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FUNCTIONAL APPROACH TO OBSERVABILITY AND CONTROLLABILITY OF LINEAR FRACTIONAL DYNAMICAL SYSTEMS

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Abstract. In this paper, a set of equivalent conditions for observability and controllability of linear fractional dynamical systems represented by the fractional differential equation in the sense of Caputo fractional derivative of order $\alpha \in (0, 1]$ are established by using the tools of linear bounded operators. Examples are included to illustrate the theoretical results.

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1. Introduction

An increasing interest in issues related to fractional dynamical systems oriented towards the field of control theory can be observed in the literature. The study of the observability and controllability of the fractional dynamical systems are two important issues for many applied problems. It is well known that the problem of controllability of dynamical systems are widely used in analysis and the design of control system. Controllability is one of the structural properties of dynamical systems and it means that it is possible to steer a dynamical system from an arbitrary initial state to arbitrary final state using the set of admissible controls. Observability is a measure for how well internal states of a system can be inferred by knowledge of its external outputs. The observability and controllability of a system are mathematical duals.

Study on observability and controllability of fractional dynamical systems provide important issues for many applied problems because the use of fractional order derivatives and integrals in control theory leads to better results than those of integer order ones. To the best of our knowledge, in recent years, a few contributions have been devoted to the observability and controllability results for fractional dynamical systems. Matignon and D'Andréa-Novel [20] discussed the observability and controllability of linear fractional dynamical systems using Grammian matrix and rank condition. Vinagre et al. [27] discussed the fundamental ideas of controllability for fractional order systems. Battayab and Djennoune [8] studied the observability and controllability of fractional dynamical systems using rank condition whereas Chen et al. [9] focused on the robust controllability for uncertain fractional-order linear time-invariant systems in state-space form. Guermah et al. [13] presented the results on both observability and controllability of discrete-time fractional order systems. Mozyrska and Torres [22, 23] obtained controllability results and modified energy control for fractional linear control systems in the sense of Riemann-Liouville and Caputo fractional derivatives. More recently, Balachandran et al. [3, 4] investigated the observability and controllability of linear and nonlinear fractional dynamical systems by using Grammian matrix and Mittag-Leffler matrix function.

Using Grammian matrix technique, we can study the controllability of linear time variant or time invariant fractional dynamical systems. But rank condition technique is used to test only whether the given linear time invariant system is controllable or not. Even if the system is controllable, we cannot find the control input $u(t)$. In order to find the control input $u(t)$, we must use the Grammian matrix technique. Without giving any information about how to obtain the observability Grammian for linear fractional-order systems and controllability Grammian, control function for linear fractional-order control systems used by Matignan in [20] is still unsolved among all fractional-order control researchers, see for example, Balachandran and Kokila [1], Balachandran et al. [2], Guo [14], Zhang [28], Zhou et al. [29] and Zhou et al. [30], the authors defined the controllability Grammian which does not satisfy the traditional definition of Grammian appearing in Bensoussan et al. [6]. Motivated by this fact, using the concept of linear bounded operators we derive a set of equivalent conditions for the observability and controllability of the linear fractional dynamical system in finite dimensional spaces. The observability and controllability Grammian matrices are obtained by using the tools of linear bounded operators. Our results are exactly fulfilled by the above problem. Examples are constructed to verify the results.

2. Preliminaries

In this section we include some definitions and properties which are needed to establish our results.

Definition 2.1. *Let $[a, b]$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I^\alpha_{a+}f$ and $I^\alpha_{b-}f$ of order $\alpha > 0$, $n-1 < \alpha \leq n$ and $n \in \mathbb{N}$ is defined as*

$$\begin{aligned} I^\alpha_{a+}f(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\mathrm{d}s, \quad x > a \\ I^\alpha_{b-}f(t) &= \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)\mathrm{d}s, \quad x < b, \end{aligned}$$

where $\Gamma(\cdot)$ is the Euler gamma function and $f(t)$ a suitable function.

Definition 2.2. *The Riemann-Liouville fractional derivatives ${}^{RL}D^\alpha_{a+}f$ and ${}^{RL}D^\alpha_{b-}f$ of order $\alpha > 0$, $n-1 < \alpha \leq n$ and $n \in \mathbb{N}$ is defined as*

$$\begin{aligned} {}^{RL}D^\alpha_{a+}f(t) &= \frac{d^n}{dt^n}(I^{n-\alpha}_{a+}f)(x), \quad x > a \\ {}^{RL}D^\alpha_{b-}f(t) &= (-1)^n\frac{d^n}{dt^n}(I^{n-\alpha}_{b-}f)(x), \quad x < b, \end{aligned}$$

where the function $f(t)$ is a suitable function.

The interpretation of the initial conditions in terms of the fractional derivatives many times is not obvious. In modeling of real process the associated conditions to them are usually expressed boundary or initial conditions involving values of the solution of the model and their derivatives of integer order in certain points given by the corresponding problem. In order to meet this physical requirement, an alternative definition of fractional derivative was introduced by Caputo [10] in 1967 defined the fractional derivative in the following way, (see, for example, [12, 17]).

Definition 2.3. *The Caputo fractional derivatives* ${}^CD^{\alpha}_{a+}f$ *and* ${}^CD^{\alpha}_{b-}f$ *of order* $\alpha > 0$*,* $n-1 < \alpha \leq n$*, is defined as*

$$\begin{aligned} {}^CD^{\alpha}_{a+}f(t) &= \left(I^{n-\alpha}_{a+}D^n f\right)(x) \\ {}^CD^{\alpha}_{b-}f(t) &= (-1)^n \left(I^{n-\alpha}_{b-}D^n f\right)(x), \end{aligned}$$

where the function $f(t)$ *has absolutely continuous derivatives up to order* $n-1$*.*

We shall state some properties of the Caputo fractional differential operators.

