

DIFFERENTIAL GEOMETRY OF TRANSVERSAL INTERSECTION CURVES OF HYPERSURFACES IN $\mathbb{R}^5$

MOHAMD SALEEM LONE¹, OSMAR ALÉSSIO², MOHAMMED JAMALI³,
MOHAMMAD HASAN SHAHID⁴

¹DEPARTMENT OF MATHEMATICS, CENTRAL UNIVERSITY OF JAMMU, INDIA

E-MAIL: SALEEMRAJA2008@GMAIL.COM

²UNIVERSIDADE FEDERAL DO TRIÂNGULO MINEIRO-UFTM, UBERABA, MG,
BRASIL

E-MAIL: OSMAR@MATHEMATICA.UFTM.EDU.BR

³DEPARTMENT OF MATHEMATICS, AL-FALAH UNIVERSITY, HARYANA 121004,
INDIA

E-MAIL: JAMALIDBD@YAHOO.CO.IN

⁴DEPARTMENT OF MATHEMATICS, JAMIA MILLIA ISLAMIA, NEW DELHI 110025,
INDIA

E-MAIL: HASAN_JMI@YAHOO.COM

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ABSTRACT. In this paper, we present the algorithms for calculating the differential geometric quantities $\{t, n, b_1, b_2, b_3, \kappa_1, \kappa_2, \kappa_3, \kappa_4\}$, geodesic curvature and geodesic torsion of the transversal intersection curve of four hypersurfaces (given by parametric representation) in Euclidean space $\mathbb{R}^5$. In transversal intersection, the normals of the surfaces at the intersection point are linearly independent, while as in nontransversal intersection, the normals of the surfaces at the intersection point are linearly dependent.

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1. INTRODUCTION

The surface-surface intersection problem is a fundamental process needed in modeling shapes in CAD/CAM system. It is useful in designing of complex objects and animations. The two types of surfaces mostly used in geometric designing are parametric and implicit surfaces. For that reason, different methods have been given for either parametric-parametric or implicit-implicit surface intersection curves in $\mathbb{R}^3$. Differential geometry of a parametric curve in $\mathbb{R}^3$ can be found in textbooks such as Struik [18], Willmore [21], whereas differential geometry of parametric curves in $\mathbb{R}^n$ can be found in the textbook such as in klingenber [13] and in the contemporary literature on Geometric Modeling [8]. On the other hand, for the differential geometry of intersection curves, there exists a little literature. Willmore [21] and Aléssio [5] presented algorithms to obtain the unit tangent, unit principal normal, unit binormal, curvature and torsion of the transversal intersection curve of two implicit surfaces. Ye and Maekawa [22] presented algorithms for computing the differential geometric properties of both transversal and tangential intersection curves of two surfaces. Aléssio [3] formulated the algorithms for obtaining the geometric properties of intersection curves of three implicit hypersurfaces in $\mathbb{R}^4$. Based on the work of Aléssio [3], Mustufa Döldül [7] worked with three parametric hypersurfaces in $\mathbb{R}^4$ to derive the algorithms for differential geometric properties of transversal intersection. To obtain the first geodesic curvature ($\kappa_{1g}^{S_i}$) and the first geodesic torsion ($\tau_{1g}^{S_i}$) for the transversal intersection curve of 4 parametric hypersurfaces in $\mathbb{R}^5$, we need to derive the Darboux frame $\{U_1^{M_i}, \dots, U_5^{M_i}\}$. The Darboux frame is obtained by using the Gram-Schmidt orthogonalization process.

S. Yilmaz and M. Turgut [23] obtained the Frenet apparatus of a curve in Euclidean curve in Euclidean space $\mathbb{R}^5$. M. Turgut, et al.[19] derived the Frenet apparatus of non-null curves in Lorentzian space $\mathbb{L}^5$ and in [24] S. Yilmaz and M. Turgut derived the same for Partially Null and Pseudo Null Curves in Minkowski Space-time. Looking at their importance in physics, curves in Euclidean space in $\mathbb{R}^5$ and $\mathbb{L}^5$ are greater importance [12, 20, 14]. In this connection we have not seen any paper on discussion of intersection curves of parametric hypersurfaces in Euclidean space $\mathbb{R}^5$. Thus in this paper, we extend the methods of Mustufa Döldül [7] to obtain the Frenet frame $\{t, n, b_1, b_2, b_3\}$ and curvatures $\{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ of transversal intersection curve of four parametric hypersurfaces in $\mathbb{R}^5$. For practical implications, we

present an example in section 5. Moreover, all the calculations in example section are done using Matlab/Mathematica.

2. PRELIMINARIES

Definition 2.1. Let $\{e_1, e_2, e_3, e_4, e_5\}$ be the standard basis of five dimensional Euclidean space $\mathbb{R}^5$. Then the vector product of the vectors $x = \sum_{i=1}^5 x_i e_i$, $y = \sum_{i=1}^5 y_i e_i$, $z = \sum_{i=1}^5 z_i e_i$ and $w = \sum_{i=1}^5 w_i e_i$ is defined by

$$(1) \quad x \otimes y \otimes z \otimes w = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ w_1 & w_2 & w_3 & w_4 & w_5 \end{vmatrix}.$$

The vector product $x \otimes y \otimes z \otimes w$ yields a vector that is orthogonal to x, y, z, w .

let $R \subset \mathbb{R}^5$ be a regular hypersurface given by $\Phi = \Phi(u_1, u_2, u_3, u_4)$ and $\alpha : I \subset \mathbb{R} \rightarrow \Phi$ be an arbitrary curve with arc length parametrisation. If $\{t, n, b_1, b_2, b_3\}$ is the Frenet Frame along α , then we have

$$(2) \quad \begin{cases} t' = \kappa_1 n, \\ n' = -\kappa_1 t + \kappa_2 b_1, \\ b_1' = -\kappa_2 n + \kappa_3 b_2, \\ b_2' = -\kappa_3 b_1 + \kappa_4 b_3, \\ b_3' = -\kappa_4 b_2, \end{cases}$$

where t, n, b_1, b_2 and b_3 denote the tangent, the principal normal, the first binormal, the second binormal and third binormal vector fields. The normal vector n is the normalised acceleration vector α'' . The unit vector b_1 is determined such that n' can be decomposed into two components, a tangent one in the direction of t and a normal one in the direction of b_1 . The unit vector b_2 is determined such that b_1' can be decomposed into two components - a normal and another in the direction of b_2 . The unit vector b_3 is the unique unit vector field perpendicular to four dimensional subspace $\{t, n, b_1, b_2\}$. The functions $\kappa_1, \kappa_2, \kappa_3$ and κ_4 are the first, second, third and fourth curvatures of $\alpha(s)$. The first, second, third and fourth curvatures measure how rapidly the curve pulls away in a neighbourhood of s , from the tangent line, from planar curve, from three dimensional curve and from the four

dimensional curve at s , respectively.

