

## THE AVERAGE SHADOWING PROPERTY AND CHAOS FOR CONTINUOUS FLOWS

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**ABSTRACT.** Let  $X$  be a compact metric space and  $\varphi : R \times X \rightarrow X$  be a continuous flow. In this paper, we prove that if  $\varphi$  has the average shadowing property and the almost periodic points of  $\varphi$  are dense in  $X$ , then  $\varphi \times \varphi$  is topologically ergodic. As a corollary, we obtain that if a Lyapunov stable flow  $\varphi$  has the average-shadowing property, then  $X$  is a singleton.

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### 1. INTRODUCTION

Chaos or complexity is an important property and an important study topic of dynamical systems. In 1975, Li and Yorke introduced the term chaos for interval maps [10], today known as Li-Yorke chaos. Since then, people from different fields try to describe the behaviors by providing definitions of chaos according to their understanding of the subject. There have appeared many important definitions of types of chaos [3]. The different notions of chaos at least share a common motivating intuition of unpredictability. It is well known that sensitive dependence on

initial conditions characterizes the unpredictability of chaotic phenomenon and was popularized by the meteorologist Ed Lorenz through the so-called butterfly effect. A system is called Auslander -Yorke chaos if it is topologically transitive and has sensitive dependence on initial conditions [3].

The pseudo-orbit tracing property is one of the most important notions in dynamical systems [5], which is closely related to stability and chaos of systems, see, for instance [14-16] and is an essential part of stability and ergodic theory [1]. In order to explore the complexity of a dynamical system, concepts of various shadowing properties were introduced [12,13]. Blank introduced the notion of the average-shadowing property in studying chaotic dynamical systems, which is a good tool to characterize Anosov diffeomorphisms [4]. The research on the average-shadowing property also have been arousing a great deal of interest from many researchers and mathematicians [ 6-9, 11, 12].

Recently, Gu discussed the relation among the average-shadowing property, Lyapunov stability and ergodicity for a flow [6,7]. It was showed in [7] that a Lyapunov stable flow with the average-shadowing property is topologically ergodic. We remark that a Lyapunov stable flow is simple. Now a natural question arises: which flow with the average-shadowing property is chaotic? In this paper, we try to study this question and show that if a flow  $\varphi$  has the average-shadowing property and the set of almost periodic points of  $\varphi$  is dense in  $X$ , then  $\varphi \times \varphi$  is topologically ergodic. In particular,  $\varphi$  is Auslander-Yorke chaotic. As a corollary, we obtain that if  $\varphi$  is Lyapunov stable and  $X$  is not a singleton, then  $\varphi$  does not have the average-shadowing property. We remark that similar results are known for discrete-time dynamical systems [11]. Our results show that a system with the average-shadowing property may be complex.

## 2. PRELIMINARIES

In this section, we recall some basic concepts and give out some notations from dynamical systems.

Let  $(X, d)$  be a compact metric space with metric  $d$ , write  $R = (-\infty, \infty)$ . Let  $\varphi : R \times X \rightarrow X$  be a continuous flow, that is,  $\varphi : R \times X \rightarrow X$  is a continuous map and satisfies the following conditions:

- (1)  $\varphi(0, x) = x$  for any  $x \in X$ ;

(2)  $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$  for any  $x \in X$  and any  $s, t \in R$ .

For  $x \in X$ , the set  $\gamma(x, \varphi) = \{\varphi(t, x) : t \in R\}$  is called the orbit of  $\varphi$  through  $x$ . For  $t \in R$ , we write  $\varphi_t : X \rightarrow X$ ,  $\varphi_t(x) = \varphi(t, x)$  for any  $x \in X$ . A point  $x \in X$  is called a recurrent point of  $\varphi$  if there is  $t_n \rightarrow \infty$  such that  $\varphi(t_n, x) \rightarrow x$ .

A set  $A \subset R$  is said to be relatively dense if there is  $L > 0$  such that  $A \cap (t, t+L) \neq \emptyset$  for every  $t \in R$ . A point  $x \in X$  is called an almost periodic point of  $\varphi$  if for any neighborhood  $U$  of  $x$ , there is a relatively dense subset  $A \subset R$  such that  $\varphi(t, x) \in U$  for every  $t \in A$ . We denote the set of almost periodic points of  $\varphi$  by  $AP(\varphi)$ .

