Journal of Dynamical Systems & Geometric Theories
ISSN : 1726-037X (Print) 2169-0057 (Online)
Vol. 15(2) November 2017, pp. 131-146
DOI : 10.1080/1726037X.2017.1390192

Taylor & Francis
Taylor & Francis Group

CHAOTIC DYNAMICAL SYSTEMS ON SYMBOLIC SPACES

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(Received: 01 April 2016, Accepted: 09 July 2017)

Abstract. We try to find chaotic dynamical systems by extending some simple continuous functions defined on $\mathbb{Q}_p$, to functions defined on $A^{\mathbb{Z}}$, where $A = \{0, 1, 2, \cdots, p-1\}$. A combination of the powers of the shift map with addition of a constant in $\mathbb{Q}_p$, gives rise to chaotic dynamical systems, conjugate to powers of the shift map. By extending the addition in $\mathbb{Z}_p$, to the whole of $A^{\mathbb{Z}}$, and combining with powers of the shift map, we get an expansive map with the same topological entropy as that of powers of the shift. We also obtain two more positively expansive maps with positive entropy.

AMS Classification: 54H20, 37B10
Keywords: Discrete dynamical systems; chaotic; transitive; entropy

1. Introduction

A dynamical system is a pair (X, f) where f is a continuous self map $f : X \to X$ of a topological space X. Topological Dynamics is the study of iterations $f^n : X \to X$. The orbits $\{f^n(x)\}_{n\geq 0}$ of elements x of X are of particular interest. We denote the set of rational numbers by $\mathbb{Q}$, the set of integers by $\mathbb{Z}$ and set of non

negative integers by $\mathbb{N}$.
Chaotic maps f on a cantor-like set, preferably with some arithmetic structure, and with f given by a simple formula, are considered as a source of randomness suited for machine computation. In [2] Woodcock and Smart discuss some discrete dynamical systems based on p-adic numbers. In [4], measurable dynamics of some p-adic functions including translation and multiplication in $\mathbb{Q}_p$, the field of p-adic numbers are studied. Our aim is to extend the addition operation on $\mathbb{Z}_p$, the ring of p-adic integers, that is addition mod p with a carry, to $A^{\mathbb{Z}}$, where $A = \{0, 1, 2 \cdots, p-1\}$, to get new chaotic dynamical systems.

2. The field of p-adic numbers

Consider the field $\mathbb{Q}$ of rational numbers. For any fixed prime p, let p^α be the highest power of p that divides an integer a not equal to zero. Then $\|a\|_p$ is $p^{-\alpha}$. For a rational number $\frac{a}{b}$, $\|\frac{a}{b}\|_p$ is $\frac{\|a\|_p}{\|b\|_p}$ and $\|0\|_p$ is defined to be zero. With respect to this norm, $\mathbb{Q}$ is a normed field, but not complete. Its completion is denoted by $\mathbb{Q}_p$ [7]. The p-adic metric takes only discrete values, namely integer powers of p. Every p-adic number x can be canonically represented as $\sum_{i=-\infty}^{i=m} d_i p^{-i}$, where m is in $\mathbb{Z}$ and $d_i \in \{0,\ 1,\ 2,\ 3, \cdots, p-1\}$, and $d_m \neq 0$. This can be viewed as a *doubly infinite* series in powers of p, where the "digits" d_i are all zero for $i > m$. The p-adic norm of this number is p^m. It is also written as $x = \cdots d_{-2}\, d_{-1}\, \overbrace{d_0}^{0^{th}} .\, d_1\, d_2 \cdots \underbrace{d_m}_{(\neq 0)}\, 0\, 0 \cdots$
The numbers for which $m \leq 0$ are called *p-adic integers.* The set of p-adic integers is denoted by $\mathbb{Z}_p$. Note that this contains all the positive integers, represented by a finite sum, which is same as the base-p expansion of a positive integer. It contains negative integers also, but the sum is infinite. Indeed, we can check that $-\sum_{i=-\infty}^{i=m} d_i p^{-i} = \sum_{i=-\infty}^{i=m-1} (p-1-d_i)p^{-i} + (p-d_m)p^{-m}$, where d_m is the last nonzero digit. Therefore for any $x \in \mathbb{Q}_p$, either x or $-x$ must have infinitely many nonzero digits (towards left) in the canonical p-adic expansion. It is easy to see that a number x in $\mathbb{Q}_p$ is a rational number if and only if its canonical p-adic expansion is eventually periodic [7].
The arithmetic operations in $\mathbb{Q}_p$ are similar to arithmetic operations on natural numbers written in base p. Algorithms of addition and subtraction are pursued from right to left indefinitely, after aligning the "point" after the digit in the zeroth

place. Multiplication also proceeds from right to left indefinitely, in the usual way. "Long division" proceeds from right to left, unlike the usual long division of natural numbers [7]. We use this representaion of the p-adic numbers and these arithmetic operations to get new chaotic functions on $A^{\mathbb{Z}}$.