Now, using the Frenet Frame we have the derivatives of α as

$$(3) \quad \alpha' = t, \quad \alpha'' = t' = \kappa_1 n, \quad \alpha''' = -\kappa_1^2 t + \kappa_1' n + \kappa_1 \kappa_2 b_1,$$

$$(4) \quad \alpha^{(4)} = -3\kappa_1 \kappa_1' t + (-\kappa_1^3 + \kappa_1'' - \kappa_1 \kappa_2^2) n + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') b_1 \\ + \kappa_1 \kappa_2 \kappa_3 b_2,$$

$$(5) \quad \alpha^{(5)} = (-3(\kappa_1')^2 - 4\kappa_1 \kappa_1'' + \kappa_1^4 + \kappa_1^2 \kappa_2^2) t + (-6\kappa_1^2 \kappa_1' + \kappa_1''' - \kappa_1' \kappa_2^2 - 3\kappa_1 \kappa_2 \kappa_2' \\ - 2\kappa_1' \kappa_2^2) n + (\kappa_1^3 \kappa_2 - \kappa_1 \kappa_2^3 + 3\kappa_1'' \kappa_2 + 3\kappa_1' \kappa_2' + \kappa_1 \kappa_2'' - \kappa_1 \kappa_2 \kappa_3^2) b_1 \\ (3\kappa_1' \kappa_2 \kappa_3 + 2\kappa_1 \kappa_2' \kappa_3 + 2\kappa_1 \kappa_2 \kappa_3' + \kappa_1 \kappa_2 \kappa_3') b_2 + \kappa_1 \kappa_2 \kappa_3 \kappa_4 b_3.$$

Also since Φ is regular, the partial derivatives $\Phi_1, \Phi_2, \Phi_3, \Phi_4$, where $(\Phi_i = \frac{\partial \Phi}{\partial u_i})$ are linearly independent at every point of Φ , i.e., $\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4 \neq 0$. Thus, the unit normal vector of Φ is given by

$$N = \frac{\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4}{\|\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \otimes \Phi_4\|}.$$

Furthermore, the third, second, and first binormal vectors of the curve are given by

$$(6) \quad b_3 = \frac{\alpha' \otimes \alpha'' \otimes \alpha''' \otimes \alpha^{(4)}}{\|\alpha' \otimes \alpha'' \otimes \alpha''' \otimes \alpha^{(4)}\|}, b_2 = \frac{b_3 \otimes \alpha' \otimes \alpha'' \otimes \alpha'''}{\|b_3 \otimes \alpha' \otimes \alpha'' \otimes \alpha'''\|}, b_1 = \frac{b_2 \otimes b_3 \otimes \alpha' \otimes \alpha''}{\|b_2 \otimes b_3 \otimes \alpha' \otimes \alpha''\|}$$

and the curvatures are obtained by

$$(7) \quad \kappa_1 = \|\alpha''\|, \quad \kappa_2 = \frac{\langle \alpha''', b_1 \rangle}{\kappa_1}, \quad \kappa_3 = \frac{\langle \alpha^{(4)}, b_2 \rangle}{\kappa_1 \kappa_2}, \quad \kappa_4 = \frac{\langle \alpha^{(5)}, b_3 \rangle}{\kappa_1 \kappa_2 \kappa_3}.$$

On the other hand, since the curve $\alpha(s)$ lies on Φ , we may write

$$\alpha(s) = \Phi(u_1(s), u_2(s), u_3(s), u_4(s)).$$

Then, we have

$$(8) \quad \alpha'(s) = \sum_{i=1}^4 \Phi_i u_i',$$

$$(9) \quad \alpha''(s) = \sum_{i=1}^4 \Phi_i u_i'' + \sum_{i,j=1}^4 \Phi_{ij} u_i' u_j',$$

$$(10) \quad \alpha'''(s) = \sum_{i=1}^4 \Phi_i u_i''' + 3 \sum_{i,j=1}^4 \Phi_{ij} u_i'' u_j' + \sum_{i,j,k=1}^4 \Phi_{ijk} u_i' u_j' u_k',$$

$$\alpha^{(4)}(s) = \sum_{i=1}^4 \Phi_i u_i^{(4)} + 4 \sum_{i,j=1}^4 \Phi_{ij} u_i''' u_j' + 3 \sum_{i,j=1}^4 \Phi_{ij} u_i'' u_j'' + 6 \sum_{i,j,k=1}^4 \Phi_{ijk} u_i'' u_j' u_k' +$$

$$(11) \quad \sum_{i,j,k,l=1}^4 \Phi_{ijkl} u_i' u_j' u_k' u_l',$$

$$\alpha^{(5)} = \sum_{i=1}^4 \Phi_i u_i^{(5)} + 5 \sum_{i,j=1}^4 \Phi_{ij} u_i^{(4)} u_j' + 10 \sum_{i,j=1}^4 \Phi_{ij} u_i''' u_j'' + 10 \sum_{i,j,k=1}^4 \Phi_{ijk} u_i''' u_j' u_k'$$

$$+ 15 \sum_{i,j,k=1}^4 \Phi_{ijk} u_i'' u_j'' u_k' + 10 \sum_{i,j,k,l=1}^4 \Phi_{ijk} u_i'' u_j' u_k' u_l'$$

$$(12) \quad + \sum_{i,j,k,l,r=1}^4 \Phi_{ijklr} u_i' u_j' u_k' u_l' u_r'.$$

3. THE CURVATURE OF THE TRANSVERSAL INTERSECTION OF HYPERSURFACES

Let R_1, R_2, R_3 and R_4 be four regular transversally intersecting hypersurfaces given by parametric equations $\Phi^i = \Phi^i(u_1^i, u_2^i, u_3^i, u_4^i)$, ($i = 1, 2, 3, 4$). Then the unit normal vector of these hypersurfaces are

$$N_i = \frac{\Phi_1^i \otimes \Phi_2^i \otimes \Phi_3^i \otimes \Phi_4^i}{\|\Phi_1^i \otimes \Phi_2^i \otimes \Phi_3^i \otimes \Phi_4^i\|}, \quad i = 1, 2, 3, 4.$$

