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HAMILTONIAN DYNAMICAL SYSTEMS AND GEOMETRY OF SURFACES IN 3-D

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Abstract. Hamiltonian vector field, Poisson vector field and the gradient of Hamiltonian function defines Darboux frame along an integral curve of a Hamiltonian dynamical system on a surface whose normal vector field corresponds to the Poisson structure for a given Hamiltonian system. We show that the existence of compatible Poisson structures determined by the normal legs of the Darboux frame is resolved to the characteristic equation for the Weingarten map. We also show that a Hamiltonian dynamical system in three dimensions has bi-Hamiltonian representation determined by the normal legs of Frenet-Serret triad if and only if an integral curve of Hamiltonian vector field is both a geodesic and a line of curvature.

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1. Introduction

A Hamiltonian dynamical system is a system of first order ordinary differential equations which is completely determined by Poisson structure and Hamiltonian

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function [18, 24]. A Poisson structure on a manifold is defined by a bi-vector subject to the Jacobi identity and due to the skew-symmetry it is necessarily degenerate in odd dimensions. The first non-trivial example of degenerate Poisson structure occurs in three dimensions. Differential geometry of three dimensional dynamical systems, their Hamiltonian and bi-Hamiltonian formulations have an importance in literature [1, 4, 6, 9–13, 15, 17, 21]. In three dimensions the Jacobi identity for a Poisson structure represents the holonomicity of the corresponding vector field and hence it defines a codimension one foliation such that an integral curve of Hamiltonian dynamical system lies on a leaf of this foliation. From the point of integrability, existence of two functionally independent Hamiltonians implies that an integral curve is described by the intersection of level surfaces of these functions and system of equations reduces to a single differential equation which is integrable with a constant solution up to a rectification. However, in its own right, the intersection of two implicit surfaces in three dimensions leads to a geometric classification [25]. Many works in literature are devoted to study of tangential or transversal intersection of surfaces in three dimensions [2,7,14,16,22,26]. In this sense, transversal intersection of two surfaces corresponding to conserved Hamiltonians leads to a bi-Hamiltonian structure and as we show in this paper that a bi-Hamiltonian representation of a dynamical system is controlled by the geometric features of an integral curve as an embedded submanifold and determination of the compatible Poisson structures depends on the intrinsic and extrinsic curvatures of corresponding surface(s).

By virtue of this point of view we devoted this work to study local geometric notions of Hamiltonian systems to understand their Hamiltonian and bi-Hamiltonian formulations in terms of the geometric characteristics of their integral curves within the geometry of surfaces. Recently, the authors have defined Darboux frame along an integral curve of a Hamiltonian dynamical system on a surface whose normal vector field corresponds to the Poisson structure and showed that the geodesic curvature and the geodesic torsion depend on the Poisson structure [5]. In this paper we prove that the existence of the compatible Poisson structures determined by the normal legs of Darboux frame is resolved to the characteristic equation for Weingarten map. Equivalently, it is also determined by a differential equation with an algebraic constraint which tells us that an integral curve of a dynamical system is a line of curvature on the underlying surface. As a consequence, we show that a

Hamiltonian dynamical system in three dimensions have bi-Hamiltonian representation determined by the normal legs of Frenet-Serret triad if and only if an integral curve of this system is both geodesic and a line of curvature.

2. Preliminaries

In terms of a local coordinates (x^1, x^2, x^3) for $\mathbb{R}^3$, an autonomous dynamical system associated to a vector field $v = v^i \partial_{x^i}$ is given by the system of autonomous ordinary differential equations:

$$\dot{x}^i = v^i(x), \ x = (x^1, x^2, x^3). \tag{1}$$

Here overdot denotes the derivative with respect to time t. A dynamical system $\dot{x} = v$ is said to be Hamiltonian if it can be written in the form:

$$\dot{x} = v(x) = \Omega(x)(dH(x)), \tag{2}$$

where $H(x)$ is the Hamiltonian function and $\Omega(x)$ is the Poisson bi-vector. Poisson bi-vector is a skew-symmetric contravariant two-tensor satisfying the Jacobi identity

