty)$ the following statements hold:
(i) If $m_l(H) = \infty$, then $m(H) = \infty$;
(ii) If $\overline{dens}(H) > 0$, then $m(H) = \infty$;
(iii) If $\overline{\log dens}(H) > 0$, then $m_l(H) = \infty$.

Now, we introduce the definition of the $[p,q]$-order as follows.

Definition 1.2 ([14], [17], [18]) Let p,q be integers such that $p \geq q \geq 1$ and let f be a meromorphic function. The $[p,q]$-order of f is defined by

$$\rho_{[p,q]}(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q r}.$$

For an entire function f, we also define

$$\rho_{[p,q]}(f) = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

Remark 1.1 By Definition 1.2, we note that $\rho_{[1,1]}(f) = \rho(f), \rho_{[2,1]}(f) = \rho_2(f)$ and $\rho_{[p,1]}(f) = \rho_p(f)$.

Now, by Definition 1.2, we can get the definition of the lower $[p,q]$-order for entire and meromorphic functions.

Definition 1.3 ([26]) Let p, q be integers such that $p \geq q \geq 1$ and let f be a meromorphic function. The lower $[p,q]$-order of f is defined by

$$\mu_{[p,q]}(f) = \liminf_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q r}.$$

For an entire function f, we also define

$$\mu_{[p,q]}(f) = \liminf_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

Definition 1.4 ([17], [18]) Let p, q be integers such that $p \geq q \geq 1$ and let f be a meromorphic function. The $[p,q]$-convergence exponent of the sequence of a-points of f is defined by

$$\lambda_{[p,q]}(f-a) = \lambda_{[p,q]}(f,a) = \limsup_{r\to+\infty} \frac{\log_p N\left(r, \frac{1}{f-a}\right)}{\log_q r},$$

where $N(r,a) = N(r,a,f) = N\left(r, \frac{1}{f-a}\right)$. If $a = 0$, then the $[p,q]$-convergence exponent of the zero-sequence of f is defined by

$$\lambda_{[p,q]}(f) := \limsup_{r\to+\infty} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log_q r},$$

where $N\left(r, \frac{1}{f}\right)$ is the integrated counting of zeros of f in $\{z : |z| \leq r\}$. If $a = \infty$, then the $[p,q]$-convergence exponent of the pole-sequence of f is defined by

$$\lambda_{[p,q]}\left(\frac{1}{f}\right) := \limsup_{r\to+\infty} \frac{\log_p N(r,f)}{\log_q r}.$$

Similarly, the $[p,q]$-convergence exponent of distinct zero-sequence of f is defined by

$$\overline{\lambda}_{[p,q]}(f) = \overline{\lambda}_{[p,q]}(f) = \limsup_{r\to+\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log_q r},$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the integrated counting of distinct zeros of f in $\{z : |z| \leq r\}$, and the $[p, q]$-convergence exponent of distinct pole-sequence of f is defined by

$$\overline{\lambda}_{[p,q]}\left(\frac{1}{f}\right) := \limsup_{r\to+\infty} \frac{\log_p \overline{N}(r, f)}{\log_q r}.$$

The theory of complex linear differential equations has been developed since 1960s. Many authors have investigated the complex linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \tag{1.1}$$

and

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z), \tag{1.2}$$

where $A_0(z) \not\equiv 0, A_1(z), \cdots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ are entire functions of finite order and $k \geq 2$ is an integer. When the coefficients $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ and $F(z)$ are entire functions, it is well-known that all solutions of (1.1) and (1.2) are entire functions, and that if some coefficients of (1.1) are transcendental then (1.1) has at least one solution with infinite order. We refer to [16] for the literature on the growth of entire solutions of (1.1) and (1.2). There are also many papers considering the hyper and iterated order of solutions of complex linear differential equations (see [4], [6], [11], [13], [15], [20], [21], [22], [23]). In recent years, some authors investigated the properties of solutions of higher order linear differential equations with entire and meromorphic coefficients of $[p, q]$-order in the complex plane and with analytic and meromorphic coefficients of $[p, q]$-order in the unit disc (see e.g. [2], [3], [7], [17], [18], [26]. As far as we known, Liu, Tu and Shi [18] firstly introduced the concept of $[p, q]$-order for the case $p \geq q \geq 1$ to investigate the entire solutions of (1.1) and (1.2), and obtained some results by using the $[p, q]$-type of $A_0(z)$ to dominate the $[p, q]$-types of other coefficients, and got the result about $\rho_{[p+1,q]}(f)$ as follows.

Theorem A ([18]) *Let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be entire functions satisfying*

$$\max\left\{\rho_{[p,q]}(A_j),\ j \neq s\right\} < \rho_{[p,q]}(A_s) < +\infty.$$

Then every solution $f(z)$ of (1.1) satisfies $\rho_{[p+1,q]}(f) \leq \rho_{[p,q]}(A_s)$. Furthermore, at least one solution of (1.1) satisfies $\rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s)$.

Theorem B ([18]) *Let* $A_j(z)\,(j=0,1,\cdots,k-1)$ *be entire functions satisfying*

$$\max\left\{\rho_{[p,q]}(A_j), j=1,\cdots,k-1\right\}<\rho_{[p,q]}(A_0)<+\infty.$$

Then every nontrivial solution $f(z)$ *of* (1.1) *satisfies* $\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0)$.

Theorem C ([18]) *Let* $A_j(z)\,(j=0,1,\cdots,k-1)$ *and* $F(z)\not\equiv 0$ *be entire functions satisfying*

$$\max\left\{\rho_{[p,q]}(A_j),\ \rho_{[p+1,q]}(F),\ j=1,\cdots,k-1\right\}<\rho_{[p,q]}(A_0)<+\infty.$$

Then every solution $f(z)$ *of* (1.2) *satisfies* $\overline{\lambda}_{[p+1,q]}(f)=\lambda_{[p+1,q]}(f)=\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0)$, *with at most one exceptional solution* f_0 *satisfying*

$$\rho_{[p+1,q]}(f_0)<\rho_{[p,q]}(A_0).$$

2. Main results

In this paper, we consider for $k\geq 2$ the homogeneous and the non-homogeneous linear differential equations

$$A_k(z)f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0, \tag{1.3}$$

$$A_k(z)f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F(z), \tag{1.4}$$

where $A_0(z)\not\equiv 0,\ A_1(z),\cdots,A_k(z)\not\equiv 0$ and $F(z)\not\equiv 0$ are entire functions. It is well-known that if A_k is a nonconstant entire function, then equation (1.3) or (1.4) can possess meromorphic solutions. For instance the equation

$$(z-1)^2 f''+(z-1)^2(2e^z-1)f'+\left((z-1)^2e^{2z}+z^2-3z\right)f$$
$$=\left[4(z-1)e^{2z}-4e^z+z-1\right]e^{e^z}+(z-1)^2e^{2z}+z^2-3z$$

has a meromorphic solution

$$f(z)=\frac{e^{e^z}+z-1}{z-1}.$$

The main purpose of this paper is to improve and extend the above results by considering the $[p,q]$-order. The present article may be understood as an extension and improvement of the recent articles of Long and Zhu [19] and of the authors [20] from iterated $p-$order to $[p,q]$-order. In fact we will prove the following results.