A set  $Y \subset X$  is said to be invariant for  $\varphi$  if  $\varphi(t, Y) \subset Y$  for all  $t \in R$ . A subset  $M$  of  $X$  is called minimal for  $\varphi$  if  $M$  is non-empty, closed, invariant and there is no proper subset of  $M$  with these three properties. If  $X$  is minimal, we say that  $\varphi$  is minimal. It is easy to see that  $\varphi$  is minimal if and only if  $\overline{\gamma(x, \varphi)} = X$  for each  $x \in X$ . It is well known that  $x \in X$  is an almost periodic point if and only if  $\overline{\gamma(x, \varphi)}$  is minimal [2].

A point  $x \in X$  is called a Lyapunov stable point of  $\varphi$  if for any  $\varepsilon > 0$ , there is  $\delta > 0$  such that  $d(x, y) < \delta$  implies  $d(\varphi(t, x), \varphi(t, y)) < \varepsilon$  for each  $t \in R$ . A flow  $\varphi$  is called Lyapunov stable or equicontinuous if every point in  $X$  is Lyapunov stable. A flow  $\varphi$  is called sensitive dependent on initial conditions (sensitive for short) if it does not have a Lyapunov stable point. It is well known that if  $\varphi$  is Lyapunov stable, then every point in  $X$  is almost periodic [Lemma 3, P37, 2].

A flow  $\varphi$  is called topologically transitive if for any non-empty open subsets  $U, V$  of  $X$  there is some  $t \in R$  such that  $\varphi(t, U) \cap V \neq \emptyset$ . It is not difficult to check that if  $\varphi$  is topologically transitive, then there is some  $x \in X$  such that  $\overline{\gamma(x, \varphi)} = X$ .

A flow  $\varphi$  is said to be topologically ergodic if

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \chi(\varphi(s, U) \cap V) ds > 0$$

for any two non-empty open subsets  $U$  and  $V$  of  $X$ , where  $\chi(U) = 1$  when  $U \neq \emptyset$  and  $\chi(U) = 0$  when  $U = \emptyset$ .

It is obvious that topological ergodicity implies topological transitivity.

For a flow  $\varphi : R \times X \rightarrow X$ , we can define the product flow  $\varphi \times \varphi : R \times (X \times X) \rightarrow X \times X$  with  $\varphi \times \varphi(t, (x, y)) = (\varphi(t, x), \varphi(t, y))$  for any  $x, y \in X$  and any  $t \in R$ . The metric  $\rho$  on  $X \times X$  is  $\rho((x, y), (x', y')) = \max \{d(x, x'), d(y, y')\}$  for

every  $x, x', y, y' \in X$ . We say that  $\varphi$  is weakly mixing if  $\varphi \times \varphi$  is topologically transitive.

Now we recall the concept of the average-shadowing property for a continuous flow.

Given  $\delta > 0$  and  $T > 0$ . A bi-sequence  $(\{x_i\}_{-\infty < i < \infty}, \{t_i\}_{-\infty < i < \infty})$  is called a  $(\delta, T)$ -average-pseudo-orbit of  $\varphi$  if  $t_i \geq T$  for every  $-\infty < i < \infty$  and there is a positive integer  $N = N(\delta)$  such that

$$\frac{1}{n} \sum_{i=0}^{n-1} d(\varphi(t_{k+i}, x_{k+i}), x_{k+i+1}) < \delta$$

for every  $n \geq N$  and every  $-\infty < k < \infty$ .

For  $\varepsilon > 0$ , a bi-sequence  $(\{x_i\}_{-\infty < i < \infty}, \{t_i\}_{-\infty < i < \infty})$  is said to be  $\varepsilon$ -shadowed in average by the orbit of  $\varphi$  through  $x$  if

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} d(\varphi(t, x), \varphi(t - s_i, x_i)) dt < \varepsilon,$$

where  $s_0 = 0, s_n = \sum_{i=0}^{n-1} t_i, n = 1, 2, \dots$ .

A flow  $\varphi$  is said to have the average-shadowing property if for any  $\varepsilon > 0$  there is  $\delta > 0$  such that every  $(\delta, 1)$ -average-pseudo-orbit of  $\varphi$  can be  $\varepsilon$ -shadowed in average by some orbit of  $\varphi$ .