$$[\Omega(x), \Omega(x)] = 0, \tag{3}$$

which is given by the Schouten-Nijenhuis bracket [24]. System of equations (2) are called Hamilton's equations. In terms of the local coordinate basis $\partial_i = \partial_{x^i}$, the Poisson bi-vector is interpreted as

$$\Omega = \Omega^{ij} \partial_i \wedge \partial_j.$$

Bi-vector Ω determines a Poisson structure on the manifold by

$$\{f, g\} = \Omega(df, dg). \tag{4}$$

A manifold with a Poisson structure is called a Poisson manifold. Since the Poisson bi-vector is skew-symmetric, it is degenerate for odd dimensions. One can associate a vector field to a Poisson bi-vector by the isomorphism between tangent and cotangent bundles of $\mathbb{R}^3$ determined by the Euclidean metric $\delta = \delta_{ij} dx^i dx^j$

$$J^k = \delta^{kl} \epsilon_{ijl} \Omega^{ij}, \tag{5}$$

where ϵ_{ijl} is totally anti-symmetric Levi-Civita tensor. The vector field

$$J = J^k \partial_k \tag{6}$$

is called the Poisson vector field associated to the Poisson structure [1,12] and given explicitly by $J = (\Omega^{23}, \Omega^{31}, \Omega^{12})$. With this notation, Hamilton's equations (2) and the Jacobi identity (3) are given respectively by

$$v = J \times \nabla H, \tag{7}$$

and

$$J \cdot \text{curl } J = 0. \tag{8}$$

From the last equation we see that the Jacobi identity for a Poisson structure exhibits the holonomicity of the vector field J and hence determines a codimension one foliation in three dimensions [3,19]. In this case, vector field J is locally written as $J = \varphi \nabla \psi$ and an integral curve of (2) lies locally on a leaf of this foliation which corresponds to a level surface of function ψ. On a trajectory of a Hamiltonian dynamical system one can locally define Frenet-Serret frame which is determined by the well known Frenet-Serret equations:

$$\frac{d}{dt}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & \nu\kappa & 0 \\ -\nu\kappa & 0 & \nu\tau \\ 0 & -\nu\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}, \tag{9}$$

where $\nu = ds/dt$ is the derivative of the arc-length.

$$s = \int_a^t \|v\| du. \tag{10}$$

The Frenet-Serret and a coordinate basis $\partial_i = \partial_{x^i}$ are the same up to an orthogonal transformation, by means of the directional derivative operators $\delta/\delta s$, $\delta/\delta n$ and $\delta/\delta b$ in the direction of tangent, normal and binormal of a curve, directional derivatives in the direction of the normal legs $\mathbf{n}$ and $\mathbf{b}$) are given as follows [20]:

$$\frac{\delta}{\delta n}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & \theta_{ns} & \mathcal{H}_b + \tau \\ -\theta_{ns} & 0 & -\text{div } \mathbf{b} \\ -(\mathcal{H}_b + \tau) & \text{div } \mathbf{b} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} \tag{11}$$

$$\frac{\delta}{\delta b}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & -(\mathcal{H}_n + \tau) & \theta_{bs} \\ \mathcal{H}_n + \tau & 0 & \kappa + \text{div } \mathbf{n} \\ -\theta_{bs} & -(\kappa + \text{div } \mathbf{n}) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \tag{12}$$

The cartesian gradient operator is given in Frenet-Serret basis as

$$\nabla = \mathbf{t}\frac{\delta}{\delta s} + \mathbf{n}\frac{\delta}{\delta n} + \mathbf{b}\frac{\delta}{\delta b}, \tag{13}$$

and therefore the divergence and the curl are given by

$$\begin{aligned} \text{div } \mathbf{t} &= \nabla \cdot \mathbf{t} = \theta_{ns} + \theta_{bs} \\ \text{div } \mathbf{n} &= \nabla \cdot \mathbf{n} = -\kappa + \mathbf{b} \cdot \frac{\delta \mathbf{n}}{\delta b} \\ \text{div } \mathbf{b} &= \nabla \cdot \mathbf{b} = \mathbf{b} \cdot \frac{\delta \mathbf{n}}{\delta n} \end{aligned} \tag{14}$$

and

$$\begin{aligned} \text{curl } \mathbf{t} &= \mathcal{H}_s \mathbf{t} + \kappa \mathbf{b} \\ \text{curl } \mathbf{n} &= -(\text{div } \mathbf{b})\mathbf{t} + \mathcal{H}_n \mathbf{n} + \theta_{ns}\mathbf{b} \\ \text{curl } \mathbf{b} &= (\kappa + \text{div } \mathbf{n})\mathbf{t} - \theta_{bs}\mathbf{n} + \mathcal{H}_b \mathbf{b}. \end{aligned} \tag{15}$$