Properties 2.1. *For* $\alpha,\ \beta > 0$ *and* f *as a suitable function* (see, for instance, [16, 24])*, we have*

(1) ${}^CD^{\alpha}_{a+}f(t) = I^{1-\alpha}_{a+}Df(t), \quad 0 < \alpha < 1.$
(2) ${}^CD^{\alpha}_{a+}\,{}^CD^{\beta}_{a+}f(t) \neq {}^CD^{\beta}_{a+}\,{}^CD^{\alpha}_{a+}f(t) \neq {}^CD^{\alpha+\beta}_{a+}f(t).$
(3) *Caputo derivative of a constant is zero.*
(4) $\left({}^CD^{\alpha}_{a+}I^{\alpha}_{a+}f\right)(x) = f(x).$
(5) *Let* $n-1 < \alpha \leq n$*. If* $f(x) \in C([a,b],\mathbb{R}^n)$*, then*

$$\begin{aligned} (I^{\alpha}_{a+}\,{}^CD^{\alpha}_{a+}f)(x) &= f(x) - \sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(x-a)^k \\ (I^{\alpha}_{b-}\,{}^CD^{\alpha}_{b-}f)(x) &= f(x) - \sum_{k=0}^{n-1}\frac{(-1)^k f^k(b)}{k!}(b-x)^k. \end{aligned}$$

In particular, if $0 < \alpha < 1$ *and* $f(x) \in C[a,b]$*, then*

$$(I^{\alpha}_{a+}\,{}^CD^{\alpha}_{a+}f)(x) = f(x) - f(a); \quad (I^{\alpha}_{b-}\,{}^CD^{\alpha}_{b-}f)(x) = f(x) - f(b).$$

We observe from the above that the Caputo fractional differential operators, in general, do not possess neither semi group nor commutative properties which are inherent to the derivatives on integer order. Just this fact was the motivation to Miller and Ross and other author to work with the concept of sequential fractional differential equations (see, for example, Chapter 7 of the book by Kilbas et al. [16]).

Definition 2.4. *Let $n \in \mathbb{N}$. Then we can consider the following sequential fractional derivative for suitable functions $y(x)$*

$$y^{(k\alpha}(x) := \left(\mathrm{D}^{k\alpha}y\right)(x) = \left(\mathrm{D}^{\alpha}\mathrm{D}^{(k-1)\alpha}y\right)(x), \tag{1}$$

where $k = 1, \cdots, n$, $\left(\mathrm{D}^0 y\right)(x) = y(x)$, and D^{α} is any fractional differential operator, for example it could be ${}^{C}D^{\alpha}_{a+}$.

The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations. That is, contains the limit values of integer-order derivative of unknown functions at the terminal $t = 0$ and also the Laplace transform of the Caputo derivative allows utilization of initial values of classical integer-order derivatives with known physical interpretation.

However, we must point out here that for modeling some real processes it is necessary to use boundary or initial conditions involving fractional integral or derivatives. See for example the paper by Bisquert[7]. In such kind of case we can consider the interpretation of such kind of conditions given by Kilbas et al. in Lemma 9 of [15], which for suitable functions establish that

$$\left({}^{RL}I^{\alpha}_{a+}f\right)(0) = \lim_{t\to a} t^{1-\alpha}f(t), \tag{2}$$

where $0 < \alpha < 1$.

Definition 2.5. *The Mittag-Leffler function $E_{\alpha,\beta}(z)$ is a complex function which depends on two complex parameter. It is defined by* (see, for example, [16, 19])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\ \alpha, \beta \geq 0 \tag{3}$$

converges for all values of the argument z. When $\beta = 1, E_{\alpha,1}(z) = E_{\alpha}(z)$. Thus the Mittag-Leffler function is an entire function. For a $n \times n$ matrix A, the matrix extension of the above Mittag-Leffler function is

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}.$$

In general $E_{\alpha,1}(A) = E_{\alpha}(A)$. If $\alpha > 0$ and A is $n \times n$ matrix then

$${}^{C}D^{\alpha}_{a+}E_{\alpha}(A(t-a)^{\alpha}) = AE_{\alpha}(A(t-a)^{\alpha}); \quad {}^{C}D^{\alpha}_{b-}E_{\alpha}(A(b-t)^{\alpha}) = AE_{\alpha}(A(b-t)^{\alpha}).$$

Consider the following fractional differential equation in the sense of Caputo derivative

$$
\begin{aligned}
{}^{C}D^{\alpha}_{t_0+}x(t) &= Ax(t) + f(t), \quad t \in [t_0, t_1], \\
x(t_0) &= x_0,
\end{aligned}
\tag{4}
$$

where $0 < \alpha < 1$, $x \in \mathbb{R}^n$, A is a $n \times n$ matrix and $f : [t_0, t_1] \to \mathbb{R}^n$ is a continuous function. Using the method of successive approximation technique, one can easily get the solution of the system (4) (see [16])

$$
x(t) = E_{\alpha}(A(t-t_0)^{\alpha})x_0 + \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) f(s) \mathrm{d}s. \tag{5}
$$

Note that, if the initial condition x_0 is zero, then the solution of the fractional differential equation with Caputo sense is same as the Riemann-Liouville sense. Throughout the paper we use the fractional derivative in the Caputo sense of order $0 < \alpha < 1$.