Theorem 2.1 *Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ (or $m_l(\{|z| : z \in H\}) = \infty$), p, q be integers such that $p \geq q \geq 1$ and let $A_j(z)$ $(j = 0, 1, \cdots, k)$, be entire functions such that $A_k \not\equiv 0$. Suppose there exists an integer s, $0 \leq s \leq k$ such that*

$$\max_{j=0,1,\cdots,k,\ j\neq s} \left\{\rho_{[p,q]}(A_j)\right\} < \mu_{[p,q]}(A_s) \leq \rho_{[p,q]}(A_s) = \rho < +\infty,$$

and for some real constant μ satisfying $0 \leq \mu < \rho$, we have for any given ε $(0 < \varepsilon < \rho - \mu)$ sufficiently small,

$$|A_j(z)| \leq \exp_{p+1}\left\{\mu \log_q r\right\},\ j \neq s, j = 0, 1, \cdots, k,$$

$$|A_s(z)| \geq \exp_{p+1}\left\{(\rho - \varepsilon)\log_q r\right\},$$

as $z \to \infty$ for $z \in H$. Then every non-transcendental solution $f(z) \not\equiv 0$ of (1.3) is a polynomial with $\deg f \leq s - 1$ and every transcendental meromorphic solution $f(z)$, of (1.3) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$, satisfies $\rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s)$.

When $A_k(z) \equiv 1$, we obtain the following corollary for entire solutions.

Corollary 2.1 *Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ (or $m_l(\{|z| : z \in H\}) = \infty$), p, q be integers such that $p \geq q \geq 1$ and let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$, be entire functions. Suppose there exists an integer s, $0 \leq s \leq k - 1$ such that*

$$\max_{j=0,1,\cdots,k-1,\ j\neq s} \left\{\rho_{[p,q]}(A_j)\right\} < \mu_{[p,q]}(A_s) \leq \rho_{[p,q]}(A_s) = \rho < +\infty,$$

and for some real constant μ satisfying $0 \leq \mu < \rho$, we have for any given ε $(0 < \varepsilon < \rho - \mu)$ sufficiently small,

$$|A_j(z)| \leq \exp_{p+1}\left\{\mu \log_q r\right\},\ j \neq s, j = 0, 1, \cdots, k-1,$$

$$|A_s(z)| \geq \exp_{p+1}\left\{(\rho - \varepsilon)\log_q r\right\},$$

as $z \to \infty$ for $z \in H$. Then every non-transcendental solution $f(z) \not\equiv 0$ of (1.1) is a polynomial with $\deg f \leq s - 1$ and every transcendental solution $f(z)$, of (1.1) satisfies $\rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s)$.

Corollary 2.2 *Let $A_j(z)$ $(j = 0, 1, \cdots, k)$, H satisfy all of the hypotheses of Theorem 2.1, and let $\varphi(z)$ be a transcendental meromorphic function satisfying*

$\rho_{[p+1,q]}(\varphi) < \rho_{[p,q]}(A_s)$. *Then every transcendental meromorphic solution* $f(z)$ *with* $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu_{[p,q]}(f)$ *of equation* (1.3) *satisfies*

$$\overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi) = \rho_{[p+1,q]}(f-\varphi) = \rho_{[p,q]}(A_s).$$

Considering nonhomogeneous linear differential equation (1.4), we obtain the following results.

Theorem 2.2 *Let* H *be a set of complex numbers satisfying* $\overline{\log dens}\{|z| : z \in H\} > 0$ (*or* $m_l(\{|z| : z \in H\}) = \infty$) *and let* $A_j(z)$ $(j = 0, 1, \cdots, k)$, $F(z) \not\equiv 0$ *be entire functions such that* $A_k \not\equiv 0$. *Suppose there exists an integer* s, $0 \leq s \leq k$ *such that*

$$\max_{j=0,1,\cdots,k,\ j\neq s}\left\{\rho_{[p,q]}(A_j), \rho_{[p,q]}(F)\right\} < \mu_{[p,q]}(A_s) \leq \rho_{[p,q]}(A_s) = \rho < +\infty,$$

and for some real constant μ *satisfying* $0 \leq \mu < \rho$, *we have for any given* ε $(0 < \varepsilon < \rho - \mu)$ *sufficiently small,*

$$|A_j(z)| \leq \exp_{p+1}\left\{\mu \log_q r\right\},\ j \neq s, j = 0, 1, \cdots, k,$$

$$|A_s(z)| \geq \exp_{p+1}\left\{(\rho - \varepsilon)\log_q r\right\},$$

as $z \to \infty$ *for* $z \in H$. *Then every non-transcendental solution* $f(z)$ *of* (1.4) *is a polynomial with* $\deg f \leq s - 1$ *and every transcendental meromorphic solution* $f(z)$, *of* (1.4) *such that* $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$ *satisfies* $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s)$.

When $A_k(z) \equiv 1$, we obtain the following corollary for entire solutions.

Corollary 2.3 *Let* H *be a set of complex numbers satisfying* $\overline{\log dens}\{|z| : z \in H\} > 0$ (*or* $m_l(\{|z| : z \in H\}) = \infty$) *and let* $A_j(z)$ $(j = 0, 1, \cdots, k-1)$, $F(z) \not\equiv 0$ *be entire functions. Suppose there exists an integer* s, $0 \leq s \leq k-1$ *such that*

$$\max_{j=0,1,\cdots,k-1,\ j\neq s}\left\{\rho_{[p,q]}(A_j),\ \rho_{[p,q]}(F)\right\} < \mu_{[p,q]}(A_s) \leq \rho_{[p,q]}(A_s) = \rho < +\infty,$$

and for some real constant μ *satisfying* $0 \leq \mu < \rho$, *we have for any given* ε $(0 < \varepsilon < \rho - \mu)$ *sufficiently small,*

$$|A_j(z)| \leq \exp_{p+1}\left\{\mu \log_q r\right\},\ j \neq s, j = 0, 1, \cdots, k-1,$$

$$|A_s(z)| \geq \exp_{p+1}\left\{(\rho - \varepsilon)\log_q r\right\},$$

as $z \to \infty$ for $z \in H$. Then every non-transcendental solution $f(z)$ of (1.2) is a polynomial with $\deg f \leq s-1$ and every transcendental solution $f(z)$ of (1.2) satisfies $\rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s)$.