Here the geometric quantities θ_{ns} and θ_{bs} are defined by

$$\theta_{ns} = \mathbf{n} \cdot \frac{\delta \mathbf{t}}{\delta n},\ \theta_{bs} = \mathbf{b} \cdot \frac{\delta \mathbf{t}}{\delta b}. \tag{16}$$

The value of the non-holonomicity for each of the vector fields $\mathbf{t},\ \mathbf{n},\ \mathbf{b}$ are determined by

$$\begin{aligned} \mathcal{H}_s &= \mathbf{t} \cdot \text{curl } \mathbf{t} = \mathbf{b} \cdot \frac{\delta \mathbf{t}}{\delta n} - \mathbf{n} \cdot \frac{\delta \mathbf{t}}{\delta b} \\ \mathcal{H}_n &= \mathbf{n} \cdot \text{curl } \mathbf{n} = \mathbf{t} \cdot \frac{\delta \mathbf{n}}{\delta b} - \tau \\ \mathcal{H}_b &= \mathbf{b} \cdot \text{curl } \mathbf{b} = -\tau - \mathbf{t} \cdot \frac{\delta \mathbf{b}}{\delta n}. \end{aligned} \tag{17}$$

For a Hamiltonian function we have $\nabla H \cdot v = 0$, i.e. $\delta_s H = 0$ by (7) and (13). On that account, Hamiltonian function is of the form $H = H(n, b)$ and its gradient is given in terms of the Frenet-Serret frame by

$$\nabla H = \frac{\delta H}{\delta n}\mathbf{n} + \frac{\delta H}{\delta b}\mathbf{b}. \tag{18}$$

Since J is orthogonal to v, it can be written as

$$J = \alpha \mathbf{n} + \beta \mathbf{b}. \tag{19}$$

In [1], it is shown that the expression of the Jacobi identity $J \cdot \text{curl } J = 0$ for the Poisson structure J reduces to the Riccati-type pde in three dimensions:

$$\delta_s \mu = \mathcal{H}_n + \mu \mathcal{H}_{nb} + \mu^2 \mathcal{H}_b, \ \mu = \beta/\alpha, \tag{20}$$

The values

$$\mathcal{H}_n = \mathbf{n} \cdot \text{curl } \mathbf{n}, \ \mathcal{H}_b = \mathbf{b} \cdot \text{curl } \mathbf{b} \ \mathcal{H}_{nb} = \mathbf{n} \cdot \text{curl } \mathbf{b} + \mathbf{b} \cdot \text{curl } \mathbf{n}, \tag{21}$$

completely determine the value of non-holonomicity of the complexified vector field [3]

$$\xi = \mathbf{n} + i\mathbf{b}. \tag{22}$$

Two Poisson structures are said to be compatible if any linear combination of them is also a Poisson structure [1, 11, 12]. A vector field is said to be bi-Hamiltonian if there exist two compatible Poisson structures J_1, J_2 such that vector field is Hamiltonian with respect to both of these structures, that is, it is written of the form $v = J_1 \times \nabla H_2 = J_2 \times \nabla H_1$. If a vector field admits a bi-Hamiltonian representation then it can be written (locally) as $v = \phi \nabla H_1 \times \nabla H_2$ [1, 12]. It is also suitable to note here that Jacobi identity for a Poisson structure has an conformal invariance in three dimensions. That is, for a smooth function f the Jacobi identity for J transforms under transformation $J \mapsto fJ$ as $f^2(J \cdot \text{curl } J)$. In other words, if J is a Poisson structure then so is fJ . Therefore without loss of generality we can consider the Poisson vector field of the form