Since the intersection is transversal, the normal vectors of these hypersurfaces at the intersection points are linearly independent, i.e., $N_1 \otimes N_2 \otimes N_3 \otimes N_4 \neq 0$. It is assumed that the intersection is a smooth curve say α , in $\mathbb{R}^5$. Let the intersection curve α be parameterised by arc length function s . Then, at the intersection point $\alpha(s_0) = p$, the unit tangent vector t of the intersection curve α can be found by the vector product of the normal vectors at p , i.e.,

$$(13) \quad t = \frac{N_1 \otimes N_2 \otimes N_3 \otimes N_4}{\|N_1 \otimes N_2 \otimes N_3 \otimes N_4\|}.$$

3.1. First curvature of the transversal intersection curve. In this subsection, we calculate first curvature of the intersection curve at p . Since t' is orthogonal to t , we can write

$$(14) \quad \alpha'' = t' = \sum_{i=1}^4 a_i N_i, \quad a_i \in \mathbb{R}, \quad i = 1, 2, 3, 4.$$

Thus, we need to calculate the scalars a_i to find α'' . On taking the dot product with both sides of (14), we get

$$(15) \quad \begin{aligned} \langle N_1, N_1 \rangle a_1 + \langle N_2, N_1 \rangle a_2 + \langle N_3, N_1 \rangle a_3 + \langle N_4, N_1 \rangle a_4 &= \kappa_n^1, \\ \langle N_1, N_2 \rangle a_1 + \langle N_2, N_2 \rangle a_2 + \langle N_3, N_2 \rangle a_3 + \langle N_4, N_2 \rangle a_4 &= \kappa_n^2, \\ \langle N_1, N_3 \rangle a_1 + \langle N_2, N_3 \rangle a_2 + \langle N_3, N_3 \rangle a_3 + \langle N_4, N_3 \rangle a_4 &= \kappa_n^3, \\ \langle N_1, N_4 \rangle a_1 + \langle N_2, N_4 \rangle a_2 + \langle N_3, N_4 \rangle a_3 + \langle N_4, N_4 \rangle a_4 &= \kappa_n^4, \end{aligned}$$

where $\kappa_n^i = \langle t', N_i \rangle$, $i = 1, 2, 3, 4$ and $\langle \cdot, \cdot \rangle$ is the scalar product. For obtaining a_i 's, we must compute $\kappa_n^1, \kappa_n^2, \kappa_n^3, \kappa_n^4$. On using (9), we find

$$(16) \quad \begin{aligned} \kappa_n^i &= \Pi_{11}^i (u_1^{i'})^2 + \Pi_{22}^i (u_2^{i'})^2 + \Pi_{33}^i (u_3^{i'})^2 + \Pi_{44}^i (u_4^{i'})^2 + 2(\Pi_{12}^i u_1^{i'} u_2^{i'} \\ &\quad + \Pi_{13}^i u_1^{i'} u_3^{i'} + \Pi_{14}^i u_1^{i'} u_4^{i'} + \Pi_{23}^i u_2^{i'} u_3^{i'} + \Pi_{24}^i u_2^{i'} u_4^{i'} + \Pi_{34}^i u_3^{i'} u_4^{i'}), \end{aligned}$$

where Π_{lm}^i , ($l, m = 1, 2, 3, 4$) are the second fundamental form coefficients of the hypersurfaces Φ^i . Since the unit tangent is known from (13) and $\|\Phi_1^i \otimes \Phi_2^i \otimes \Phi_3^i \otimes \Phi_4^i\| \neq 0$, the scalar multiplication on both sides of (8) with $\Phi_1^i, \Phi_2^i, \Phi_3^i$ and Φ_4^i , respectively yields a linear system of four equations as

$$(17) \quad \begin{aligned} \Upsilon_{11}^i u_1^{i'} + \Upsilon_{12}^i u_2^{i'} + \Upsilon_{13}^i u_3^{i'} + \Upsilon_{14}^i u_4^{i'} &= \langle t, \Phi_1^i \rangle, \\ \Upsilon_{21}^i u_1^{i'} + \Upsilon_{22}^i u_2^{i'} + \Upsilon_{23}^i u_3^{i'} + \Upsilon_{24}^i u_4^{i'} &= \langle t, \Phi_2^i \rangle, \\ \Upsilon_{31}^i u_1^{i'} + \Upsilon_{32}^i u_2^{i'} + \Upsilon_{33}^i u_3^{i'} + \Upsilon_{34}^i u_4^{i'} &= \langle t, \Phi_3^i \rangle, \\ \Upsilon_{41}^i u_1^{i'} + \Upsilon_{42}^i u_2^{i'} + \Upsilon_{43}^i u_3^{i'} + \Upsilon_{44}^i u_4^{i'} &= \langle t, \Phi_4^i \rangle, \end{aligned}$$

with respect to u_1', u_2', u_3' and u_4' , where Υ_{lm}^i , $l, m = 1, 2, 3, 4$ are the first fundamental form coefficients of the hypersurface Φ^i . Substituting the solutions of these systems into (16) gives κ_n^i , $i = 1, 2, 3, 4$. Then using matlab/mathematica the system of linear equations in (15) can be solved for a_1, a_2, a_3 and a_4 . Thus the first curvature of the intersection curve at p is obtained from (14) and the first equation of (7).