Corollary 2.4 *Let $A_j(z)$ $(j = 0, 1, \cdots, k)$, $F(z)$, H satisfy all of the hypothesis of Theorem 2.2, and let $\varphi(z)$ be a transcendental meromorphic function satisfying $\rho_{[p+1,q]}(\varphi) < \rho_{[p,q]}(A_s)$. Then every transcendental meromorphic solution $f(z)$ with $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu_{[p,q]}(f)$ of equation (1.4) satisfies*

$$\overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi) = \rho_{[p+1,q]}(f-\varphi) = \rho_{[p,q]}(A_s).$$

3. Preliminary lemmas

Our proofs depend mainly upon the following lemmas.

Lemma 3.1 ([9]) *Let f be a transcendental meromorphic function in the plane, and let $\alpha > 1$ be a given constant. Then there exist a set $E_1 \subset (1, +\infty)$ that has a finite logarithmic measure, and a constant $B > 0$ depending only on α and (m, n) $(m, n \in \{0, 1, \cdots, k\})$ $m < n$ such that for all z with $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leq B\left(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right)^{n-m}.$$

By using similar proof of Lemma 3.5 in [22], we can easily extend Lemma 3.6 in [26] to the case $\rho_{[p,q]}(f) = \rho_{[p,q]}(g) = +\infty$.

Lemma 3.2 *Let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$, $d(z)$ are entire functions satisfying $\mu_{[p,q]}(g) = \mu_{[p,q]}(f) = \mu \leq \rho_{[p,q]}(f) = \rho_{[p,q]}(g) \leq +\infty$ and $\lambda_{[p,q]}(d) = \rho_{[p,q]}(d) = \lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu$. Then there exists a set $E_2 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E_2$ and $|g(z)| = M(r, g)$, we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n (1 + o(1)), \ n \in \mathbb{N},$$

where $\nu_g(r)$ denote the central index of g.

Lemma 3.3 ([18]) *Let $f(z)$ be an entire function of $[p,q]$-order and let $\nu_f(r)$ be the central index of f. Then*

$$\rho_{[p,q]}(f)=\limsup_{r\to+\infty}\frac{\log_p \nu_f(r)}{\log_q r}.$$

Lemma 3.4 ([1], [10]) *Let $\varphi:[0,+\infty)\to\mathbb{R}$ and $\psi:[0,+\infty)\to\mathbb{R}$ be monotone nondecreasing functions such that (i) $\varphi(r)\le\psi(r)$ outside of an exceptional set of finite linear measure, or (ii) $\varphi(r)\le\psi(r)$, for all $r\notin(E_3\cup[0,1])$, where E_3 is a set of finite logarithmic measure. Let $\varkappa>1$ be a given constant. Then there exists an $r_1=r_1(\varkappa)>0$ such that $\varphi(r)\le\psi(\varkappa r)$ for all $r>r_1$.*

By using similar proof of Lemma 2.5 in [11], we can easily extend Lemma 3.3 in [26] to the case $\rho_{[p,q]}(f)=\rho_{[p,q]}(g)=+\infty$.

Lemma 3.5 *Let $f(z)=\frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$, $d(z)$ are entire functions satisfying $\mu_{[p,q]}(g)=\mu_{[p,q]}(f)=\mu\le\rho_{[p,q]}(f)=\rho_{[p,q]}(g)\le+\infty$* and *$\lambda_{[p,q]}(d)=\rho_{[p,q]}(d)=\lambda_{[p,q]}\left(\frac{1}{f}\right)<\mu$. Then there exists a set $E_4\subset(1,+\infty)$ of finite logarithmic measure such that for all $|z|=r\notin([0,1]\cup E_4)$ and $|g(z)|=M(r,g)$, we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right|\le r^{2s},\quad(s\in\mathbb{N}).$$

By using similar proof of Lemma 3.7 in [21], we can easily obtain the following lemma.

Lemma 3.6 ([7]) *Let $f(z)$ be an entire function such that $\rho_{(p,q)}(f)<+\infty$. Then, there exists a set $E_5\subset(1,+\infty)$ of r of finite linear measure such that for any given $\varepsilon>0$, we have*

$$\exp\left\{-\exp_p\left\{(\rho_{(p,q)}(f)+\varepsilon)\log_q r\right\}\right\}\le|f(z)|$$
$$\le\exp_{p+1}\left\{(\rho_{(p,q)}(f)+\varepsilon)\log_q r\right\}\quad(r\notin E_5).$$

Lemma 3.7 ([12]) *Let f be a meromorphic function and let $k\in\mathbb{N}$. Then*

$$m\left(r,\frac{f^{(k)}}{f}\right)=S(r,f),$$

where $S(r,f)=O(\log T(r,f)+\log r)$, possibly outside a set $E_6\subset(0,\infty)$ with a finite linear measure.

Lemma 3.8 *Let $A_j(z)$ $(j = 0, 1, \cdots, k)$, $A_k(z)\,(\not\equiv 0)$, $F(z)\,(\not\equiv 0)$ be meromorphic functions and let $f(z)$ be a meromorphic solution of (1.4) of infinite $[p,q]-$order satisfying the following condition*

$$b = \max\left\{\rho_{[p+1,q]}(F),\ \rho_{[p+1,q]}(A_j)\,(j = 0, 1, \cdots, k)\right\} < \rho_{[p+1,q]}(f).$$

Then

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f).$$

Proof, We assume $f(z)$ is a meromorphic solution of (1.4) with infinite $[p,q]$ order. By (1.4), it is easy to see that if f has a zero at z_0 of order $\alpha\,(\alpha > k)$, and A_0, $A_1, \cdots, A_k\,(\not\equiv 0)$ are all analytic at z_0, then F must have a zero at z_0 of order at least $\alpha - k$. Hence,

$$n\left(r, \frac{1}{f}\right) \le k\overline{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k} n(r, A_j),$$

and

$$N\left(r, \frac{1}{f}\right) \le k\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k} N(r, A_j). \tag{3.1}$$