3. Observability Results

Consider the fractional order linear time invariant system

$$
{}^{C}D^{\alpha}_{t_0+}x(t) = Ax(t), \; t \in [t_0, t_1], \tag{6}
$$

where $x \in \mathbb{R}^n$ and A is a $n \times n$ matrix. Along with (6) we have a linear observation

$$
y(t) = Cx(t), \tag{7}
$$

where $y \in \mathbb{R}^m$ and C is an $m \times n$ matrix.

Definition 3.1. *The system* (6) *with a linear observation* (7) *is said to be observable over a time period* $[t_0, t_1]$ *if it is possible to determine uniquely the initial state* $x(t_0) = x_0$ *from the knowledge of the output* $y(t)$ *over* $[t_0, t_1]$. *i.e, The complete state of the system is known if initial state* x_0 *is known.*

The solution of (6) is $x(t) = E_{\alpha}(A(t-t_0)^{\alpha})x_0$. Thus

$$
y(t) = CE_{\alpha}(A(t-t_0)^{\alpha})x_0, \quad t \in [t_0, t_1].
$$

We have the following characterization regarding observability of (6) with a linear observation (7) in terms of linear bounded operators.

Define a linear operator $\mathscr{P}_\alpha : \mathbb{R}^n \to L^2([t_0, t_1]; \mathbb{R}^m)$ by

$$(\mathscr{P}_\alpha x_0)(t) = CE_\alpha(A(t-t_0)^\alpha)x_0. \tag{8}$$

Thus, $y(t) = (\mathscr{P}_\alpha x_0)(t)$, $t \in [t_0, t_1]$. Let us compute the adjoint of $\mathscr{P}_\alpha$: for every $x_0 \in \mathbb{R}^n, w \in L^2([t_0, t_1]; \mathbb{R}^m)$, we have

$$\begin{aligned}\langle(\mathscr{P}_\alpha x_0)(t), w(t)\rangle_{L^2([t_0,t_1],\mathbb{R}^n)} &= \int_{t_0}^{t_1} \langle CE_\alpha(A(t-t_0)^\alpha)x_0, w(t)\rangle_{\mathbb{R}^n)}\mathrm{dt} \\ &= \langle x_0, \int_{t_0}^{t_1} E_\alpha(A^*(t-t_0)^\alpha)C^*w(t)\mathrm{dt}\rangle_{\mathbb{R}^n} \\ &= \langle(x_0, \mathscr{P}_\alpha^* w(t)\rangle_{\mathbb{R}^n}.\end{aligned}$$

Thus $(\mathscr{P}_\alpha^* w)(t) = \displaystyle\int_{t_0}^{t_1} E_\alpha(A^*(t-t_0)^\alpha)C^*w(t)\mathrm{dt}$. The following Theorem gives the observability of the system (6) with a linear observation (7) in terms of the non-singularity of the matrix $M_\alpha(t_0, t_1)$.

Theorem 3.1. *The following statements are equivalent:*

(i) The system (6) *with linear observation* (7) *is observable.*

(ii) The operator $\mathscr{P}_\alpha : \mathbb{R}^n \to L^2([t_0, t_1]; \mathbb{R}^m)$ is one-one.

(iii) The operator $\mathscr{P}_\alpha^* : L^2([t_0, t_1]; \mathbb{R}^m) \to \mathbb{R}^n$ is onto.

(iv) The operator $\mathscr{P}_\alpha^*\mathscr{P}_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is onto. (Observe that, $\mathscr{P}_\alpha^*\mathscr{P}_\alpha$ is a bounded linear operator and hence can be realized by an $n \times n$ matrix.)

Proof. For the homogeneous system given by $y(t)$ is known. Solving $y(t) = CE_\alpha(A(t-t_0)^\alpha)x_0$ for x_0 is equivalent to solving the operator equation

$$\mathscr{P}_\alpha x_0 = y, \tag{9}$$

where $\mathscr{P}_\alpha$ is defined by (8). Premultiplying (9) by $\mathscr{P}_\alpha^*$, we get

$$\mathscr{P}_\alpha^*\mathscr{P}_\alpha x_0 = \mathscr{P}_\alpha^* y. \tag{10}$$

Observe that $M_\alpha(t_0, t_1) = \mathscr{P}_\alpha^*\mathscr{P}_\alpha$. If right hand side of (10) is known, x_0 is determined up to an additive constant lying in the null space of $\mathscr{P}_\alpha^*\mathscr{P}_\alpha$. Here

$$M_\alpha(t_0, t_1) = \int_{t_0}^{t_1} E_\alpha(A^*(t-t_0)^\alpha)C^*CE_\alpha(A(t-t_0)^\alpha)\mathrm{dt} \tag{11}$$

is observability Grammian. If $\mathscr{P}_\alpha^*\mathscr{P}_\alpha$ is invertible, then it is clear that $\mathscr{P}_\alpha^*$ is one-one and hence the system (6) with a linear observation (7) is observable.