Remark 3.1. If the normal vectors are mutually orthogonal to each other at the intersection point, then the first curvature is given by $\kappa_1 = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$.

3.2. Second curvature of the transversal intersection curve. To compute the second curvature, we have to find the third derivative of the intersection curve α .

Since $N_i, i = 1, 2, 3, 4$ are orthogonal to t , we may write $c_1N_1 + c_2N_2 + c_3N_3 + c_4N_4$ instead of terms $\kappa_1'n + \kappa_1\kappa_2b_1$ in α''' , i.e.,

$$(18) \quad \alpha''' = -\kappa_1^2 t + c_1N_1 + c_2N_2 + c_3N_3 + c_4N_4.$$

Taking scalar product on both sides of (18) with N_i , we obtain

$$(19) \quad \begin{aligned} \langle N_1, N_1 \rangle c_1 + \langle N_2, N_1 \rangle c_2 + \langle N_3, N_1 \rangle c_3 + \langle N_4, N_1 \rangle c_4 &= \mu_1, \\ \langle N_1, N_2 \rangle c_1 + \langle N_2, N_2 \rangle c_2 + \langle N_3, N_2 \rangle c_3 + \langle N_4, N_2 \rangle c_4 &= \mu_2, \\ \langle N_1, N_3 \rangle c_1 + \langle N_2, N_3 \rangle c_2 + \langle N_3, N_3 \rangle c_3 + \langle N_4, N_3 \rangle c_4 &= \mu_3, \\ \langle N_1, N_4 \rangle c_1 + \langle N_2, N_4 \rangle c_2 + \langle N_3, N_4 \rangle c_3 + \langle N_4, N_4 \rangle c_4 &= \mu_4, \end{aligned}$$

where $\mu_i = \langle \alpha''', N_i \rangle, i = 1, 2, 3, 4$ and $c_i \in \mathbb{R}$. Now, we have to find the unknown scalars μ_i . Using (10), we have

$$(20) \quad \mu_r = 3 \sum_{i,j=1}^4 \langle \Phi_{ij}^r, N_r \rangle u_i'' u_j' + \sum_{i,j,k=1}^4 \langle \Phi_{ijk}^r, N_r \rangle u_i' u_j' u_k'.$$

Since the components u_i' are known from the system (17), it is straight to find u_i'' . To obtain u_i'' , we use (9) and write

$$(21) \quad \Delta_r = \sum_{i,j=1}^4 \Phi_{ij}^r u_i' u_j', \quad r = 1, 2, 3, 4.$$

Then, we have

$$(22) \quad \begin{aligned} \Upsilon_{11}^i u_1^{i''} + \Upsilon_{12}^i u_2^{i''} + \Upsilon_{13}^i u_3^{i''} + \Upsilon_{14}^i u_4^{i''} &= \langle t' - \Delta_i, \Phi_1^i \rangle, \\ \Upsilon_{21}^i u_1^{i''} + \Upsilon_{22}^i u_2^{i''} + \Upsilon_{23}^i u_3^{i''} + \Upsilon_{24}^i u_4^{i''} &= \langle t' - \Delta_i, \Phi_2^i \rangle, \\ \Upsilon_{31}^i u_1^{i''} + \Upsilon_{32}^i u_2^{i''} + \Upsilon_{33}^i u_3^{i''} + \Upsilon_{34}^i u_4^{i''} &= \langle t' - \Delta_i, \Phi_3^i \rangle, \\ \Upsilon_{41}^i u_1^{i''} + \Upsilon_{42}^i u_2^{i''} + \Upsilon_{43}^i u_3^{i''} + \Upsilon_{44}^i u_4^{i''} &= \langle t' - \Delta_i, \Phi_4^i \rangle, \end{aligned}$$

which gives required derivatives. Thus from (20), we find the values of μ_i , which finally helps us to find the value of c_i in system (19). Thus, on finding b_1 , the second

curvature can be found from the second equation of (7).

From (3) we get $\kappa_1' = \langle \alpha''', n \rangle$.

3.3. Third curvature of the transversal intersection curve. To find the third curvature, we need to find the fourth derivative of the intersection curve α at p . Since N_i is orthogonal to t , we may write $d_1N_1 + d_2N_2 + d_3N_3 + d_4N_4$ instead of $(-\kappa_1^3 + \kappa_1'' - \kappa_1\kappa_2^2)n + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')b_1 + \kappa_1\kappa_2\kappa_3b_2$ in $\alpha^{(4)}$, i.e.,

$$(23) \quad \alpha^{(4)} = -3\kappa_1\kappa_1't + d_1N_1 + d_2N_2 + d_3N_3 + d_4N_4.$$

Taking scalar product of (23) with N_1, N_2, N_3 and N_4 , we get the following respectively,

$$(24) \quad \begin{aligned} \langle N_1, N_1 \rangle d_1 + \langle N_2, N_1 \rangle d_2 + \langle N_3, N_1 \rangle d_3 + \langle N_4, N_1 \rangle d_4 &= \xi_1, \\ \langle N_1, N_2 \rangle d_1 + \langle N_2, N_2 \rangle d_2 + \langle N_3, N_2 \rangle d_3 + \langle N_4, N_2 \rangle d_4 &= \xi_2, \\ \langle N_1, N_3 \rangle d_1 + \langle N_2, N_3 \rangle d_2 + \langle N_3, N_3 \rangle d_3 + \langle N_4, N_3 \rangle d_4 &= \xi_3, \\ \langle N_1, N_4 \rangle d_1 + \langle N_2, N_4 \rangle d_2 + \langle N_3, N_4 \rangle d_3 + \langle N_4, N_4 \rangle d_4 &= \xi_4, \end{aligned}$$

where $\xi_i = \langle \alpha^{(4)}, N_i \rangle$, $i = 1, 2, 3, 4$ and $d_i \in \mathbb{R}$.