### 3. THE MAIN RESULTS AND PROOFS

The following lemma is due to Zhou [17], here we give a simpler proof for completeness.

**Lemma 3.1.** Let  $X$  be a compact metric space with metric  $d$  and  $\varphi : R \times X \rightarrow X$  be a continuous flow. If  $x \in X$  is a recurrent point of  $\varphi$ , then for any  $\varepsilon > 0$  there are  $\delta > 0$  and  $\Delta t > 0$  such that for any  $t > 0$ ,  $\varphi(t, x) \in B(x, \delta)$  implies  $\varphi((t - \Delta t, t + \Delta t), x) \subset B(x, \varepsilon)$ , where  $B(a, \varepsilon) = \{b \in X : d(a, b) < \varepsilon\}$ .

**Proof.** For given  $\varepsilon > 0$ , since  $\varphi(0, x) = x$ , by continuity of  $\varphi$  there is  $\Delta t > 0$  such that  $d(\varphi(t, x), x) < \frac{\varepsilon}{2}$  for all  $t \in (-\Delta t, \Delta t)$ . For  $\Delta t > 0$ , there is  $\delta > 0$  such that  $d(u, v) < \delta$  implies  $d(\varphi(t, u), \varphi(t, v)) < \frac{\varepsilon}{2}$  for all  $t \in [-\Delta t, \Delta t]$  by continuity of  $\varphi$  and compactness of  $X$ . Since  $x \in X$  is a recurrent point of  $\varphi$ , there is  $t > 0$  such that  $\varphi(t, x) \in B(x, \delta)$ .

For any  $t > 0$ , if  $\varphi(t, x) \in B(x, \delta)$ , i.e,  $d(\varphi(t, x), x) < \delta$ , then  $d(\varphi(t + \tau, x), \varphi(\tau, x)) < \frac{\varepsilon}{2}$  for all  $\tau \in (-\Delta t, \Delta t)$ . Therefore  $d(\varphi(t + \tau, x), x) \leq d(\varphi(t + \tau, x), \varphi(\tau, x)) + d(\varphi(\tau, x), x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$  for all  $\tau \in (-\Delta t, \Delta t)$ . Hence  $\varphi((t - \Delta t, t + \Delta t), x) \subset B(x, \varepsilon)$ .

The proof of Lemma 3.1 is completed.  $\square$

**Lemma 3.2.** Let  $X$  be a compact metric space with metric  $d$  and  $\varphi : R \times X \rightarrow X$  be a continuous flow. If  $\varphi$  is topologically transitive and the set of the almost periodic points of  $\varphi$  is dense in  $X$ , then  $\varphi$  is topologically ergodic.

**Proof.** Let  $U, V$  be two non-empty open subsets of  $X$ , we will prove

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \chi(\varphi(s, U) \cap V) ds > 0.$$

Since  $\varphi$  is topologically transitive, there is  $t_0 \in R$  such that  $\varphi(t_0, U) \cap V \neq \emptyset$ . Hence,  $W = U \cap \varphi_{t_0}^{-1}(V)$  is a non-empty open set. By the assumption, there is  $x \in W \cap AP(\varphi)$ . We may pick  $\varepsilon > 0$  such that  $B(x, \varepsilon) \subset W$ . By Lemma 3.1, there are  $\delta > 0$  and  $\Delta t > 0$  such that for any  $t > 0$ ,  $\varphi(t, x) \in B(x, \delta)$  implies  $\varphi((t - \Delta t, t + \Delta t), x) \subset B(x, \varepsilon)$ . Since  $x \in X$  is an almost periodic point of  $\varphi$ , the set  $A = \{t \in R : \varphi(t, x) \in B(x, \delta)\}$  is a relatively dense subset of  $R$ . Therefore there is  $L > 0$  such that for any  $t \in R$ ,  $(t, t + L) \cap A \neq \emptyset$ . Given a positive integer  $n$ , there is  $t_i \in ((i - 1)L, iL)$  such that  $\varphi(t_i, x) \in B(x, \delta)$ ,  $i = 1, 2, \dots, n$ . Hence  $\varphi((t_i - \Delta t, t_i + \Delta t), x) \subset B(x, \varepsilon) \subset W$  for any  $i : 1 \leq i \leq n$ . It is not difficult to check that if  $\varphi(r, x) \in W$ , then  $\varphi(r + t_0, U) \cap V \neq \emptyset$ . Hence, for any  $r \in (t_0 + t_i - \Delta t, t_0 + t_i + \Delta t)$ ,  $\varphi(r, U) \cap V \neq \emptyset$ ,  $i = 1, 2, \dots, n$ . Hence we have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \chi(\varphi(s, U) \cap V) ds \\ & \geq \limsup_{n \rightarrow \infty} \frac{1}{nL} \int_0^{nL} \chi(\varphi(s, U) \cap V) ds \\ & \geq \limsup_{n \rightarrow \infty} \frac{1}{nL} \left( \sum_{i=1}^n \int_{t_0+t_i-\Delta t}^{t_0+t_i+\Delta t} \chi(\varphi(s, U) \cap V) ds - \int_{nL}^{t_0+t_n+\Delta t} \chi(\varphi(s, U) \cap V) ds \right) \\ & \geq \limsup_{n \rightarrow \infty} \frac{1}{nL} (2n\Delta t) = \frac{2\Delta t}{L} > 0 \end{aligned}$$