Now (1.4) can be rewritten as

$$\frac{1}{f} = \frac{1}{F}\left(A_k\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z)\right). \tag{3.2}$$

By Lemma 3.7 and (3.2), for $|z| = r$ outside a set $E_6 \subset (0,\infty)$ of finite linear measure, we have

$$m\left(r, \frac{1}{f}\right) \le m\left(r, \frac{1}{F}\right) + \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=0}^{k} m(r, A_j)$$

$$\le m\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k} m(r, A_j) + O(\log rT(r,f)). \tag{3.3}$$

Therefore, by (3.1) and (3.3), we have

$$T(r,f) = T(r,\frac{1}{f}) + O(1)$$

$$\le T(r,F) + \sum_{j=0}^{k} T(r, A_j) + k\overline{N}\left(r, \frac{1}{f}\right) + O(\log rT(r,f)) \tag{3.4}$$

for all sufficiently large $r \notin E_6$. For sufficiently large r, we have

$$O\left(\log rT\left(r,f\right)\right) \leq \frac{1}{2}T(r,f). \tag{3.5}$$

By using the definition of the $[p,q]$ order, for any given $\varepsilon \left(0 < 2\varepsilon < \rho_{[p+1,q]}\left(f\right) - b\right)$, and for sufficiently large r, we have

$$T\left(r,F\right) \leq \exp_{p+1}\left\{\left(b+\varepsilon\right)\log_q r\right\} = o\left(1\right)\exp_{p+1}\left\{\left(\rho_{[p+1,q]}\left(f\right)-\varepsilon\right)\log_q r\right\}, \tag{3.6}$$

$$T\left(r,A_j\right) \leq \exp_{p+1}\left\{\left(b+\varepsilon\right)\log_q r\right\}$$

$$= o\left(1\right)\exp_{p+1}\left\{\left(\rho_{[p+1,q]}\left(f\right)-\varepsilon\right)\log_q r\right\}, j=0,1,,\cdots,k. \tag{3.7}$$

By $(3.4), (3.5), (3.6)$ and (3.7), for $r \notin E_6$ sufficiently large, we obtain

$$T(r,f) \leq 2k\overline{N}\left(r,\frac{1}{f}\right) + o\left(1\right)\exp_{p+1}\left\{\left(\rho_{[p+1,q]}\left(f\right)-\varepsilon\right)\log_q r\right\}. \tag{3.8}$$

Hence for any f, by Lemma 3.4 and (3.8), we have

$$\rho_{[p+1,q]}\left(f\right) \leq \overline{\lambda}_{[p+1,q]}\left(f\right)$$

then

$$\rho_{[p+1,q]}\left(f\right) \leq \overline{\lambda}_{[p+1,q]}\left(f\right) \leq \lambda_{[p+1,q]}\left(f\right).$$

Since by definition, we have $\overline{\lambda}_{[p+1,q]}\left(f\right) \leq \lambda_{[p+1,q]}\left(f\right) \leq \rho_{[p+1,q]}\left(f\right)$, then

$$\overline{\lambda}_{[p+1,q]}\left(f\right) = \lambda_{[p+1,q]}\left(f\right) = \rho_{[p+1,q]}\left(f\right).$$

Lemma 3.9 *Let $f\left(z\right) = \frac{g(z)}{d(z)}$ be a meromorphic function, where $g\left(z\right)$, $d\left(z\right)$ are entire functions. If $0 \leq \rho_{[p,q]}\left(d\right) < \mu_{[p,q]}\left(f\right)$, then $\mu_{[p,q]}\left(g\right) = \mu_{[p,q]}\left(f\right)$ and $\rho_{[p,q]}\left(g\right) = \rho_{[p,q]}\left(f\right)$. Moreover, if $\rho_{[p,q]}\left(f\right) = +\infty$, then $\rho_{[p+1,q]}\left(g\right) = \rho_{[p+1,q]}\left(f\right)$.*

Proof. Case 1. $\rho_{[p,q]}\left(f\right) < +\infty$. By definition of the $[p,q]$-order, there exists an increasing sequence $\{r_n\}$, $(r_n \to +\infty)$ and a positive integer n_0 such that for all $n > n_0$ and for any given $\varepsilon \in \left(0, \frac{\rho_{[p,q]}(f)-\rho_{[p,q]}(d)}{2}\right)$ (as $0 \leq \rho_{[p,q]}\left(d\right) < \mu_{[p,q]}\left(f\right) \leq \rho_{[p,q]}\left(f\right)$), we have

$$T\left(r_n,f\right) \geq \exp_p\left\{\left(\rho_{[p,q]}\left(f\right)-\varepsilon\right)\log_q\left(r_n\right)\right\}, \tag{3.9}$$

and

$$T\left(r_n,d\right) \leq \exp_p\left\{\left(\rho_{[p,q]}\left(d\right)+\varepsilon\right)\log_q\left(r_n\right)\right\}. \tag{3.10}$$

Since $T(r,f) \leq T(r,g) + T(r,d) + O(1)$, we get, for all sufficiently large n,

$$\exp_p\left\{\left(\rho_{[p,q]}(f) - \varepsilon\right)\log_q(r_n)\right\} \leq T(r_n, g)$$

$$+\exp_p\left\{\left(\rho_{[p,q]}(d) + \varepsilon\right)\log_q(r_n)\right\} + O(1). \tag{3.11}$$

Since $\varepsilon \in \left(0, \frac{\rho_{[p,q]}(f) - \rho_{[p,q]}(d)}{2}\right)$, then (3.11) becomes

$$\exp\left\{(1 - o(1))\exp_{p-1}\left\{\left(\rho_{[p,q]}(f) - \varepsilon\right)\log_q(r_n)\right\}\right\} \leq T(r_n, g) + O(1),$$

for all sufficiently large n. Hence $\rho_{[p,q]}(f) \leq \rho_{[p,q]}(g)$. On the other hand, since $T(r,g) \leq T(r,f) + T(r,d)$, and $\rho_{[p,q]}(d) < \rho_{[p,q]}(f)$, so we obtain $\rho_{[p,q]}(g) \leq \rho_{[p,q]}(f)$. Therefore, we get $\rho_{[p,q]}(g) = \rho_{[p,q]}(f)$. By using the similar way above and the definition of lower $[p,q]$-order $\mu_{[p,q]}(f)$ and $\mu_{[p,q]}(g)$, we can prove $\mu_{[p,q]}(g) = \mu_{[p,q]}(f)$.