Now to find d_i , we have to find ξ_i . For that taking the scalar product of $\alpha^{(4)}$ with N_i , we obtain

$$(25) \quad \begin{aligned} \xi_r = & 4 \sum_{i,j=1}^4 \langle \Phi_{ij}^r, N_r \rangle u_i''' u_j' + 3 \sum_{i,j=1}^4 \langle \Phi_{ij}^r, N_r \rangle u_i'' u_j'' + 6 \sum_{i,j,k=1}^4 \langle \Phi_{ijk}^r, N_r \rangle u_i'' u_j' u_k' + \\ & \sum_{i,j,k,l=1}^4 \langle \Phi_{ijkl}^r, N_r \rangle u_i' u_j' u_k' u_l'. \end{aligned}$$

Since u_i', u_i'' are already known, so to find ξ_r we have to find u_i''' . These derivatives can be found by taking the scalar product of both side of (10) with $\Phi_1^i, \Phi_2^i, \Phi_3^i$ and Φ_4^i , respectively.

As now $\alpha', \alpha'', \alpha'''$ and $\alpha^{(4)}$ are at our disposal, we can find b_3, b_2 and b_1 from (6). On using the binormal vector b_1 and (18), the second curvature of the intersection curve at p is obtained from the second equation of (7). Now, since κ_1, κ_2 and b_2 are already known, the third curvature can be now found from the third equation of (7).

3.4. Fourth curvature of the transversal intersection curve. To obtain the fourth curvature κ_4 , it is required to find the fifth derivative of the intersection curve of α at p . Similar to third and fourth derivative of the curve α , we may write

$$(26) \quad \alpha^{(5)} = \{-3(\kappa_1')^2 - 4\kappa_1\kappa_2'' + \kappa_1^{(4)} + \kappa_1^2\kappa_2^2\}t + m_1N_1 + m_2N_2 + m_3N_4 + n_4N_4,$$

where the system of equations for unknowns is

$$(27) \quad \begin{aligned} \langle N_1, N_1 \rangle m_1 + \langle N_2, N_1 \rangle m_2 + \langle N_3, N_1 \rangle m_3 + \langle N_4, N_1 \rangle m_4 &= \eta_1, \\ \langle N_1, N_2 \rangle m_1 + \langle N_2, N_2 \rangle m_2 + \langle N_3, N_2 \rangle m_3 + \langle N_4, N_2 \rangle m_4 &= \eta_2, \\ \langle N_1, N_3 \rangle m_1 + \langle N_2, N_3 \rangle m_2 + \langle N_3, N_3 \rangle m_3 + \langle N_4, N_3 \rangle m_4 &= \eta_3, \\ \langle N_1, N_4 \rangle m_1 + \langle N_2, N_4 \rangle m_2 + \langle N_3, N_4 \rangle m_3 + \langle N_4, N_4 \rangle m_4 &= \eta_4, \end{aligned}$$

and $\eta_i = \langle \alpha^{(5)}, N_i \rangle$. Projecting (12) onto the unit vector N_i , respectively, we obtain η_i depending on $u_i^{(4)}$ besides u_i''' , u_2'' and u_1' . Except $u_i^{(4)}$ all are known, so to find $u_i^{(4)}$ taking the scalar product of (11) with Φ_1^i , Φ_2^i , Φ_3^i , Φ_4^i , respectively. Consequently, the fourth curvature of the intersection curve can be found from the last equation of (7). Now, since all the quantities are found in the subsections (3.1) – (3.4), we can easily compute the Frenet vectors from (2).

4. DARBOUX FRAME, FIRST GEODESIC CURVATURE AND FIRST, SECOND AND THIRD GEODESIC TORSION.

In this section, we derive the Darboux frame $\{U_1^i, U_2^i, U_3^i, U_4^i, U_5^i\}$, the first geodesic curvature (κ_{1g}^i) and the first, second and third geodesic torsion (τ_{jg}^i), $j = 1, 2, 3$, for the transversal intersection curve of 4 parametric hypersurfaces in $\mathbb{R}^5$.

Definition 4.1. Let M_i be a regular hypersurface in $\mathbb{R}^5$ and α be a curve on M_i . Then for each i , $1 \leq i \leq 4$, the function $\kappa_{ig} : I \rightarrow \mathbb{R}$, defined for $s \in I$ by

$$\kappa_{ig}(s) = \langle U_i'(s), U_{i+1}(s) \rangle$$

is called the i^{th} **geodesic curvature** of the curve α at $\alpha(s)$.

4.1. Darboux Frame. We are able to obtain a natural frame for the intersection curve-hypersurface pair $(\alpha(s), \mathbf{M}_i)$, i.e., the frame $\{U_1^i, U_2^i, U_3^i, U_4^i, U_5^i\}$ by using the Gram-Schmidt orthogonalization process. By assumption the sets $\{\alpha'(s), \alpha''(s), \alpha'''(s), \alpha^{(4)}(s)\}$ and $\{N_1(p), N_2(p), N_3(p), N_4(p)\}$ are linearly independent. Fixing $U_1^i = \alpha'(p)$ and $U_5^i = N_i$, $1 \leq i \leq 4$, the natural frame (Darboux frame) $\{U_1^i, U_2^i, U_3^i, U_4^i, U_5^i\}$ is obtained as

$$\begin{aligned} U_1^i &= \alpha'(p), \\ U_5^i &= N_i, \\ U_2^i &= \frac{-\langle \alpha''(p), N_i \rangle N_i - \langle \alpha''(p), U_1^i \rangle U_1^i + \alpha''(p)}{\|-\langle \alpha''(p), N_i \rangle N_i - \langle \alpha''(p), U_1^i \rangle U_1^i + \alpha''(p)\|}, \\ U_3^i &= \frac{-\langle \alpha'''(p), N_i \rangle N_i - \langle \alpha'''(p), U_2^i \rangle U_2^i - \langle \alpha'''(p), U_1^i \rangle U_1^i + \alpha'''(p)}{\|-\langle \alpha'''(p), N_i \rangle N_i - \langle \alpha'''(p), U_2^i \rangle U_2^i - \langle \alpha'''(p), U_1^i \rangle U_1^i + \alpha'''(p)\|}, \\ U_4^i &= \frac{-\langle \alpha^{(4)}(p), N_i \rangle N_i - \langle \alpha^{(4)}(p), U_3^i \rangle U_3^i - \langle \alpha^{(4)}(p), U_2^i \rangle U_2^i - \langle \alpha^{(4)}(p), U_1^i \rangle U_1^i + \alpha^{(4)}(p)}{\|-\langle \alpha^{(4)}(p), N_i \rangle N_i - \langle \alpha^{(4)}(p), U_3^i \rangle U_3^i - \langle \alpha^{(4)}(p), U_2^i \rangle U_2^i - \langle \alpha^{(4)}(p), U_1^i \rangle U_1^i + \alpha^{(4)}(p)\|}. \end{aligned}$$