Thus,  $\varphi$  is topologically ergodic.  $\square$

With the help of Lemma 3.2 and using the idea of the proof of Theorem 3.4 in [18], we obtain the following result.

**Theorem 3.3.** Let  $X$  be a compact metric space with metric  $d$  and  $\varphi : R \times X \rightarrow X$  be a continuous flow. If  $\varphi$  has the average-shadowing property and the set of the almost periodic points of  $\varphi$  is dense in  $X$ , then  $\varphi$  is topologically ergodic.

**Proof.** By Lemma 3.2, it suffices to prove that  $\varphi$  is topologically transitive.

Suppose that  $U$  and  $V$  are two nonempty open subsets of  $X$ . By the assumption we can choose  $x \in U \cap AP(\varphi)$ ,  $y \in V \cap AP(\varphi)$  and  $\varepsilon > 0$  such that  $B(x, \varepsilon) \subset U$ ,  $B(y, \varepsilon) \subset V$ . We have  $J_x = \{t \in R : \varphi(t, x) \in B(x, \frac{\varepsilon}{2})\}$  and  $J_y = \{t \in R : \varphi(t, y) \in B(y, \frac{\varepsilon}{2})\}$  which are two relatively dense subsets of  $R$ . Therefore there are  $L_1 > 0$  and  $L_2 > 0$  such that  $J_x \cap (t, t + L_1) \neq \emptyset$ ,  $J_y \cap (t, t + L_2) \neq \emptyset$  for any  $t \in R$ . Let  $L = \max\{L_1, L_2\} > 0$ , by compactness of  $X$  and continuity of  $\varphi$  there is  $\delta > 0$  such that  $d(u, v) < \delta$  implies  $d(\varphi(t, u), \varphi(t, v)) < \frac{\varepsilon}{2}$  for every  $t \in [-L - 1, L + 1]$ .

Since  $\varphi$  has the average-shadowing property, for  $\frac{\delta}{2} > 0$ , there is  $\delta_1 : 0 < \delta_1 < \frac{\delta}{2}$  such that every  $(\delta_1, 1)$ -average-pseudo-orbit of  $\varphi$  can be  $\frac{\delta}{2}$ -shadowed in average by some orbit of  $\varphi$ .

Choose a positive integer  $N_0$  such that  $\frac{3D}{N_0} < \delta_1$ , where  $D = \text{diam}(X)$ , that is,  $D = \sup \{d(x, y) : x, y \in X\}$ . We define a bi-sequence  $(\{x_i\}_{-\infty < i < \infty}, \{t_i\}_{-\infty < i < \infty})$  (Where  $t_i = 1$  for each  $-\infty < i < \infty$ ) as follows:

$$\begin{aligned} x_i &= \varphi(i \pmod{2N_0}, x), \text{ if } i \pmod{2N_0} \in \{0, 1, 2, \dots, N_0 - 1\}; \\ x_i &= \varphi(i \pmod{2N_0} - N_0, y), \text{ if } i \pmod{2N_0} \in \{N_0, N_0 + 1, \dots, 2N_0 - 1\}. \end{aligned}$$