Case 2. $\rho_{[p,q]}(f) = +\infty$. Suppose on the contrary to the assertion that $\mu_{[p,q]}(g) < \mu_{[p,q]}(f)$. We aim for a contradiction. By the definition of lower $[p,q]$-order, there exists an increasing sequence $\{r_n\}$, $(r_n \to +\infty)$ and a positive integer n_0 such that for all $n > n_0$ and for any given $\varepsilon > 0$

$$T(r_n, g) \leq \exp_p\left\{\left(\mu_{[p,q]}(g) + \varepsilon\right)\log_q r_n\right\},$$

$$T(r_n, d) \leq \exp_p\left\{\left(\mu_{[p,q]}(d) + \varepsilon\right)\log_q r_n\right\}.$$

Since $T(r_n, f) \leq T(r_n, g) + T(r_n, d) + O(1)$, we get, for all sufficiently large n,

$$T(r_n, f) \leq \exp_p\left\{\left(\mu_{[p,q]}(g) + \varepsilon\right)\log_q r_n\right\} + \exp_p\left\{\left(\mu_{[p,q]}(d) + \varepsilon\right)\log_q r_n\right\} + O(1),$$

hence $\mu_{[p,q]}(f) \leq \max\left\{\mu_{[p,q]}(g), \mu_{[p,q]}(d)\right\}$. This is a contradiction with our assumption.

Case 3. $\mu_{[p,q]}(f) < +\infty$ and $\rho_{[p,q]}(f) = +\infty$. By using the similar way in proving Cases 1 and 2, we can prove Case 3.

Finally, we will prove $\rho_{[p+1,q]}(g) = \rho_{[p+1,q]}(f)$. Suppose that $\rho_{[p,q]}(f) = +\infty$. Then there exists an increasing sequence $\{r_n\}$, $(r_n \to +\infty)$, such that

$$\rho_{[p+1,q]}(f) = \lim_{n\to\infty} \frac{\log_{p+1} T(r_n, f)}{\log_q r_n}.$$

Combining $\rho_{[p,q]}(d) < \mu_{[p,q]}(f)$ and the definitions of the $[p,q]$-order and the lower $[p,q]$-order, we get

$$\lim_{n\to\infty} \frac{T(r_n, d)}{T(r_n, f)} = 0.$$

Then, there exists a positive integer N, such that for $n > N$

$$T(r_n, f) \leq 2T(r_n, g) + O(1).$$

Thus, $\rho_{[p+1,q]}(f) \leq \rho_{[p+1,q]}(g)$. By using the similar way in proving Case 1, since $T(r,g) \leq T(r,f) + T(r,d)$, then there exists a positive integer N, such that for $n > N$

$$T(r_n, g) \leq 2T(r_n, f).$$

Hence, $\rho_{[p+1,q]}(g) \leq \rho_{[p+1,q]}(f)$. Therefore $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(g)$. Lemma 3.9 is thus proved.

Remark 3.1. Lemma 3.9 was proved for $p = q = 1$ by Long and Zhu in [19].

Lemma 3.10 ([5]) *Let $p \geq q \geq 1$ be integers, and let f, g be non-constant meromorphic functions of $[p,q]$-order. Then we have*

$$\rho_{[p,q]}(f+g) \leq \max\left\{\rho_{[p,q]}(f), \rho_{[p,q]}(g)\right\}$$

and

$$\rho_{[p,q]}(fg) \leq \max\left\{\rho_{[p,q]}(f), \rho_{[p,q]}(g)\right\}.$$

Furthermore, if $\rho_{[p,q]}(f) > \rho_{[p,q]}(g)$, then we obtain

$$\rho_{[p,q]}(f+g) = \rho_{[p,q]}(fg) = \rho_{[p,q]}(f).$$

4. Proof of Theorems

Proof of Theorem 2.1. Assume that $f(z) \not\equiv 0$ is a rational solution of (1.3). If either $f(z)$ is a rational function, which has a pole at z_0 of degree $m \geq 1$, or $f(z)$ is a polynomial with $\deg f \geq s$, then $f^{(s)}(z) \not\equiv 0$. By (1.3), we have

$$A_s(z) f^{(s)}(z) = -\sum_{\substack{j=0\\ j\neq s}}^{k} A_j(z) f^{(j)}(z).$$

Then

$$\rho_{[p,q]}(A_s) = \rho_{[p,q]}(A_s f^{(s)}) = \rho_{[p,q]}\left(-\sum_{j=0,\ j\neq s}^{k} A_j f^{(j)}\right)$$
$$\leq \max_{j=0,1,\cdots,k,\ j\neq s}\left\{\rho_{[p,q]}\left(A_j\right)\right\},$$

which is a contradiction. Therefore, $f(z)$ must be a polynomial with $\deg f \leq s-1$.

Now, we assume that $f(z)$ is a transcendental meromorphic solution of (1.3) such that $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$. By (1.3), we have

$$|A_s| \leq \left|\frac{f}{f^{(s)}}\right|\left(|A_0| + \sum_{\substack{j=1\\ j\neq s}}^{k}|A_j|\left|\frac{f^{(j)}}{f}\right|\right). \tag{4.1}$$

Using Lemma 3.1, there exists a set $E_1 \subset (1,+\infty)$ with $m_l(E_1) < \infty$ and a constant $B > 0$, such that for all z satisfying $|z| = r \notin ([0,1] \cup E_1)$

$$\left|\frac{f^{(j)}\left(z\right)}{f\left(z\right)}\right| \leq B\left[T\left(2r,f\right)\right]^{k+1},\ j = 1,2,\cdots,k,\ j \neq s. \tag{4.2}$$

By Lemma 3.5, there exists a set $E_4 \subset (1,+\infty)$ of finite logarithmic measure such that for all $|z| = r \notin ([0,1] \cup E_4)$ and $|g\left(z\right)| = M\left(r,g\right)$ and for r sufficiently large

$$\left|\frac{f\left(z\right)}{f^{(s)}\left(z\right)}\right| \leq r^{2s}, \tag{4.3}$$

where $g\left(z\right)$ is an entire function satisfying $\mu_{[p,q]}\left(g\right) = \mu_{[p,q]}\left(f\right) = \mu \leq \rho_{[p,q]}\left(f\right) = \rho_{[p,q]}\left(g\right) \leq +\infty$. By the hypotheses of Theorem 2.1, there exists a set H with $\overline{\log dens}\{|z| : z \in H\} > 0$ (or $m_l(\{|z| : z \in H\}) = \infty$) such that for all $z \in H$ as $z \to \infty$, we have

$$|A_j\left(z\right)| \leq \exp_{p+1}\left\{\mu \log_q r\right\},\ j \neq s, j = 0,1,\cdots,k, \tag{4.4}$$

$$|A_s\left(z\right)| \geq \exp_{p+1}\left\{(\rho-\varepsilon)\log_q r\right\}. \tag{4.5}$$