Conversely if $\mathscr{P}_\alpha$ is one-one then $\mathscr{P}_\alpha^*$ is onto, so is $\mathscr{P}_\alpha^*\mathscr{P}_\alpha$ both one-one and onto and hence $R(\mathscr{P}_\alpha^*\mathscr{P}_\alpha) = [N(\mathscr{P}_\alpha^*\mathscr{P}_\alpha)]^\perp = \mathbb{R}^n$. This implies that $\mathscr{P}_\alpha^*\mathscr{P}_\alpha$ is nonsingular. For such an observable system, the initial state $x(t_0)$ is given by

$$x(t_0) = \left(\mathscr{P}_\alpha^*\mathscr{P}_\alpha\right)^{-1} \mathscr{P}_\alpha^* y = [M_\alpha(t_0,t_1)]^{-1}\left[\int_{t_0}^{t_1} E_\alpha(A^*(t-t_0)^\alpha)C^*y(t)\mathrm{d}t\right].$$

This proves the Theorem. □

Theorem 3.2. *The system* (6) *with a linear observation* (7) *is observable if and only if*

$$y(t) = Cx(t) = 0, \quad \forall t \in [t_0, t_1],$$

implies

$$x(t) = 0, \quad \forall t \in [t_0, t_1].$$

Proof. The proof is obvious from Definition 3.1. □

Theorem 3.3. *The system* (6) *with linear observation* (7) *is observable if and only if the columns of* $CE_\alpha(A(t-t_0)^\alpha)$ *are linearly independent.*

Proof. Recall that the solution of (6) with initial condition $x(t_0) = x_0$ is given by $x(t) = E_\alpha(A(t-t_0)^\alpha)x_0$. Hence, the observability condition given in Theorem 3.2 is satisfied if and only if columns of $CE_\alpha(A(t-t_0)^\alpha)$ are linearly independent. □

Example 3.1. *Consider the composite fractional relaxation equation represented by the following fractional differential equation* [18]*:*

$$\dot{x}(t) + 2\,{}^C D_{0+}^{1/2} x(t) + x(t) = 0, \;\; t \in [0,5] \tag{12}$$

with linear observation $y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$.

Let us take the auxiliary variables as $x_1 = x$ and $x_2 = {}^C D^{1/2} x$. Then

$$\begin{aligned} {}^C D_{0+}^{1/2} x_1 &= x_2, \\ {}^C D_{0+}^{1/2} x_2 &= -2x_2 - x_1. \end{aligned}$$

The system of equations can be written in the matrix form as

$$ {}^C D_{0+}^{1/2} X(t) = AX(t),\; t \in [0,5], \tag{13}$$

$$ Y(t) = CX(t) \tag{14}$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. The Mittag-Leffler matrix function for given matrix A is $E_{1/2}(At^{1/2}) = \begin{bmatrix} N_1(t) & N_2(t) \\ N_3(t) & N_4(t) \end{bmatrix}$ with

$$\begin{aligned}
N_1(t) &= -2\sqrt{\frac{t}{\pi}} + (1+2t)\, E_{1/2}\left(-t^{1/2}\right) + 2\left[2\sqrt{\frac{t}{\pi}} - 2tE_{1/2}\left(-t^{1/2}\right)\right] \\
N_2(t) &= 2\sqrt{\frac{t}{\pi}} - 2tE_{1/2}\left(-t^{1/2}\right) \\
N_3(t) &= -2\sqrt{\frac{t}{\pi}} + 2tE_{1/2}\left(-t^{1/2}\right) \\
N_4(t) &= -2\sqrt{\frac{t}{\pi}} + (1+2t)\, E_{1/2}\left(-t^{1/2}\right).
\end{aligned}$$

The operator $(\mathscr{P}_{1/2}v)(t) = CE_{1/2}(At^{1/2})v = \begin{bmatrix} N_1(t) & N_2(t) \end{bmatrix} v, \quad v \in \mathbb{R}^n$. Then

$$\begin{aligned}
\mathscr{P}^*_{1/2}\mathscr{P}_{1/2} &= \int_0^5 E_{1/2}(A^* t^{1/2}) C^* C E_{1/2}(At^{1/2}) \mathrm{d}t \\
&= \int_0^5 \begin{bmatrix} N_1^2(t) & N_1(t)N_2(t) \\ N_1(t)N_3(t) & N_2(t)N_3(t) \end{bmatrix} \mathrm{d}t \\
&= \begin{bmatrix} 1.8234 & 0.7084 \\ 0.7084 & 0.2858 \end{bmatrix} > 0.
\end{aligned}$$

Hence, the system (12) with a linear observation $y(t)$ is observable.

4. Controllability Results

Consider the linear fractional dynamical system represented by the fractional differential equation of the form

$$\begin{aligned}
&{}^C D^{\alpha}_{t_0+} x(t) = Ax(t) + Bu(t),\ t \in [t_0, t_1], \\
&x(t_0) = x_0,
\end{aligned} \tag{15}$$

where, the state vector x, the control vector u and the matrices A and B are defined as above. Using the method of successive approximation, the solution of (15) can be written as

$$x(t) = E_\alpha(A(t-t_0)^\alpha)x_0 + \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)Bu(s)\mathrm{ds}, \tag{16}$$

The function $E_{\alpha,\alpha}(At^\alpha)$ is continuous and it satisfies $\|E_{\alpha,\alpha}(At^\alpha)\| \leq M$ for all $t \in [t_0, t_1]$.

Definition 4.1. *The system* (15) *is said to be controllable over* $[t_0, t_1]$ *if for each pair of vectors* $x_0, x_1 \in \mathbb{R}^n$*, there exists a control* $u \in L^2([t_0, t_1]; \mathbb{R}^m)$ *such that the solution of* (15) *with* $x(t_0) = x_0$ *also satisfies* $x(t_1) = x_1$*.*

The control $u(t)$ is said to steer the trajectory from the initial state x_0 to the final state x_1. So the controllability of (15) is equivalent to finding the function $u(t)$ such that

$$x_1 = x(t_1) = E_\alpha(A(t_1-t_0)^\alpha)x_0 + \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(A(t_1-s)^\alpha)Bu(s)\mathrm{ds}.$$

Equivalently, the system (15) is controllable if and only if there exists a control u such that

$$x_1 - E_\alpha(A(t_1-t_0)^\alpha)x_0 = \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(A(t_1-s)^\alpha)Bu(s)\mathrm{ds}. \tag{17}$$

We have the following characterization regarding controllability of (15) in terms of linear bounded operators.