The j^{th} geodesic curvature τ_{jg}^i associated with i^{th} hypersurface is

$$(28) \quad \kappa_{jg}^i = \left\langle (U_j^i)', U_{j+1}^i \right\rangle = -\left\langle (U_{j+1}^i)', U_j^i \right\rangle, \quad j \in \{1, 2, 3\}.$$

The j^{th} geodesic torsion τ_{jg}^i associated with i^{th} hypersurface is

$$(29) \quad \tau_{jg}^i = -\left\langle (U_5^i)', U_{j+1}^i \right\rangle = \left\langle (U_{j+1}^i)', U_5^i \right\rangle, \quad j \in \{1, 2, 3\}.$$

4.2. First geodesic and j^{th} torsion geodesic Formulas. Using the darbox frame, we have following theorem for geodesic curvature and geodesic torsion.

Theorem 4.1. First Geodesic Curvature and the j -th geodesic torsion of the intersection curve of 4 parametric hypersurfaces are

$$(30) \quad \kappa_{1g}^i = \sum_{j=1}^4 a_j \langle N_j, U_2^i \rangle,$$

$$(31) \quad \tau_{jg}^i = -\left\langle \frac{(\overline{N}_i)_{u_1} u_1' + (\overline{N}_i)_{u_2} u_2' + (\overline{N}_i)_{u_3} u_3' + (\overline{N}_i)_{u_4} u_4'}{\|\Phi_1^i \times \Phi_2^i \times \Phi_3^i \times \Phi_4^i\|}, U_{j+1}^i \right\rangle, \quad j = 1, 2, 3.$$

Proof. For first geodesic curvature, we have

$$\kappa_{1g}^i = \left\langle (U_1^i)', U_2^i \right\rangle = \left\langle \alpha''(s), U_2^i \right\rangle.$$

Now using (14), (30) follows easily. For the j^{th} geodesic torsion, we need to differentiate $U_5^i = N_i = \frac{\Phi_1^i \times \Phi_2^i \times \Phi_3^i \times \Phi_4^i}{\|\Phi_1^i \times \Phi_2^i \times \Phi_3^i \times \Phi_4^i\|}$. If we define $N_i = \frac{\overline{N}_i}{\|\overline{N}_i\|}$, where $\overline{N}_i =$

$\Phi_1^i \times \Phi_2^i \times \Phi_3^i \times \Phi_4^i$, we derive

$$(32) \quad \frac{d}{ds} (U_5^i(s)) = \frac{dN_i}{ds} = \frac{\frac{d\overline{N}_i}{ds} \|\overline{N}_i\| - \overline{N}_i \frac{d(\|\overline{N}_i\|)}{ds}}{\|\overline{N}_i\|^2}.$$

Now (29) can be written as

$$\tau_{jg}^i = - \left\langle \frac{dN_i}{ds}, U_{j+1}^i \right\rangle, j = 1, 2, 3.$$

Using (32), we get

$$\begin{aligned} \tau_{jg}^i &= - \left\langle \frac{\frac{d\overline{N}_i}{ds} \|\overline{N}_i\|}{\|\overline{N}_i\|^2} - \frac{\overline{N}_i \frac{d(\|\overline{N}_i\|)}{ds}}{\|\overline{N}_i\|^2}, U_{j+1}^i \right\rangle, j = 1, 2, 3 \\ &= - \left\langle \frac{\frac{d\overline{N}_i}{ds} \|\overline{N}_i\|}{\|\overline{N}_i\|^2}, U_{j+1}^i \right\rangle, j = 1, 2, 3 \text{ because } \langle \overline{N}_i, U_{j+1}^i \rangle = 0, j = 1, 2, 3 \end{aligned}$$

or,

$$\tau_{jg}^i = - \left\langle \frac{\frac{d\overline{N}_i}{ds}}{\|\overline{N}_i\|}, U_{j+1}^i \right\rangle, j = 1, 2, 3$$

where

$$\begin{aligned} \frac{d\overline{N}_i}{ds} &= (\overline{N}_i)_{u_1} u_1' + (\overline{N}_i)_{u_2} u_2' + (\overline{N}_i)_{u_3} u_3' + (\overline{N}_i)_{u_4} u_4', \\ (\overline{N}_i)_{u_i} &= \Phi_{1i}^i \times \Phi_2^i \times \Phi_3^i \times \Phi_4^i + \Phi_1^i \times \Phi_{2i}^i \times \Phi_3^i \times \Phi_4^i + \Phi_1^i \times \Phi_2^i \times \Phi_{3i}^i \times \Phi_4^i \\ &\quad + \Phi_1^i \times \Phi_2^i \times \Phi_3^i \times \Phi_{4i}^i. \end{aligned}$$