$$J = \mathbf{n} + \mu \mathbf{b}. \tag{23}$$

2.1. **Darboux Frame on an Integral Curve.** For a given Hamiltonian dynamical system in three dimensions, the authors constructed an orthonormal frame, so called Darboux frame, along an integral curve on a leaf of the foliation determined by a Poisson structure. This frame is composed of Hamiltonian vector field, Poisson vector field and the gradient of a Hamiltonian function [5]. For a Hamiltonian dynamical system $v = J \times \nabla H$ with Poisson structure $J = \mathbf{n} + \mu \mathbf{b}$, this frame is

obtained by applying Gram-Schmidt process and given by

$$\begin{aligned} e_1 &= \frac{v}{\|v\|} \\ e_2 &= \frac{1}{\sqrt{1+\mu^2}} J \\ e_3 &= e_2 \times e_1 = \frac{1}{\sqrt{1+\mu^2}} J^{\perp}, \end{aligned} \tag{24}$$

where $J^{\perp} = \mu\mathbf{n} - \mathbf{b}$. Here e_2 defines a Poisson structure for the Hamiltonian system under consideration and so it satisfies the Jacobi identity:

$$e_2 \cdot \text{curl } e_2 = 0. \tag{25}$$

Accordingly, for an integral curve of the Hamiltonian vector field, we have the following on a leaf of the foliation or on a member family of surfaces determined by the Poisson structure [5]:

Proposition 1. *Let $v = e_2 \times \nabla H$ be a Hamiltonian dynamical system with Poisson vector field $e_2 = \frac{1}{\sqrt{1+\mu^2}}(\mathbf{n} + \mu\mathbf{b})$* and let $\mathcal{S}$ be a member of the foliation determined by the Poisson structure e_2. Assume that an integral curve of v is re-parametrized by its arc-length. Then the orthonormal frame (e_1, e_2, e_3) defines Darboux frame on $\mathcal{S}$ and along an integral curve satisfies the following equations:

$$\frac{d}{ds}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa_n & \kappa_g \\ -\kappa_n & 0 & \tau_g \\ -\kappa_g & -\tau_g & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{26}$$

where

$$\kappa_n = \frac{\kappa}{\sqrt{1+\mu^2}},\ \kappa_g = \frac{\kappa\mu}{\sqrt{1+\mu^2}},\ \tau_g = -\left(\tau + \frac{\dot{\mu}}{1+\mu^2}\right),\ \dot{\mu} = \delta_s\mu. \tag{27}$$

The quantities κ_n, κ_g and τ_g describe respectively the normal curvature, geodesic curvature and geodesic torsion of an integral curve of the Hamiltonian dynamical system (2). So the followings are immediate.

Corollary 2. *An integral curve ($\kappa > 0$) of the Hamiltonian dynamical system*

$$\dot{x} = e_2 \times \nabla H, \tag{28}$$

is a geodesic curve on $\mathcal{S}$ with normal vector field $e_2 = \frac{1}{\sqrt{1+\mu^2}}(\mathbf{n} + \mu\mathbf{b})$ iff $\mu = 0$.

Corollary 3. *Let* $e_2 = \frac{1}{\sqrt{1+\mu^2}}(\mathbf{n} + \mu\mathbf{b})$ be a Poisson structure constructed from a first integral μ, that is, $\delta_s\mu = 0$. Then an integral curve of the Hamiltonian dynamical system (28) with Poisson structure e_2 *is a line of curvature iff it is planar, i.e.,* $\tau = 0$.

3. Main Results

In order to give a bi-Hamiltonian representation to (7) we shall state and prove the following theorem exhibiting holonomicity of the $J^\perp$ in terms of the Gaussian curvature, mean curvature and the normal curvature in the direction of Hamiltonian vector field of a given Hamiltonian dynamical system. As we will touch upon, this is equivalent to say that the existence of the compatible Poisson structure determined by the normal legs of Darboux frame is interpreted as the characteristic equation for the Weingarten map or the shape operator.