That is, the terms of the sequence from  $i = 0$  to  $2N_0 - 1$  are

$$x_0 = x, x_1 = \varphi(1, x), \dots, x_{N_0-1} = \varphi(N_0 - 1, x),$$

$$x_{N_0} = y, x_{N_0+1} = \varphi(1, y), \dots, x_{2N_0-1} = \varphi(N_0 - 1, y).$$

It is easy to check that for every  $n \geq N_0$  and every  $-\infty < k < \infty$ ,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(\varphi(1, x_{k+i}), x_{k+i+1}) < \frac{[\frac{n}{N_0}] \times 3D}{n} \leq \frac{3D}{N_0} < \delta_1.$$

Thus the bi-sequence  $(\{x_i\}_{-\infty < i < \infty}, \{t_i\}_{-\infty < i < \infty})$  is a  $(\delta_1, 1)$ -average-pseudo-orbit of  $\varphi$ . Hence, it can be  $\frac{\delta}{2}$ -shadowed in average by the orbit of  $\varphi$  through some point  $\omega \in X$ , that is,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_i^{i+1} d(\varphi(t, \omega), \varphi(t-i, x_i)) dt < \frac{\delta}{2}. \quad (3.1)$$

We have the following claim.

**Claim.** (1) There are infinitely many positive integers  $i$  and real numbers  $t_i \in [i, i+1]$  such that

$$x_i \in \{x, \varphi(1, x), \dots, \varphi(N_0 - 1, x)\} \text{ and } d(\varphi(t_i, \omega), \varphi(t_i - i, x_i)) < \delta.$$

(2) There are infinitely many positive integers  $j$  and real numbers  $t_j \in [j, j+1]$  such that

$$x_j \in \{y, \varphi(1, y), \dots, \varphi(N_0 - 1, y)\} \text{ and } d(\varphi(t_j, \omega), \varphi(t_j - j, x_j)) < \delta.$$

**Proof of Claim.** Without loss of generality we only prove (1).

If (1) is not true, then there is a positive integer  $N$  such that for all  $i \geq N$ , whenever  $x_i \in \{x, \varphi(1, x), \dots, \varphi(N_0 - 1, x)\}$ , we have  $d(\varphi(t, \omega), \varphi(t-i, x_i)) \geq \delta$  for any  $t \in [i, i+1]$ .

So,

$$\int_i^{i+1} d(\varphi(t, \omega), \varphi(t-i, x_i)) dt \geq \delta$$

for all  $i \geq N$ .

Thus, it is gotten that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_i^{i+1} d(\varphi(t, \omega), \varphi(t-i, x_i)) dt \geq \frac{\delta}{2}.$$

Which contradicts (3.1). The proof of the claim is completed.

According to the claim, we can pick  $j_0 > i_0 + L$  and  $t_{i_0} \in [i_0, i_0 + 1], t_{j_0} \in [j_0, j_0 + 1]$ ,  $x_{i_0} = \varphi(i_1, x)$ ,  $x_{j_0} = \varphi(j_1, y)$  and  $0 \leq i_1, j_1 < N_0$  such that

$$d(\varphi(t_{i_0}, \omega), \varphi(t_{i_0} - i_0, x_{i_0})) < \delta \text{ and } d(\varphi(t_{j_0}, \omega), \varphi(t_{j_0} - j_0, x_{j_0})) < \delta.$$

That is,

$$d(\varphi(t_{i_0}, \omega), \varphi(t_{i_0} - i_0 + i_1, x)) < \delta \quad (3.2)$$

and

$$d(\varphi(t_{j_0}, \omega), \varphi(t_{j_0} - j_0 + j_1, y)) < \delta \quad (3.3)$$

Since  $J_x \cap (i_1, i_1 + L) \neq \emptyset$ ,  $J_y \cap (j_1, j_1 + L) \neq \emptyset$ , there are  $0 \leq i_2, j_2 \leq L$  such that  $d(\varphi(i_1 + i_2, x), x) < \frac{\varepsilon}{2}$  and  $d(\varphi(j_1 + j_2, y), y) < \frac{\varepsilon}{2}$