Set $H_1 = \{|z| : z \in H\}\setminus([0,1] \cup E_1 \cup E_4)$, so $m_l\left(H_1\right) = \infty$. It follows from (4.1), (4.2), (4.3), (4.4) and (4.5) that for all z satisfying $|z| = r \in H_1$ and $|g(z)| = M(r,g)$,

$$\exp_{p+1}\left\{(\rho-\varepsilon)\log_q r\right\} \leq kBr^{2s}\left(T\left(2r,f\right)\right)^{k+1}\exp_{p+1}\left\{\mu\log_q r\right\},$$

using $0 \leq \mu < \rho$, we have for any given ε $(0 < \varepsilon < \rho - \mu)$ sufficiently small

$$\exp\left((1-o\left(1\right))\exp_p\left\{(\rho-\varepsilon)\log_q r\right\}\right) \leq kBr^{2s}\left(T\left(2r,f\right)\right)^{k+1}. \tag{4.6}$$

By (4.6) and Lemma 3.4, we get

$$\rho_{[p,q]}(A_s) = \rho \le \limsup_{r\to+\infty} \frac{\log_{p+1} T(r,f)}{\log_q r} = \rho_{[p+1,q]}(f).$$

On the other hand, by the hypotheses of Theorem 2.1, for sufficiently large r, we have

$$|A_j(z)| \le \exp_{p+1}\left\{(\rho+\varepsilon)\log_q r\right\},\ j = 0,1,\cdots,k, \tag{4.7}$$

and by Lemma 3.6, for any given $\varepsilon > 0$, there exists a set $E_5 \subset (1,+\infty)$ of finite linear measure (and so of finite logarithmic measure), such that for all z satisfying $|z| = r \notin E_5$ we obtain

$$\begin{aligned}|A_k(z)| &\ge \exp\left\{-\exp_p\left\{(\rho_{(p,q)}(A_k)+\varepsilon)\log_q r\right\}\right\}\\ &\ge \exp\left\{-\exp_p\left\{(\rho+\varepsilon)\log_q r\right\}\right\}.\end{aligned} \tag{4.8}$$

It follows by (1.3) that

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \le \frac{1}{|A_k(z)|}\left(\sum_{j=1}^{k-1}|A_j(z)|\left|\frac{f^{(j)}(z)}{f(z)}\right| + |A_0(z)|\right). \tag{4.9}$$

By Hadamard factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$, $d(z)$ are entire functions satisfying $\mu_{[p,q]}(g) = \mu_{[p,q]}(f) = \mu \le \rho_{[p,q]}(f) = \rho_{[p,q]}(g) \le +\infty$ and $\lambda_{[p,q]}(d) = \rho_{[p,q]}(d) = \lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu$. By Lemma 3.2, there exists a set $E_2 \subset (1,+\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0,1]\cup E_2$ and $|g(z)| = M(r,g)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1+o(1)),\ j = 1,\cdots,k. \tag{4.10}$$

By substituting (4.7), (4.8) and (4.10) into (4.9), we obtain

$$\begin{aligned}&\left|\frac{\nu_g(r)}{z}\right|^k |1+o(1)|\\ &\le \frac{1}{\exp\left\{-\exp_p\left\{(\rho+\varepsilon)\log_q r\right\}\right\}}\left\{\sum_{j=1}^{k-1}\left|\frac{\nu_g(r)}{z}\right|^j |1+o(1)| + 1\right\}\\ &\quad\times \exp_{p+1}\left\{(\rho+\varepsilon)\log_q r\right\}\\ &= \left\{\sum_{j=1}^{k-1}\left|\frac{\nu_g(r)}{z}\right|^j |1+o(1)| + 1\right\}\exp\left\{2\exp_p\left\{(\rho+\varepsilon)\log_q r\right\}\right\}.\end{aligned}$$

Hence

$$|\nu_g(r)|\,|1+o(1)| \leq kr^k\,|1+o(1)| \exp\left\{2\exp_p\left\{(\rho+\varepsilon)\log_q r\right\}\right\}, \tag{4.11}$$

for all z satisfying $|z| = r \notin ([0,1] \cup E_2 \cup E_5)$ and $|g(z)| = M(r,g)$, $r \to +\infty$. By (4.11) and Lemma 3.4, we get

$$\limsup_{r\to+\infty} \frac{\log_{p+1} \nu_g(r)}{\log_q r} \leq \rho + \varepsilon. \tag{4.12}$$

Since $\varepsilon > 0$ is arbitrary, by (4.12) and Lemma 3.3, we obtain $\rho_{[p+1,q]}(g) \leq \rho$. Since $\rho_{[p,q]}(d) < \mu_{[p,q]}(f)$, so by Lemma 3.9 we have $\rho_{[p+1,q]}(g) = \rho_{[p+1,q]}(f)$. This and the fact that $\rho_{[p,q]}(A_s) = \rho \leq \rho_{[p+1,q]}(f)$ yield $\rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s)$. Theorem 2.1 is thus proved.

Proof of Corollary 2.2. Setting $h = f - \varphi$ such that φ is a transcendental meromorphic function with $\rho_{[p+1,q]}(\varphi) < \rho_{[p,q]}(A_s)$. By Theorem 2.1, we have $\rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s)$. By Lemma 3.10, we get $\rho_{[p+1,q]}(h) = \rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s)$. By substituting $f = h + \varphi$ into (1.3), we obtain

$$A_k(z)\,h^{(k)} + A_{k-1}(z)\,h^{(k-1)} + \cdots + A_1(z)\,h' + A_0(z)\,h$$
$$= -\left(A_k(z)\,\varphi^{(k)} + A_{k-1}(z)\,\varphi^{(k-1)} + \cdots + A_1(z)\,\varphi' + A_0(z)\,\varphi\right).$$