Theorem 4. *Let* $\dot{x} = e_2 \times \nabla H$ *be a Hamiltonian system with Poisson structure* $e_2 = \frac{1}{\sqrt{1+\mu^2}}(\mathbf{n} + \mu\mathbf{b})$ and $\mathcal{H}_s \neq 0$. Then $J^\perp = \mu\mathbf{n} - \mathbf{b}$ describes also a Poisson structure if and only if

$$K + \kappa_n \left(\text{div } e_2 + \kappa_n\right) = 0 \tag{29}$$

where K is the Gaussian curvature of the surface $\mathcal{S}$ with unit normal e_2.

Proof. In order to prove this statement we use the formula for the Gaussian curvature [23]

$$K = [e_2, \text{curl } \mathbf{t}, \text{curl } e_3] = e_2 \cdot \text{curl } \mathbf{t} \times \text{curl } e_3. \tag{30}$$

Using the property

$$\text{curl } (\phi\xi) = \nabla\phi \times \xi + \phi\text{curl } \xi \tag{31}$$

of the curl operator we compute curl e_3 as

$$\text{curl } e_3 = \nabla\left(\frac{1}{\sqrt{1+\mu^2}}\right) \times (\mu\mathbf{n} - \mathbf{b}) + \frac{1}{\sqrt{1+\mu^2}}\left(\nabla\mu \times \mathbf{n} + \mu\text{curl } \mathbf{n} - \text{curl } \mathbf{b}\right) \tag{32}$$

In (15), curl $\mathbf{t}$ is given by

$$\text{curl } \mathbf{t} = \mathcal{H}_s\mathbf{t} + \kappa\mathbf{b}. \tag{33}$$

Also with the help of the other formulas in (15) we obtain

$$\begin{aligned}\text{curl } e_3 &= \left[\frac{\mu}{(1+\mu^2)^{\frac{3}{2}}}(\delta_n\mu+\mu\delta_b\mu)-\frac{1}{\sqrt{1+\mu^2}}(\mu\text{div }\mathbf{b}+\kappa+\text{div }\mathbf{n}+\delta_b\mu)\right]\mathbf{t}\\ &+ \left[\frac{-\mu\delta_s\mu}{(1+\mu^2)^{\frac{3}{2}}}+\frac{1}{\sqrt{1+\mu^2}}(\mu\mathcal{H}_n+\theta_{bs})\right]\mathbf{n}\\ &+ \left[\frac{-\mu\delta_s\mu}{(1+\mu^2)^{\frac{3}{2}}}+\frac{1}{\sqrt{1+\mu^2}}(\delta_s\mu+\mu\theta_{ns}-\mathcal{H}_b)\right]\mathbf{b}.\end{aligned}$$

As a result we find

$$\begin{aligned}K &= \frac{\kappa\mu}{(1+\mu^2)^2}(\delta_n\mu+\mu\delta_b\mu)+\frac{1}{1+\mu^2}[\mu\mathcal{H}_s(\mu\mathcal{H}_n+\theta_{bs})-\mathcal{H}_s(\delta_s\mu+\mu\theta_{ns}-\mathcal{H}_b)\\ &- \kappa(\mu\text{div }\mathbf{b}+(\kappa+\text{div }\mathbf{n})+\delta_b\mu)].\end{aligned}$$

Since μ satisfies the Riccati equation (20), it follows that

$$\begin{aligned}K &= \frac{\kappa\mu}{(1+\mu^2)^2}(\delta_n\mu+\mu\delta_b\mu)+\frac{1}{1+\mu^2}[(1-\mu^2)\mathcal{H}_s(\mathcal{H}_b-\mathcal{H}_n)-2\mu\mathcal{H}_s\mathcal{H}_{nb}\\ &- \kappa(\delta_b\mu+\mu\text{div }\mathbf{b}+(\kappa+\text{div }\mathbf{n}))].\end{aligned}$$

On the other hand, it is found by a direct calculation that

$$\text{div } e_2 = -\frac{\mu}{(1+\mu^2)^{\frac{3}{2}}}(\delta_n\mu+\mu\delta_b\mu)+\frac{1}{\sqrt{1+\mu^2}}[\text{div }\mathbf{n}+\delta_b\mu+\mu\text{div }\mathbf{b}]. \tag{34}$$