It is easy to check that  $i_2 + i_0 - t_{i_0}, j_2 + j_0 - t_{j_0} \in [-L - 1, L + 1]$ . Hence by (3.2) and (3.3), we obtain that

$$d(\varphi(t_{i_0} + i_2 + i_0 - t_{i_0}, \omega), \varphi(i_1 + i_2, x)) < \frac{\varepsilon}{2}$$

and

$$d(\varphi(t_{j_0} + j_2 + j_0 - t_{j_0}, \omega), \varphi(j_1 + j_2, y)) < \frac{\varepsilon}{2}.$$

Write  $k = t_{i_0} + i_2 + i_0 - t_{i_0}, r = t_{j_0} + j_2 + j_0 - t_{j_0}$ . Therefore,  $d(\varphi(k, \omega), x) < \varepsilon$  and  $d(\varphi(r, \omega), y) < \varepsilon$ , which implies that  $\varphi(r - k, U) \cap V \neq \emptyset$ . Since  $U, V$  are arbitrary,  $\varphi$  is topologically transitive. The proof of Theorem 3.3 is finished.  $\square$

The proof of the following lemma is easy, we omit the proof.

**Lemma 3.4.** Let  $X$  be a compact metric space with metric  $d$  and  $\varphi : R \times X \rightarrow X$  be a continuous flow. If  $\varphi$  has the average-shadowing property, then so does  $\varphi \times \varphi$ .

Let  $X, Y$  be compact metric spaces,  $\varphi : R \times X \rightarrow X$  and  $\psi : R \times Y \rightarrow Y$  be continuous flows. If there is a continuous, surjective map  $\pi : X \rightarrow Y$  satisfying  $\pi \circ \varphi = \psi \circ \pi$ , i.e.,  $\pi(\varphi(t, x)) = \psi(t, \pi(x))$  for any  $t \in R, x \in X$ , then  $\pi$  is said to be a factor map from  $\varphi$  to  $\psi$ . It is well known that  $\pi(AP(\varphi)) = AP(\psi)$ .

**Lemma 3.5.** Let  $X$  be a compact metric space with metric  $d$  and  $\varphi : R \times X \rightarrow X$  be a continuous flow. If the almost periodic points of  $\varphi$  are dense in  $X$ , then the almost periodic points of  $\varphi \times \varphi$  are dense in  $X \times X$ .

**Proof.** Since  $x \in X$  is an almost periodic point of  $\varphi$  if and only if  $\overline{\gamma(t, x)}$  is a minimal set of  $\varphi$ , we quickly deduce to the case when  $\varphi$  is minimal. It is well known that there is an almost periodic point  $(x, y) \in X \times X$  of  $\varphi \times \varphi$ . It is easy to verify that for any  $t, s \in R$ ,  $\varphi_t \times \varphi_s$  is a factor map from  $\varphi \times \varphi$  to  $\varphi \times \varphi$ . Thus, for any  $t, s \in R$ ,  $(\varphi(t, x), \varphi(s, y)) \in AP(\varphi \times \varphi)$ . Since  $\varphi$  is minimal,  $\overline{\gamma(t, x)} = X$  and  $\overline{\gamma(t, y)} = X$ . Hence for any non-empty open subsets  $U, V \subset X$ , there are  $t, s \in R$  such that  $(\varphi(t, x), \varphi(s, y)) \in (U \times V) \cap AP(\varphi \times \varphi)$ . Therefore  $\overline{AP(\varphi \times \varphi)} = X \times X$ .  $\square$

**Lemma 3.6.** Let  $X$  be a compact metric space with metric  $d$  and  $\varphi : R \times X \rightarrow X$  be a continuous flow. If  $\varphi$  is weakly mixing and  $X$  is not a singleton, then  $\varphi$  is sensitive.