□

5. EXAMPLE

Let M_1, M_2, M_3 and M_4 be the hypersurfaces given by

$$\begin{aligned} X^1(u_1, u_2, u_3, u_4) &= (\sqrt{2} \sin u_3 \cos u_2 \cos u_1, \sqrt{2} \sin u_3 \sin u_2 \sin u_1, \\ &\quad \sqrt{2} \sin u_3 \cos u_2, \sqrt{2} u_4 \cos u_3, \frac{1}{\sqrt{2}} \sin u_3), \end{aligned}$$

$$X^2(u_1, u_2, u_3, u_4) = (u_3 \cos u_1 \cos u_2, u_3 \sin u_1 \cos u_2, u_3 \sin u_2, u_3, \cos u_2 \cos u_4),$$

$$X^3(u_1, u_2, u_3, u_4) = (u_3 \cos u_1 \cos u_2, u_3 \sin u_1 \cos u_2, u_4 \sin u_1, u_3, \sin u_1 \sin u_2),$$

$$X^4(u_1, u_2, u_3, u_4) = \left(\frac{1}{2} + \frac{1}{2} \cos u_1, \frac{1}{2} \sin u_1, u_2, u_3, \frac{u_4}{2} \right),$$

respectively. We first find the Frenet vectors and the curvatures of the intersection curve at the intersection point p given by

$$\begin{aligned} p = X^1 \left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, 1 \right) &= X^2 \left(\frac{\pi}{4}, \frac{\pi}{4}, 1, \frac{\pi}{4} \right) = X^3 \left(\frac{\pi}{4}, \frac{\pi}{4}, 1, 1 \right) = X^4 \left(\frac{\pi}{2}, \frac{\sqrt{2}}{2}, 1, 1 \right) \\ &= \left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}, 1, \frac{1}{2} \right). \end{aligned}$$

The unit normals of these hypersurfaces are

$$\begin{aligned} N_1 &= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0, 0, -\sqrt{\frac{2}{3}} \right), \quad N_2 = \left(-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, -\frac{1}{2}, \frac{1}{\sqrt{2}}, 0 \right), \\ N_3 &= \left(-\frac{2}{3}, 0, 0, \frac{1}{3}, -\frac{2}{3} \right), \quad N_4 = (0, 1, 0, 0, 0). \end{aligned}$$

The non-vanishing first fundamental coefficients are calculated as

$$\begin{aligned} g_{11}^1 &= \frac{1}{2}, \quad g_{22}^1 = 1, \quad g_{33}^1 = \frac{9}{4}, \quad g_{44}^1 = 1, \quad g_{12}^1 = \frac{1}{2}, \quad g_{23}^1 = -\frac{1}{2}, \\ g_{34}^1 &= -1, \quad g_{11}^2 = \frac{1}{2}, \quad g_{22}^2 = \frac{5}{4}, \quad g_{33}^2 = 2, \quad g_{44}^2 = \frac{1}{4}, \quad g_{24}^2 = \frac{1}{4}, \\ g_{11}^3 &= \frac{5}{4}, \quad g_{22}^3 = \frac{3}{4}, \quad g_{33}^3 = \frac{3}{2}, \quad g_{44}^3 = \frac{1}{2}, \quad g_{12}^3 = \frac{1}{4}, \quad g_{14}^3 = \frac{1}{2}, \\ g_{23}^3 &= -\frac{1}{2}, \quad g_{11}^4 = \frac{1}{4}, \quad g_{22}^4 = 1, \quad g_{33}^4 = 1, \quad g_{44}^4 = \frac{1}{4}. \end{aligned}$$

The unit tangent at the intersection point is found by

$$t = \frac{N_1 \otimes N_2 \otimes N_3 \otimes N_4}{\|N_1 \otimes N_2 \otimes N_3 \otimes N_4\|} = \left(-\frac{2}{\sqrt{91}}, 0, -5\sqrt{\frac{2}{91}}, -\frac{6}{\sqrt{91}}, -\frac{1}{\sqrt{91}} \right).$$

Also, the non-vanishing second fundamental coefficients are

$$\begin{aligned} h_{11}^1 &= -\frac{1}{\sqrt{6}}, \quad h_{22}^1 = -\frac{1}{\sqrt{6}}, \quad h_{12}^1 = -\frac{1}{\sqrt{6}}, \quad h_{11}^2 = \frac{1}{2\sqrt{2}}, \\ h_{22}^2 &= \frac{1}{\sqrt{2}}, \quad h_{11}^3 = \frac{2}{3}, \quad h_{22}^3 = \frac{2}{3}, \quad h_{12}^3 = -\frac{2}{3}, \quad h_{13}^3 = \frac{1}{3}, \quad h_{11}^4 = -\frac{1}{2}. \end{aligned}$$

From the linear system of equations in (17), we obtain

$$\begin{aligned}(u_1^1)' &= -0.628971, & (u_2^1)' &= 0.838628, & (u_3^1)' &= -0.209657, & (u_4^1)' &= -0.838628, \\(u_1^2)' &= 0.209657, & (u_2^2)' &= -0.419314, & (u_3^2)' &= -0.628971, & (u_4^2)' &= 0.628971, \\(u_1^3)' &= 0.209657, & (u_2^3)' &= -0.419314, & (u_3^3)' &= -0.628971, & (u_4^3)' &= -1.25794, \\(u_1^4)' &= 0.419314, & (u_2^4)' &= -0.741249, & (u_3^4)' &= -0.628971, & (u_4^4)' &= -0.209651.\end{aligned}$$

Thus, we obtain $\kappa_n^1 = -0.879304$, $\kappa_n^2 = 0.139867$, $\kappa_n^3 = 0.117216$, $\kappa_n^4 = -0.0879121$.

Consequently, we get

$$\begin{aligned}\alpha'' &= -1.321618N_1 - 0.469981N_2 + 0.698466N_3 - 0.285472N_4 \\&= (-0.839029, -0.658857, 0.234991, -0.0995048, 0.613453).\end{aligned}$$

Thus,

$$\kappa_1 = \|\alpha''\| = 1.25679.$$

Also the unit normal vector is

$$n = (-0.524421, -0.411807, 0.146877, -0.621938, 0.383428).$$

From (21), we get

$$\begin{aligned}\Delta_1 &= (-1.05495, -1.14286, -0.279735, -0.395605, -0.021978), \\ \Delta_2 &= (-0.32967, -0.417583, 0.248653, 0, -0.549451), \\ \Delta_3 &= (-0.32967, -0.417583, -0.404061, 0, -0.197802), \\ \Delta_4 &= (0, -0.0879121, 0, 0, 0).\end{aligned}$$