Accordingly we have

$$\begin{aligned}K &= -\frac{\kappa}{\sqrt{1+\mu^2}}\text{div } e_2-\frac{\kappa^2}{1+\mu^2}\\ &+ \frac{1}{1+\mu^2}[(1-\mu^2)\mathcal{H}_s(\mathcal{H}_b-\mathcal{H}_n)-2\mu\mathcal{H}_s\mathcal{H}_{nb}].\end{aligned}$$

If we arrange the right hand side of the last equality we get

$$\begin{aligned}K &= -\frac{\kappa}{\sqrt{1+\mu^2}}\text{div } e_2-\frac{\kappa^2}{1+\mu^2}+\frac{1}{1+\mu^2}[-\mathcal{H}_s(\mu^2\mathcal{H}_b+\mu\mathcal{H}_{nb}+\mathcal{H}_n)\\ &+ \mathcal{H}_s(\mu^2\mathcal{H}_n-\mu\mathcal{H}_{nb}+\mathcal{H}_b)].\end{aligned} \tag{35}$$

Since e_2 is Poisson vector field, μ satisfies the equations(20), and the Jacobi identity for the vector field $J^\perp=\mu\mathbf{n}-\mathbf{b}$ is described by

$$J^\perp\cdot\text{curl } J^\perp = \mu^2\mathcal{H}_n-\mu\mathcal{H}_{nb}+\mathcal{H}_b-\delta_s\mu. \tag{36}$$

If we substitute (20) and (36) into the latter expression (35) for the Gaussian curvature we find

$$K = -\left(\frac{\kappa\sqrt{1+\mu^2}\text{div } e_2 + \kappa^2}{1+\mu^2}\right) + \frac{\mathcal{H}_s}{1+\mu^2}[J^\perp \cdot (\text{curl}J^\perp)]. \tag{37}$$

Using (27) we obtain

$$K = -\left(\kappa_n \text{div } e_2 + \kappa_n^2\right) + \frac{\mathcal{H}_s}{1+\mu^2}[J^\perp \cdot (\text{curl}J^\perp)]. \tag{38}$$

So the theorem follows from this identity. □

Since the Jacobi identity for fJ described by $f^2 J \cdot \text{curl}J$, equation (38) can be also given by

$$K = -\left(\kappa_n \text{div } e_2 + \kappa_n^2\right) + \mathcal{H}_s[e_3 \cdot (\text{curl}e_3)]. \tag{39}$$

The Gaussian curvature and the mean curvature are described respectively by the determinant and the trace of the Weingarten map or the shape operator S:

$$K = \det S,\ 2H = \text{tr } S \tag{40}$$

Together with the identity $-2H = \text{div } e_2$ for the mean curvature we see that equation (29) is described completely by the following algebraic equation:

$$K - 2H\kappa_n + \kappa_n^2 = 0. \tag{41}$$

This equation is the characteristic equation for the Weingarten map or the shape operator whose solutions correspond to principal values. κ_n in equation (41) represents the normal curvature in the direction of the velocity field of given Hamiltonian system and hence solution of (41) exists when Hamiltonian vector field defines a principal direction. Accordingly we have the following:

Corollary 5. *Let $v = e_2 \times \nabla H$ be a Hamiltonian dynamical system whose integral curve lies on the surface $\mathcal{S}$ with normal vector field corresponding to the Poisson vector field e_2. Then $J^\perp = \mu\mathbf{n} - \mathbf{b}$ defines also a Poisson structure for the vector field v* iff integral curve is a line of curvature.