Let $L^2_\alpha([t_0,t_1];\mathbb{R}^m) = \Big\{(t_1-t)^{\alpha-1}u(t) : u(t) \in L^2([t_0,t_1];\mathbb{R}^m)$ and $\displaystyle\int_{t_0}^{t_1}(t_1-t)^{\alpha-1}u(t)\mathrm{dt} < \infty\Big\}$ such that $\displaystyle\int_{t_0}^{t_1}(t_1-t)^{2(\alpha-1)}|u(t)|^2\mathrm{dt} < \infty$. This is a Hilbert space with respect to the inner product $\langle f_\alpha, g_\alpha\rangle = \displaystyle\int_{t_0}^{t_1} f_\alpha(t)g_\alpha(t)\mathrm{dt}$. Let us define a linear operator $\mathscr{Q}_\alpha : L^2_\alpha([t_0,t_1];\mathbb{R}^m) \to \mathbb{R}^n$ by

$$\mathscr{Q}_\alpha v_\alpha = \int_{t_0}^{t_1} E_{\alpha,\alpha}(A(t_1-s)^\alpha)Bv_\alpha(s)\mathrm{ds}. \tag{18}$$

Obviously $\mathscr{Q}_\alpha$ is a bounded linear operator, since $E_{\alpha,\alpha}(A(t_1-t)^\alpha)$ is bounded for every $t \in [t_0,t_1]$ and range space of $\mathscr{Q}_\alpha$ is a subspace of $\mathbb{R}^n$. Using the above definition of $\mathscr{Q}_\alpha$, equation (17) can be written as $\mathscr{Q}_\alpha v_\alpha = w_\alpha$, where $w_\alpha = x_1 - E_{\alpha,\alpha}(A(t_1-t_0)^\alpha)x_0$. Thus the controllability problem reduces to the problem of proving that the operator $\mathscr{Q}_\alpha$ is subjective, for every $v_\alpha \in L^2_\alpha([t_0,t_1];\mathbb{R}^m)$.

Theorem 4.1. *The following statements are equivalent:*

(i) *The system* (15) *is controllable.*

(ii) *The operator* $\mathscr{Q}_\alpha : L^2_\alpha([t_0, t_1]; \mathbb{R}^m) \to \mathbb{R}^n$ *is onto.*

(iii) *The operator* $\mathscr{Q}^*_\alpha : \mathbb{R}^n \to L^2_\alpha([t_0, t_1]; \mathbb{R}^m)$ *is one-one.*

(iv) *The operator* $\mathscr{Q}_\alpha\mathscr{Q}^*_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ *is onto. (Observe that,* $\mathscr{Q}_\alpha\mathscr{Q}^*_\alpha$ *is a bounded linear operator and hence can be realized by an* $n \times n$ *matrix.)*

Proof. Let (x_0, x_1) be any pair of vectors in $\mathbb{R}^n \times \mathbb{R}^n$. The controllability problem of (15) reduces to the determination of $u \in L^2([t_0, t_1], \mathbb{R}^m)$ such that

$$\int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}(A(t_1 - t)^\alpha) Bu(s)\mathrm{ds} = \mathrm{w}_\alpha = \mathrm{x}_1 - \mathrm{E}_\alpha(\mathrm{A}(\mathrm{t}_1 - \mathrm{t}_0)^\alpha)\mathrm{x}_0.$$

This is equivalent to solving $\mathscr{Q}_\alpha v_\alpha = w_\alpha$ for v_α in the space $L^2_\alpha([t_0, t_1]; \mathbb{R}^m)$ for a given w_α, where $\mathscr{Q}_\alpha$ is given by (18).

We first note that $\mathscr{Q}_\alpha$ is a bounded linear operator. Let $\mathscr{Q}^*_\alpha : \mathbb{R}^n \to L^2_\alpha([t_0, t_1]; \mathbb{R}^m)$ be its adjoint. This is determined as follows: For $v_\alpha \in L^2_\alpha([t_0, t_1]; \mathbb{R}^m)$ and $y \in \mathbb{R}^n$, we have

$$\begin{aligned}
\langle \mathscr{Q}_\alpha v_\alpha, y\rangle_{\mathbb{R}^n} &= \langle \int_{t_0}^{t_1} E_{\alpha,\alpha}(A(t_1 - s)^\alpha) B v_\alpha(s)\mathrm{ds}, \mathrm{y}\rangle \\
&= \int_{t_0}^{t_1} \langle v_\alpha(s), B^* E_{\alpha,\alpha}(A^*(t_1 - s)^\alpha) y\rangle \mathrm{ds} \\
&= \langle v_\alpha, \mathscr{Q}^*_\alpha y\rangle_{L^2_\alpha}.
\end{aligned}$$

This gives

$$(\mathscr{Q}^*_\alpha y)(t) = B^* E_{\alpha,\alpha}(A^*(t_1 - t)^\alpha) y.$$

To claim that $\mathscr{Q}_\alpha$ is onto, it is sufficient to prove that $\mathscr{Q}_\alpha\mathscr{Q}^*_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is onto (equivalently invertibility). That is, given any $w_\alpha \in \mathbb{R}^n$, find $y \in \mathbb{R}^n$ such that

$$\mathscr{Q}_\alpha \mathscr{Q}^*_\alpha y = w. \tag{19}$$

If (19) is solvable, then a control $u(t)$ which satisfies $\mathscr{Q}_\alpha v_\alpha = w_\alpha$ is given by $v_\alpha = \mathscr{Q}^*_\alpha y = (t_1 - t)^{\alpha-1} u(t)$. This implies that $u(t) = (t_1 - t)^{1-\alpha} \mathscr{Q}^*_\alpha y$. Computing