From the linear system of equations in (22), we obtain

$$\begin{aligned}(u_1^1)'' &= 0.106614, & (u_2^1)'' &= 0.161474, & (u_3^1)'' &= 0.89026, & (u_4^1)'' &= 1.05091, \\(u_1^2)'' &= 0.268086, & (u_2^2)'' &= 0.366087, & (u_3^2)'' &= -0.309769, & (u_4^2)'' &= -2.69189, \\(u_1^3)'' &= 0.454481, & (u_2^3)'' &= 0.795236, & (u_3^3)'' &= -0.141793, & (u_4^3)'' &= 0.450137, \\(u_1^4)'' &= 1.67806, & (u_2^4)'' &= 0.2356, & (u_3^4)'' &= -0.234991, & (u_4^4)'' &= 1.22691.\end{aligned}$$

which yields

$$\mu_1 = 0.4544, \quad \mu_2 = -0.5105, \quad \mu_3 = -1.4338, \quad \mu_4 = -1.0554.$$

Then, we have

$$\begin{aligned}\alpha''' &= -(1.25679)^2 t + 1.84188N_1 + 0.451035N_2 - 2.14772N_3 - 1.64788N_4 \\ &= (2.35545, -1.0554, 0.945301, 0.596496, 0.0935034),\end{aligned}$$

$$\kappa_1' = \langle \alpha''', n \rangle = -0.996915.$$

Using α''' , we compute

$$\begin{aligned}(u_1^1)''' &= -5.90731, \quad (u_2^1)''' = 2.14186, \quad (u_3^1)''' = 1.51393, \quad (u_4^1)''' = 2.11042, \\ (u_1^2)''' &= -3.76546, \quad (u_2^2)''' = 2.58181, \quad (u_3^2)''' = 0.76716, \quad (u_4^2)''' = -2.76882, \\ (u_1^3)''' &= -0.88196, \quad (u_2^3)''' = -3.60164, \quad (u_3^3)''' = -1.14366, \quad (u_4^3)''' = 2.21882, \\ (u_1^4)''' &= -3.33833, \quad (u_2^4)''' = 0.545801, \quad (u_3^4)''' = 0.596496, \quad (u_4^4)''' = 0.187007.\end{aligned}$$

Hence, we get

$$\xi_1 = -21.7826, \quad \xi_2 = -3.3273, \quad \xi_3 = 8.8342, \quad \xi_4 = -1.3710.$$

From (23), we have

$$\begin{aligned}\alpha^{(4)} &= -3(1.25679)(-0.996915) \left(-\frac{2}{\sqrt{91}}, 0, -5\sqrt{\frac{2}{91}}, -\frac{6}{\sqrt{91}}, -\frac{1}{\sqrt{91}} \right) \\ &\quad -41.7712N_1 - 29.6654N_2 + 34.1873N_3 + 5.19372N_4 \\ &= (-30.1443, -1.371, 12.0465, -11.945, 10.9205).\end{aligned}$$

Thus

$$b_3 = \frac{\alpha' \otimes \alpha'' \otimes \alpha''' \otimes \alpha^{(4)}}{\|\alpha' \otimes \alpha'' \otimes \alpha''' \otimes \alpha^{(4)}\|} = (0.235522, 0.439366, -0.161734, -0.0297594, 0.851143),$$

$$b_2 = \frac{b_3 \otimes \alpha' \otimes \alpha'' \otimes \alpha'''}{\|b_3 \otimes \alpha' \otimes \alpha'' \otimes \alpha'''\|}$$

$$= (0.0469859, -0.0544597, 0.64055, -0.763297, -0.0435799),$$

$$b_1 = \frac{b_2 \otimes b_3 \otimes \alpha' \otimes \alpha''}{\|b_2 \otimes b_3 \otimes \alpha' \otimes \alpha''\|} = (0.648319, -0.724303, -0.161293, -0.0530185, 0.16199).$$

Now curvature,

$$\kappa_2 = \frac{\langle \alpha''', b_1 \rangle}{\kappa_1} = 1.68888, \quad \kappa_3 = \frac{\langle \alpha^{(4)}, b_2 \rangle}{\kappa_1 \kappa_2} = 7.07463.$$

Also from (4), we have, $\kappa_1'' = 35.3284$.

Using $\alpha^{(4)}$, we obtain

$$\begin{aligned}(u_1^1)^{(4)} &= 44.1618, (u_2^1)^{(4)} = -21.5916, (u_3^1)^{(4)} = -4.64151, (u_4^1)^{(4)} = -16.5865, \\(u_1^2)^{(4)} &= 22.5702, (u_2^2)^{(4)} = 33.779, (u_3^2)^{(4)} = -7.12615, (u_4^2)^{(4)} = 55.62, \\(u_1^3)^{(4)} &= 26.8139, (u_2^3)^{(4)} = 22.2084, (u_3^3)^{(4)} = -4.81273, (u_4^3)^{(4)} = -9.77667, \\(u_1^4)^{(4)} &= -0.6116, (u_2^4)^{(4)} = 16.2234, (u_3^4)^{(4)} = -11.945, (u_4^4)^{(4)} = 21.841.\end{aligned}$$

Hence, we have

$$\eta_1 = 194.9662, \quad \eta_2 = -27.5036, \quad \eta_3 = -56.7000, \quad \eta_4 = 29.6466.$$

Thus, we obtain

$$\begin{aligned}\alpha^{(5)} &= (-3(1.25679)^2 - 4(1.25679)(35.3284) \\&\quad + (1.25679)^4 + (1.25679)^2(1.68888)^2)t \\&\quad + 313.53N_1 + 145.818N_2 - 210.771N_3 - 46.7969N_4 \\&= (253.719, 29.6467, 57.0616, 143.136, -97.1016).\end{aligned}$$

Thus, the fourth curvature is given by

$$\kappa_4 = \frac{\langle \alpha^{(5)}, b_3 \rangle}{\kappa_1 \kappa_2 \kappa_3} = -1.55521.$$

Now, to find κ_{1g}^1 and τ_{jg}^1 , $j = 1, 2, 3$, for hypersurface M_1 , we have from (30) and (31), $\kappa_{1g}^1 = 0.584888$, $\tau_{1g}^1 = -0.774977$, $\tau_{2g}^1 = -0.0496875$, $\tau_{3g}^1 = -0.276372$. Similarly we can find κ_{1g}^i and τ_{jg}^i for M_2, M_3 and M_4 .

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