This corrollary is equivalent to saying that μ satisfies the following equation:

$$(\mathcal{H}_n - \mathcal{H}_b)\,\mu^2 - 2\mathcal{H}_{nb}\mu - (\mathcal{H}_n - \mathcal{H}_b) = 0. \tag{42}$$

Equation (42) has always a solution since it has non-negative discriminant:

$$\Delta = 4\left[\left(\mathcal{H}_n - \mathcal{H}_b\right)^2 + \mathcal{H}_{nb}^2\right]. \tag{43}$$

This discriminant in fact corresponds to the geometric quantity which measures the stability of a closed solution of a dynamical system [3]. The existence of the Poisson structures determined by the normal legs of the Darboux frame is interpreted by differential-algebraic equation, that is, differential equation (20) with an algebraic constraint (42). On the other hand the compatibility condition for the Poisson structures corresponding to the vector fields e_2 and e_3 or J and $J^\perp$ is described by

$$J \cdot \operatorname{curl} J^\perp + J^\perp \cdot \operatorname{curl} J = 0. \tag{44}$$

This equation is equivalent to

$$\mathcal{H}_{nb}\mu^2 + 2\left(\mathcal{H}_n - \mathcal{H}_b\right)\mu - \mathcal{H}_{nb} = 0. \tag{45}$$

Clearly equation (42) and (45) have same discriminant. However, by a direct calculation it is seen here that there do not exist a common solution for (42) and (45) provided that $\mathcal{H}_n \neq \mathcal{H}_b$ and $\mathcal{H}_{nb} \neq 0$.

Proposition 6. *Let $v = J \times \nabla H$ be a Hamiltonian dynamical system with Poisson structure corresponding to the Poisson vector field $J = \mathbf{n} + \mu\mathbf{b}$.* Then $J^\perp = \mu\mathbf{n} - \mathbf{b}$ defines a Poisson structure compatible with J if μ satisfies the following equation:

$$\delta_s \mu = \mathcal{H}_n(1 + \mu^2) \tag{46}$$

Proof. Jacobi identities for J and $J^\perp$ together with (46) imply $\mathcal{H}_{nb}(1 + \mu^2) = 0$. Since μ is real, this gives $\mathcal{H}_{nb} = 0$. Substituting this into

$$\mathcal{H}_n + \mu\mathcal{H}_{nb} + \mu^2\mathcal{H}_b = \mathcal{H}_n + \mathcal{H}_n\mu^2 \tag{47}$$

results in $\mathcal{H}_n = \mathcal{H}_b$. Compatibility follows directly from (45).

□

If an integral curve of a Hamiltonian dynamical system is a line of curvature then equation (46) is already satisfied by the corollary (5), definition of $\mathcal{H}_n$ and the definition of geodesic torsion τ_g. As a consequence, we have proved the following theorem which is exhibiting the existence of a bi-Hamiltonian representation of a given dynamical system by the normal legs of Darboux frame.

Theorem 7. *An integral curve of a dynamical system*

$$\dot{x} = e_2 \times \nabla H, \tag{48}$$

is a line of curvature on surface S with normal vector field e_2 if and only if it admits a bi-Hamiltonian representation with compatible Poisson structures determined by the normal legs e_2 and e_3.

In this case $e_1 = e_3 \times e_2$ manifests the bi-Hamiltonian representation of a given dynamical system $v = \phi \nabla H_1 \times \nabla H_2$ with corresponding Poisson vector fields $J_i = (-1)^{i+1}\phi \nabla H_i$ and Hamiltonian functions H_2 and H_1 for $i = 1, 2$. Also trivial solutions of (42) and (45) correspond to an interesting case for which compatible Poisson structures are determined by $\mathbf{n}$ and $\mathbf{b}$. Indeed, in the case of $\mu = 0$, J reduces to $\mathbf{n}$ and $J^{\perp}$ reduces to $-\mathbf{b}$. Since J and $J^{\perp}$ are both Poisson vector fields then we have $\mathcal{H}_n = \mathcal{H}_b = 0$. Accordingly, $\mathcal{H}_{nb} = 0$ follows directly from the Hamilton's equations (7). If $\mu = 0$, then we see from the definition of geodesic curvature κ_g that an integral curve of Hamiltonian system is a geodesic curve on the surface with normal $\mathbf{n}$. Due to the theorem (7) it is also a line of curvature on this surface.

Corollary 8. *A Hamiltonian dynamical system*

$$\dot{x} = e_2 \times \nabla H, \tag{49}$$

whose integral curve is a geodesic on $\mathcal{S}$ have the bi-Hamiltonian representation determined respectively by the normal legs of the Frenet-Serret triad if and only if it is also a line of curvature.

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