Set $K(z) = A_k(z)\,\varphi^{(k)} + A_{k-1}(z)\,\varphi^{(k-1)} + \cdots + A_1(z)\,\varphi' + A_0(z)\,\varphi$. Since $\rho_{[p+1,q]}(\varphi) < \rho_{[p,q]}(A_s)$, then by Theorem 2.1, we deduce that φ is not a solution of equation (1.3), implying that $K(z) \not\equiv 0$, and in this case we have $\rho_{[p+1,q]}(K) \leq \rho_{[p+1,q]}(\varphi) < \rho_{[p,q]}(A_s) = \rho_{[p+1,q]}(f)$, so

$$\max\left\{\rho_{[p+1,q]}(K),\ \rho_{[p+1,q]}(A_j)\,(j = 0, 1, \cdots, k)\right\} < \rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s)$$

and by Lemma 3.8, we obtain

$$\overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi) = \rho_{[p+1,q]}(f-\varphi) = \rho_{[p,q]}(A_s).$$

Proof of Theorem 2.2. Assume that $f(z)$ is a rational solution of (1.4). If either $f(z)$ is a rational function, which has a pole at z_0 of degree $m \geq 1$, or $f(z)$ is a polynomial with $\deg f \geq s$, then $f^{(s)}(z) \not\equiv 0$. By (1.4), we have

$$A_s(z)\,f^{(s)}(z) = F(z) - \sum_{\substack{j=0 \\ j\neq s}}^{k} A_j(z)\,f^{(j)}(z).$$

Then

$$\rho_{[p,q]}(A_s) = \rho_{[p,q]}(A_s f^{(s)}) = \rho_{[p,q]}\left(F - \sum_{j=0,\ j\neq s}^{k} A_j f^{(j)}\right)$$

$$\leq \max_{j=0,1,\cdots,k,\ j\neq s}\left\{\rho_{[p,q]}\left(A_j\right),\ \rho_{[p,q]}\left(F\right)\right\},$$

which is a contradiction. Therefore, $f(z)$ must be a polynomial with $\deg f \leq s-1$.

Now, we assume that $f(z)$ is a transcendental meromorphic solution of (1.4) such that $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$. By (1.4), we have

$$\left|\frac{f^{(k)}\left(z\right)}{f\left(z\right)}\right| \leq \frac{1}{\left|A_k\left(z\right)\right|}\left(\sum_{j=1}^{k-1}\left|A_j\left(z\right)\right|\left|\frac{f^{(j)}\left(z\right)}{f\left(z\right)}\right| + \left|A_0\left(z\right)\right| + \left|\frac{F\left(z\right)}{f\left(z\right)}\right|\right). \tag{4.13}$$

By Hadamard factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g\left(z\right)$, $d\left(z\right)$ are entire functions satisfying $\mu_{[p,q]}\left(g\right) = \mu_{[p,q]}\left(f\right) = \mu \leq \rho_{[p,q]}\left(f\right) = \rho_{[p,q]}\left(g\right) \leq +\infty$ and $\lambda_{[p,q]}\left(d\right) = \rho_{[p,q]}\left(d\right) = \lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu$. By Lemma 3.2, there exists a set E_2 of finite logarithmic measure such that for all z satisfying $|z| = r \notin \left([0,1] \cup E_2\right)$ at which $\left|g\left(z\right)\right| = M\left(r,g\right)$ we have (4.10). By the hypotheses of Theorem 2.2 and Lemma 3.6, for any given $\varepsilon > 0$, there exists a set $E_5 \subset (1,+\infty)$ of a finite linear measure, such that for all z satisfying $|z| = r \notin E_5$

$$\left|A_k\left(z\right)\right| \geq \exp\left\{-\exp_p\left\{\left(\rho_{(p,q)}\left(A_k\right)+\varepsilon\right)\log_q r\right\}\right\}$$

$$\geq \exp\left\{-\exp_p\left\{\left(\rho+\varepsilon\right)\log_q r\right\}\right\}. \tag{4.14}$$

On the other hand, for sufficiently large r, we have for $j = 0,1,\cdots,k$

$$\left|A_j\left(z\right)\right| \leq \exp_{p+1}\left\{\left(\rho+\varepsilon\right)\log_q r\right\},\ \text{and}\ \ \left|F\left(z\right)\right| \leq \exp_{p+1}\left\{\left(\rho+\varepsilon\right)\log_q r\right\}. \tag{4.15}$$

So, for any given ε $\left(0 < 2\varepsilon < \mu_{(p,q)}(g) - \rho_{(p,q)}(d)\right)$ and sufficiently large r, we obtain

$$\left|\frac{F\left(z\right)}{f\left(z\right)}\right| = \frac{\left|F\left(z\right)d\left(z\right)\right|}{|g(z)|} = \frac{\left|F\left(z\right)\right|\left|d\left(z\right)\right|}{M(r,g)}$$

$$\leq \frac{\exp_{p+1}\left\{\left(\rho+\varepsilon\right)\log_q r\right\}\exp_{p+1}\left\{\left(\rho_{(p,q)}(d)+\varepsilon\right)\log_q r\right\}}{\exp_{p+1}\left\{\left(\mu_{(p,q)}(g)-\varepsilon\right)\log_q r\right\}}$$

$$\leq \exp_{p+1}\left\{\left(\rho+\varepsilon\right)\log_q r\right\}. \tag{4.16}$$

By substituting (4.10), (4.14), (4.15) and (4.16) into (4.13), for z satisfying $|z| = r \notin ([0,1] \cup E_2 \cup E_5)$, $r \to +\infty$ and $|g(z)| = M(r,g)$, we have

$$\left|\frac{\nu_g(r)}{z}\right|^k |1+o(1)|$$

$$\leq \frac{1}{\exp\left\{-\exp_p\left\{(\rho+\varepsilon)\log_q r\right\}\right\}}\left\{\sum_{j=1}^{k-1}\left|\frac{\nu_g(r)}{z}\right|^j |1+o(1)|+2\right\}$$
$$\times \exp_{p+1}\left\{(\rho+\varepsilon)\log_q r\right\}$$
$$=\left\{\sum_{j=1}^{k-1}\left|\frac{\nu_g(r)}{z}\right|^j |1+o(1)|+2\right\}\exp\left\{2\exp_p\left\{(\rho+\varepsilon)\log_q r\right\}\right\}.$$

Hence

$$|\nu_g(r)|\,|1+o(1)| \leq (k+1)\,r^k\,|1+o(1)|\exp\left\{2\exp_p\left\{(\rho+\varepsilon)\log_q r\right\}\right\}, \tag{4.17}$$

for all z satisfying $|z| = r \notin ([0,1] \cup E_2 \cup E_5)$ and $|g(z)| = M(r,g)$, $r \to +\infty$. By (4.17) and Lemma 3.4, we get

$$\limsup_{r\to+\infty}\frac{\log_{p+1}\nu_g(r)}{\log_q r} \leq \rho+\varepsilon. \tag{4.18}$$