**Proof.** Since  $X$  is not a singleton,  $D = \sup\{d(x, y) : x, y \in X\} > 0$ . Let  $\varepsilon = \frac{D}{2} > 0$ . For any non-empty open subset  $U \subset X$ , we can choose two non-empty open subsets  $V_1, V_2 \subset X$  with  $d(\overline{V_1}, \overline{V_2}) > \varepsilon$ . As  $\varphi$  is weakly mixing, there is  $t \in R$  such that

$\varphi(t, U) \cap V_1 \neq \emptyset, \varphi(t, U) \cap V_2 \neq \emptyset$ , which implies that there are  $x, y \in U$  such that  $d(\varphi(t, x), \varphi(t, y)) > \varepsilon$ . Therefore  $\varphi$  is sensitive.  $\square$

**Theorem 3.7.** Let  $X$  be a compact metric space with metric  $d$  and  $\varphi : R \times X \rightarrow X$  be a continuous flow. If the set of almost periodic points of  $\varphi$  is dense in  $X$  and  $\varphi$  has the average-shadowing property, then  $\varphi \times \varphi$  is topologically ergodic. In particular, if additionally  $X$  is not a singleton, then  $\varphi$  is Auslander-Yorke chaotic.

**Proof.** We write  $\psi = \varphi \times \varphi$ . Since the set of almost periodic points of  $\varphi$  is dense in  $X$  and  $\varphi$  has the average-shadowing property, By Lemma 3.4 and Lemma 3.5,  $\psi$  has the average-shadowing property and the almost periodic points of  $\psi$  are dense. Therefore  $\psi$  is topologically transitive from Theorem 3.3. Hence  $\psi$  satisfies the conditions of Lemma 3.2 and  $\psi = \varphi \times \varphi$  is topologically ergodic. It is obvious that  $\varphi$  is weakly mixing. By Lemma 3.6,  $\varphi$  is sensitive when  $X$  is not a singleton. Therefore  $\varphi$  is Auslander-Yorke chaotic. The proof is finished.  $\square$

The following theorem is due to Gu, it is the main result in [7].

**Theorem 3.8.** Let  $X$  be a compact metric space with metric  $d$  and  $\varphi : R \times X \rightarrow X$  be a continuous flow. If a Lyapunov stable flow  $\varphi$  has the average-shadowing property, then  $\varphi$  is topologically ergodic.

By Theorem 3.7, we obtain the following result which shows that the conditions of Theorem 3.8 are never satisfied except when  $X$  is a singleton.

Note that if  $\varphi$  is Lyapunov stable, then every point in  $X$  is almost periodic [2, Lemma 3]. We have immediately the following Corollary.

**Corollary 3.9.** Let  $X$  be a compact metric space with metric  $d$  and  $\varphi : R \times X \rightarrow X$  be a continuous flow. If  $\varphi$  is Lyapunov stable and  $X$  is not a singleton, then  $\varphi$  does not have the average-shadowing property.

The ordered pair  $(x, y) \in X \times X$  is called distal if  $\inf\{d(\varphi(t, x), \varphi(t, y)) : t \in R\} > 0$ . A flow  $\varphi$  is said to be distal if for all ordered pairs  $(x, y) \in X \times X$  with  $x \neq y$  are distal. It is well known that if  $\varphi$  is Lyapunov stable, then it is distal. Moreover, if  $\varphi$  is distal, then every point  $x \in X$  is an almost periodic point of  $\varphi$  [2].

In fact, we can obtain the following stronger result.

**Corollary 3.10.** Let  $X$  be a compact metric space with metric  $d$  and  $\varphi : R \times X \rightarrow X$  be a continuous flow. If  $\varphi$  is distal and  $X$  is not a singleton, then  $\varphi$  does not have the average-shadowing property.

**Proof.** Suppose  $\varphi$  has the average-shadowing property. By Theorem 3.7,  $\varphi \times \varphi$  is topologically ergodic, whence  $\varphi \times \varphi$  is topologically transitive. Therefore, there is  $(x, y) \in X \times X$  such that  $\overline{\gamma((x, y), \varphi \times \varphi)} = X \times X$ . Since  $X$  is not a singleton,  $x \neq y$ . Obviously, the pair  $(x, y) \in X \times X$  is not distal, which contradicts the assumption. Thus  $\varphi$  does not have the average-shadowing property.  $\square$

#### 4. CONCLUSION

In this paper, we discuss that some dynamical properties of semi-flows with the average shadowing property and prove that if the set of almost periodic points of a semi-flow  $\varphi$  with the average shadowing property, then  $\varphi \times \varphi$  is topologically ergodic. A natural question is: whether is the result above can be strengthen? It is the topic of our further research.

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