$\mathscr{Q}_\alpha\mathscr{Q}_\alpha^*$, we get

$$\begin{aligned}
\mathscr{Q}_\alpha\mathscr{Q}_\alpha^* y &= \mathscr{Q}_\alpha(B^* E_{\alpha,\alpha}(A^*(t_1-t)^\alpha)y) \\
&= \int_{t_0}^{t_1} [E_{\alpha,\alpha}(A(t_1-s)^\alpha)BB^*E_{\alpha,\alpha}(A^*(t_1-s)^\alpha)\mathrm{ds}]\, y \\
&= W_\alpha(t_0,t_1)y.
\end{aligned}$$

The operator $\mathscr{Q}_\alpha\mathscr{Q}_\alpha^*$ is known as controllability Grammian and is denoted by $W_\alpha(t_0,t_1)$. Thus we have

$$W_\alpha(t_0,t_1) = \int_{t_0}^{t_1} E_{\alpha,\alpha}(A(t_1-s)^\alpha)BB^*E_{\alpha,\alpha}(A^*(t_1-s)^\alpha)\mathrm{ds}.$$

If the controllability Grammian $W_\alpha(t_0,t_1)$ is non-singular, we have a control $u(t)$ given by

$$\begin{aligned}
u(t) &= (t_1-t)^{1-\alpha}\mathscr{Q}_\alpha^* y \\
&= (t_1-t)^{1-\alpha}\mathscr{Q}_\alpha^*(\mathscr{Q}_\alpha\mathscr{Q}_\alpha^*)^{-1} w_\alpha \\
&= (t_1-t)^{1-\alpha}B^*E_{\alpha,\alpha}(A^*(t_1-t)^\alpha)W_\alpha^{-1}(t_0,t_1)\,[x_1 - E_\alpha(A(t_1-t_0)^\alpha)x_0]
\end{aligned}$$

which will steer the state from the initial state x_0 to the desired final state

$$\begin{aligned}
x(t_1) &= E_\alpha(A(t_1-t_0)^\alpha)x_0 + \int_{t_0}^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(A(t_1-s)^\alpha)Bu(s)\mathrm{ds} \\
&= E_\alpha(A(t_1-t_0)^\alpha)x_0 + \int_{t_0}^{t_1} E_{\alpha,\alpha}(A(t_1-s)^\alpha)BB^*E_{\alpha,\alpha}(A^*(t_1-t)^\alpha)\mathrm{ds} \\
&\quad \times W^{-1}(t_0,t_1)\,[x_1 - E_\alpha(A(t_1-t_0)^\alpha)x_0] \\
&= E_\alpha(A(t_1-t_0)^\alpha)x_0 + W_\alpha(t_0,t_1)W_\alpha^{-1}(t_0,t_1)\,[x_1 - E_\alpha(A(t_1-t_0)^\alpha)x_0] \\
&= x_1.
\end{aligned}$$

Conversely, let the system (15) be controllable. That is, $\mathscr{Q}_\alpha$ is onto. Equivalently $\mathscr{R}(\mathscr{Q}_\alpha) = \mathbb{R}^n$. Then, by Theorem 11.2 in [25], $\mathscr{N}(\mathscr{Q}_\alpha^*) = \mathscr{R}(\mathscr{Q}_\alpha)^\perp = [\mathbb{R}^n]^\perp = \{0\}$. Hence $\mathscr{Q}_\alpha^*$ is also one-one. This implies $\mathscr{Q}_\alpha\mathscr{Q}_\alpha^*$ is also one-one, which is proved as under. Assume that $\mathscr{Q}_\alpha\mathscr{Q}_\alpha^* y = 0$. This gives

$$0 = \langle \mathscr{Q}_\alpha\mathscr{Q}_\alpha^* y, y\rangle = \langle \mathscr{Q}_\alpha^* y, \mathscr{Q}_\alpha^* y\rangle = \|\mathscr{Q}_\alpha^* y\|^2$$

and hence $\mathscr{Q}_\alpha^* y = 0$, $\mathscr{Q}_\alpha^*$ is one-one implies that $y = 0$. Thus $\mathscr{N}(\mathscr{Q}_\alpha\mathscr{Q}_\alpha^*) = \{0\}$. Thus $W_\alpha(t_0,t_1) = \mathscr{Q}_\alpha\mathscr{Q}_\alpha^* : \mathbb{R}^n \to \mathbb{R}^n$ is one-one and onto implying that $\mathscr{Q}_\alpha\mathscr{Q}_\alpha^*$ is non-singular. This completes the proof. $\square$

Example 4.1. *Consider the composite linear fractional dynamical system represented by the following fractional differential equation* [18]*:*

$$\begin{aligned} \dot{y}(t) + 3{}^{C}D_{0+}^{1/2}y(t) + 2y(t) &= f(t),\ t \in [0,2], \\ y(0) &= 1, \end{aligned} \tag{20}$$

where the forcing function $f(t)$ is taken in terms of control function $u(t)$.