Since ε $\left(0 < 2\varepsilon < \mu_{(p,q)}(g) - \rho_{(p,q)}(d)\right)$ is arbitrary, by (4.18) and Lemma 3.3, we obtain $\rho_{[p+1,q]}(g) \leq \rho$. Since $\rho_{[p,q]}(d) < \mu_{[p,q]}(f)$, so by Lemma 3.9 we have $\rho_{[p+1,q]}(g) = \rho_{[p+1,q]}(f)$, hence $\rho_{[p+1,q]}(f) \leq \rho$. On the other hand, by (1.4), we have

$$|A_s| \leq \left|\frac{f}{f^{(s)}}\right|\left(\left|\frac{F(z)}{f(z)}\right| + |A_0| + \sum_{\substack{j=1\\ j\neq s}}^{k}|A_j|\left|\frac{f^{(j)}}{f}\right|\right). \tag{4.19}$$

Using Lemma 3.1, there exists a set $E_1 \subset (1,+\infty)$ with $m_l(E_1) < +\infty$ and a constant $B > 0$, such that for all z satisfying $|z| = r \notin ([0,1] \cup E_1)$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[T(2r,f)\right]^{k+1},\ j=1,2,\cdots,k,\ j\neq s. \tag{4.20}$$

By Lemma 3.5, there exists a set $E_4 \subset (1,+\infty)$ of finite logarithmic measure such that for all $|z| = r \notin ([0,1] \cup E_4)$ and $|g(z)| = M(r,g)$ and for r sufficiently large

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq r^{2s}. \tag{4.21}$$

By the hypotheses of Theorem 2.2, there exists a set H with $\overline{\log dens}\{|z| : z \in H\} > 0$ (or $m_l(\{|z| : z \in H\}) = \infty$) such that for all $z \in H$, we have $(4,4)$ and (4.5) hold. Set $\delta = \rho_{[p,q]}(F)$. Then, for any given ε $(0 < 2\varepsilon < \rho - \delta)$, we have

$$|F(z)| \leq \exp_{p+1}\left\{(\delta + \varepsilon)\log_q r\right\} \tag{4.22}$$

So, for any given ε $\left(0 < 2\varepsilon < \min\left\{\mu_{(p,q)}(g) - \rho_{(p,q)}(d), \rho - \delta\right\}\right)$ and sufficiently large r, we obtain

$$\begin{aligned}\left|\frac{F(z)}{f(z)}\right| &= \frac{|F(z)\,d(z)|}{|g(z)|} = \frac{|F(z)|\,|d(z)|}{M(r,g)}\\ &\leq \frac{\exp_{p+1}\left\{(\delta+\varepsilon)\log_q r\right\}\exp_{p+1}\left\{\left(\rho_{(p,q)}(d)+\varepsilon\right)\log_q r\right\}}{\exp_{p+1}\left\{\left(\mu_{(p,q)}(g)-\varepsilon\right)\log_q r\right\}}\\ &\leq \exp_{p+1}\left\{(\delta+\varepsilon)\log_q r\right\}. \end{aligned} \tag{4.23}$$

Set $H_1 = \{|z| : z \in H\} \setminus ([0,1] \cup E_1 \cup E_4)$, so $m_l(H_1) = \infty$. By substituting (4.4), (4.5), (4.20), (4.21), (4.22) and (4.23) into (4.19), for z satisfying $|z| = r \in H_1$, $r \to +\infty$ and $|g(z)| = M(r,g)$, we have

$$\begin{aligned}\exp_{p+1}\left\{(\rho-\varepsilon)\log_q r\right\} \leq Br^{2s}\left(T(2r,f)\right)^{k+1}\Big(&\exp_{p+1}\left\{(\delta+\varepsilon)\log_q r\right\}\\ &+k\exp_{p+1}\left\{\mu\log_q r\right\}\Big).\end{aligned} \tag{4.24}$$

It follow from (4.24) and Lemma 3.4 that $\rho \leq \rho_{[p+1,q]}(f)$. This and the fact that $\rho_{[p+1,q]}(f) \leq \rho$ yield $\rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s) = \rho$. Since

$$\max\left\{\rho_{[p+1,q]}(F),\ \rho_{[p+1,q]}(A_j)\,(j = 0, 1, \cdots, k)\right\} < \rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s),$$

then by Lemma 3.8, we obtain

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s).$$

Theorem 2.2 is thus proved.

Proof of Corollary 2.4. Setting $h = f - \varphi$ such that φ is a transcendental meromorphic function with $\rho_{[p+1,q]}(\varphi) < \rho_{[p,q]}(A_s)$. By Theorem 2.2, we have $\rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s)$. By Lemma 3.10, we get $\rho_{[p+1,q]}(h) = \rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s)$. By substituting $f = h + \varphi$ into (1.4), we get

$$\begin{aligned}&A_k(z)\,h^{(k)} + A_{k-1}(z)\,h^{(k-1)} + \cdots + A_1(z)\,h' + A_0(z)\,h\\ &= F(z) - \left(A_k(z)\,\varphi^{(k)} + A_{k-1}(z)\,\varphi^{(k-1)} + \cdots + A_1(z)\,\varphi' + A_0(z)\,\varphi\right).\end{aligned} \tag{4.25}$$

Set $G(z) = F(z) - \left(A_k(z)\varphi^{(k)} + A_{k-1}(z)\varphi^{(k-1)} + \cdots + A_1(z)\varphi' + A_0(z)\varphi\right)$. Since $\rho_{[p+1,q]}(\varphi) < \rho_{[p,q]}(A_s)$, then by Theorem 2.2, we deduce that φ is not a solution of equation (1.4), implying that $G(z) \not\equiv 0$, and in this case we have $\rho_{[p+1,q]}(G) \leq \rho_{[p+1,q]}(\varphi) < \rho_{[p,q]}(A_s) = \rho_{[p+1,q]}(f)$, so

$$\max\left\{\rho_{[p+1,q]}(G),\ \rho_{[p+1,q]}(A_j)\,(j=0,1,\cdots,k)\right\} < \rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_s)$$

and by Lemma 3.8, we obtain

$$\overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi) = \rho_{[p+1,q]}(f-\varphi) = \rho_{[p,q]}(A_s).$$

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