Using the auxiliary variables as $x_1 = y$ and $x_2 = {}^{C}D^{1/2}y$, the system (20) can be written in the matrix form as

$$ {}^{C}D_{0+}^{1/2}x(t) = Ax(t) + Bu(t),\ t \in [0,2], \tag{21}$$

$$ x(0) = x_0 \tag{22}$$

where $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x(0) = x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

The Mittag-Leffler matrix function for given matrix A is $E_{1/2}(At^{1/2}) = \begin{bmatrix} S_1(t) & S_2(t) \\ S_3(t) & S_4(t) \end{bmatrix}$ with

$$\begin{aligned} S_1(t) &= 2E_{1/2}(-t^{1/2}) - 1E_{1/2}(-2t^{1/2}); \quad S_2(t) = E_{1/2}(-t^{1/2}) - E_{1/2}(-2t^{1/2}) \\ S_3(t) &= -2\left[E_{1/2}(-t^{1/2}) - E_{1/2}(-2t^{1/2})\right]; \quad S_4(t) = -E_{1/2}(-t^{1/2}) + 2E_{1/2}(-2t^{1/2}). \end{aligned}$$

The operator $(\mathscr{Q}_{1/2}^{*}v)(t) = B^{*}E_{1/2}(A^{*}(2-t)^{1/2})v = \begin{bmatrix} S_3(2-t) & S_4(2-t) \end{bmatrix} v, \quad v \in \mathbb{R}^n$. Then

$$\begin{aligned} \mathscr{Q}_{1/2}\mathscr{Q}_{1/2}^{*} &= \int_0^2 E_{1/2}(A(2-t)^{1/2})BB^{*}E_{1/2}(A^{*}(2-t)^{1/2})\mathrm{d}t \\ &= \int_0^2 \begin{bmatrix} S_2(2-t)S_3(2-t) & S_2(2-t)S_4(2-t) \\ S_3(2-t)S_4(2-t) & S_4^2(2-t) \end{bmatrix} \mathrm{d}t \\ &= \begin{bmatrix} 0.0573 & 0.0430 \\ 0.0430 & 0.0623 \end{bmatrix} > 0. \end{aligned}$$

Hence, the system (20) is controllable on $[0,2]$. This means that, for every $x(2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$, the control function

$$\begin{aligned} u(t) &= (2-t)^{1/2}\left[(-1-S_1(2-t))(4785.9P_3(2-t)-9555.1S_4(2-t))\right. \\ &\quad \left. +(2-S_3(2-t))(-9555.1S_3(2-t)+1901.7S_4(2-t))\right] \end{aligned}$$

transfer the states of the system (20) from the initial point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to the final point $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ during the time interval $[0,2]$.

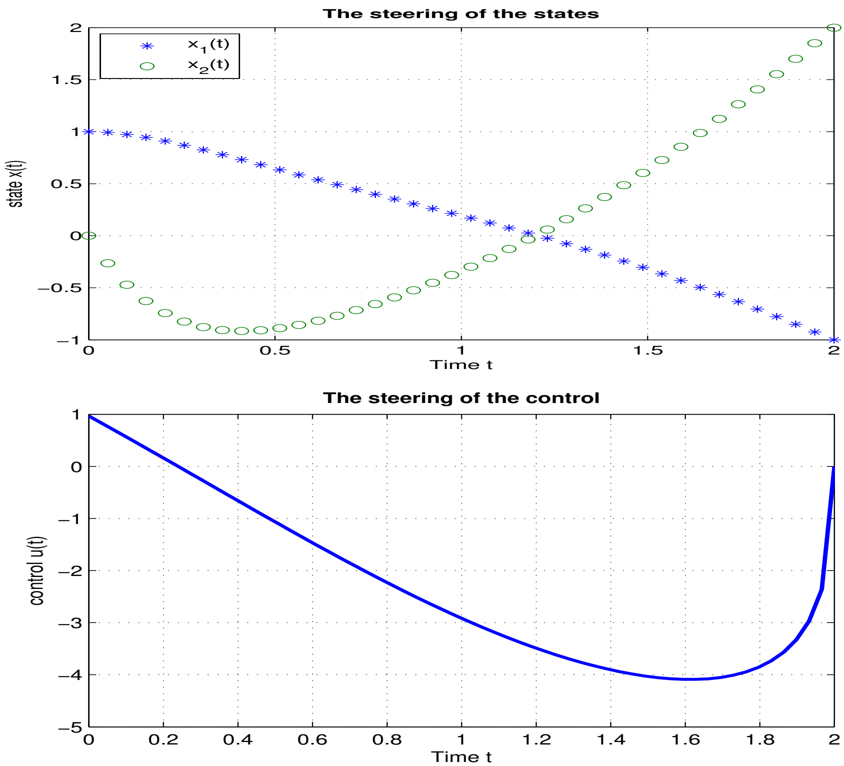

Figure 1. *The trajectories and control function of the system* (21).

Theorem 4.2. *The system* (15) *is controllable if and only if the columns of* $B^*E_\alpha(A^*(t_1-t)^\alpha)$ *are linearly independent.*

Proof. By Theorem 4.1, the controllability Grammian $\mathscr{Q}_\alpha \mathscr{Q}_\alpha^*$ is positive definite if and only if columns of $B^* E_\alpha(A^*(t_1-t)^\alpha)$ are linearly independent. □

Definition 4.2. *A bounded linear operator $\mathscr{R} : \mathbb{R}^n \to L^2_\alpha([t_0,t_1];\mathbb{R}^m)$ is called a steering operator for* (15)*, if for any $y \in \mathbb{R}^n, u(t) = (t_1-t)^{1-\alpha}(\mathscr{R}y)(t)$ steers* 0 *to y.*

An $m \times n$ matrix function $\mathscr{R}(t)$ is called a steering function, if the operator $\mathscr{R}$ defined by $(\mathscr{R}y)(t) = \mathscr{R}(t)y$ is a steering function. We observe that

(1) A bounded linear operator $\mathscr{R} : \mathbb{R}^n \to L^2_\alpha([t_0,t_1];\mathbb{R}^m)$ is a steering operator if and only if $\mathscr{Q}_\alpha \mathscr{R} = I$.

(2) An $m \times n$ matrix function $\mathscr{R}(t)$ is a steering function if and only if

$$\int_{t_0}^{t_1} E_{\alpha,\alpha}(A(t_1-s)^\alpha)B\mathscr{R}(s)\mathrm{ds} = \mathrm{I}. \tag{23}$$

We have the following characterization regarding controllability of (15) in terms of steering operator and steering function.

Theorem 4.3. *The following are equivalent*

(1) The system (15) *is controllable.*

(2) There exists a steering function $\mathscr{R}(t)$ for (15)*.*

(3) There exists a steering operator $\mathscr{R}$ for (15)*.*

Remark 4.1. *If $\mathscr{Q}_\alpha \mathscr{Q}_\alpha^*$ is invertible (that is, if the controllability Grammian matrix is non-singular) then*

$$(\mathscr{R}y)(t) = \mathscr{Q}_\alpha^*(\mathscr{Q}_\alpha \mathscr{Q}_\alpha^*)^{-1}y, y \in \mathbb{R}^n$$

is a steering operator. In this case

$$\mathscr{R}_0(t) = B^* E_{\alpha,\alpha}(A^*(t_1-t)^\alpha)W_\alpha^{-1}(t_0,t_1)$$

is a steering function.

5. Duality Result

Let $X = L_2(t_0,t_1)$ and consider the operator A given by

$${}^{C}D^{\alpha}_{t_0+}x(t) = Ax(t), \quad 0 < \alpha \le 1, \quad t \in [t_0,t_1], \tag{24}$$

where $D(A) = \{x \in X \,|\, x$ is absolutely continuous with ${}^C D^{\alpha}_{t_0+} x(t) \in L_2(t_0, t_1), x(t_0) = 0\}$. In order to discuss the duality results, the adjoint of (24) is(see, [5, 21, 26])

$$ {}^C D^{\alpha}_{T-} y(t) = A^* y(t), \tag{25} $$

where $D(A^*) = \{y \in X \,|\, y$ is absolutely continuous with ${}^C D^{\alpha}_{t_1-} y(t) \in L_2(t_0, t_1), y(t_1) = 0\}$.

Theorem 5.1. *The system* (15) *is controllable over* $[t_0, t_1]$ *if and only if the system*

$$ {}^C D^{\alpha}_{t_1-} y(t) = A^* y(t), \quad w(t) = B^* y(t) $$

is observable.

Proof. The function

$$ y(t) = E_{\alpha}(A^*(t_1 - t)^{\alpha}) v $$

is the solution of ${}^C D^{\alpha}_{t_1-} y(t) = A^* y(t)$ with $y(t_1) = v$. Hence $w(t) = B^* E_{\alpha}(A^*(t_1 - t)^{\alpha}) v$. Hence, By Theorem 3.3, the system ${}^C D^{\alpha}_{t_1-} y(t) = A^* y(t)$ with observation $w(t) = B^* y(t)$ is observable if and only if the columns of $B^* E_{\alpha}(A^*(t_1 - t)^{\alpha})$ are linearly independent functions and by Theorem 4.2, columns of $B^* E_{\alpha}(A^*(t_1 - t)^{\alpha})$ are linearly independent if and only if the system (15) is controllable over $[t_0, t_1]$. □

Example 5.1. *Consider the following fractional dynamical system*

$$ {}^C D^{\alpha}_{0+} x(t) = Ax(t) + Bu(t), \quad t \in [0, 1], \tag{26} $$

where $\alpha = 3/4$, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Then the Mittag-Leffler matrix function for given matrix A is

$$ E_{3/4}(At^{3/4}) = \begin{bmatrix} L_1 & L_2 \\ L_2 & L_1 \end{bmatrix}, $$

where

$$ L_1 = \frac{1}{2}\left[E_{3/4}(i\, t^{3/4}) + E_{3/4}(-i\, t^{3/4})\right] \text{ and } L_2 = \frac{i}{2}\left[E_{3/4}(-i\, t^{3/4}) - E_{3/4}(i\, t^{3/4})\right]. $$

The corresponding controllability Grammian matrix for this system is

$$ \begin{aligned} W_{3/4}(0,1) &= \int_0^1 E_{3/4,3/4}(A(1-\tau)^{3/4}) BB^* E_{3/4,3/4}(A^*(1-\tau)^{3/4}) \mathrm{d}\tau \\ &= \begin{bmatrix} 0.8893 & 1.1624 \\ 1.1624 & 1.6273 \end{bmatrix} > 0. \end{aligned} $$

Therefore the system (26) is controllable. The adjoint of the system (26) is

$$
\begin{aligned}
{}^{C}D_{1-}^{3/4}y(t) &= A^{*}y(t), \quad t \in [0,1], && (27)\\
w(t) &= B^{*}y(t). && (28)
\end{aligned}
$$

In this case, the observability Grammian matrix is

$$
\begin{aligned}
M_{3/4}(0,1) &= \int_0^1 E_{3/4}(A(1-\tau)^{3/4})BB^{*}E_{3/4}(A^{*}(1-\tau)^{3/4})\mathrm{d}\tau \\
&= \begin{bmatrix} 0.7483 & 1.1326 \\ 1.1326 & 1.8887 \end{bmatrix} > 0.
\end{aligned}
$$

Then the system (27)-(28) is observable, because the system (26) is controllable.

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