3. Two types of Chaos

There are many definitions of chaos and entropy. According to the definition given by R. L. Devaney [6], an infinite system is **chaotic** if it is transitive, has a dense set of periodic points and is sensitive to initial conditions.
The above basic concepts are discussed in [5], for a **Cantor space**. It is a metric space which is compact, totally disconnected and perfect. Any two Cantor spaces are homeomorphic. A **symbolic space** is a closed subset of a Cantor space. It is compact and totally disconnected. A **symbolic dynamical system**, denoted by SDS, is an ordered pair $(X,\ f)$, where X is a symbolic space and $f\ :\ X\ \to\ X$ is a continuous function.

Let A be a finite set with m elements, where $m \geq 2$.
Let A have discrete topology and $A^{\mathbb{Z}}$ have the product topology. It is induced by the metric
$d(x,y)\ = 2^{-n}\ where\ n\ =\ min\{i \geq 0,\ x_i \neq y_i,\ or\ x_{-i} \neq y_{-i}\}$. The shift map $\sigma\ :\ A^{\mathbb{Z}}\ \to\ A^{\mathbb{Z}}$ given by $\sigma(x)_i = x_{i+1}$ is continuous in this topology. $A^{\mathbb{Z}}$ is a Cantor space. The system $(A^{\mathbb{Z}},\ \sigma)$ is called the **full shift**. A **two sided subshift** is the dynamical system $(\sum, \sigma|_{\sum})$, where $\sum$ is a closed subspace of $A^{\mathbb{Z}}$ such that $\sigma(\sum) = \sum$. The full shift is chaotic according to the definition by Devaney.

4. Entropy

We define **topological entropy** using open covers as follows :
Let $\mathscr{U}$ be an open covering of X. Let $N(\mathscr{U})$ be the minimum number of elements of $\mathscr{U}$ that are needed to cover X. Since X is compact, this number exists. If $\mathscr{U}$ and $\mathscr{V}$ are finite open covers of X, their joint open cover, denoted by $\mathscr{U}\ \vee\ \mathscr{V}$ is $\{U \cap V | U \in \mathscr{U}, V \in \mathscr{V}, U \cap V \neq \emptyset\}$. If f is a continuous function from X to itself, $N_n(\mathscr{U})\ =\ N(\mathscr{U}\ \vee\ f^{-1}(\mathscr{U})\ \vee\ f^{-2}(\mathscr{U})\ \cdots\ \vee f^{-(n-1)}(\mathscr{U}))$. The **topological entropy** of the open cover $\mathscr{U}$ of X is

$$h(\mathscr{U}, f) = \lim_{n\to+\infty} \frac{logN_n(\mathscr{U})}{n}$$

The topological entropy of (X, f), denoted by $h(X, f)$ is defined to be $\sup\{h(\mathscr{U}, f)|\mathscr{U}\ finite\ open\ cover\ of\ X\}$.
If the topological entropy of (X, f) is positive, (X, f) is said to be chaotic.
To find estimates for topological entropy, some concepts of fractal dimensions are used. We make use of the upper box counting dimension. For a compact metric space (X, d), the **upper box counting dimension** is

$$D_d(X) = \overline{\lim_{\epsilon\to\ 0}} \frac{log\ b(\epsilon)}{|log\ \epsilon|} \tag{1}$$

where $b(\epsilon)$ is the number of ϵ-balls required to cover X.
Topological entropy can be defined using the capacity function as follows. Let (X, d) be a compact metric space and let f be a continuous self map of X.
For any two points x and y of X, and any $n \geq 1$, let

$$d_n^f(x, y) \ = \max_{0\leq i\leq n-1}\{d(f^i(x), f^i(y))\}$$

We denote the open ball $\{y \in X|\ d_n^f(x, y) < r\}$ by $B_f(x, r, n)$.
A set $E \subseteq X$ is ***r*-dense** with respect to d_f^n, or ***(n,r)*-dense**, if $X \subseteq \bigcup_{x\in E} B_f(x, r, n)$.
The ***r*- capacity** of d_f^n, denoted by $S_d(f, r, n)$ is the minimal cardinality of an *(n,r)*-dense set.
Consider the exponential growth rate

$$h_d(f, r) \ := \overline{\lim_{n\to\infty}} \frac{1}{n}log\ S_d(f, r, n) \tag{2}$$

We define

$$h_d(f) \ := \lim_{r\to 0}\ h_d(f, r) \tag{3}$$

For equivalent metrics d and d', $h_d(f) = h_{d'}(f)$, and both may be denoted by $h(X, f)$, which is the topological entropy.
The topological entropy of the the full-shift $(A^{\mathbb{Z}}, \sigma)$ is $log\ p$, and so for $(A^{\mathbb{Z}}, \sigma^k)$ it is $k\ log\ p$.

5. Topological Conjugacy

In order to identify dynamical systems with similar behaviour, the concept of **topological conjugacy** is used. A system (X, f) is said to be topologically conjugate to (Y, g) if there exists a homeomorphism $h : X \to Y$ such that $h^{-1}gh = f$.

Theorem 1. *Let (X, f) and (Y, g) be dynamical systems and h be a homeomorphism from X to Y such that $h^{-1}gh = f$. Let x be any point of X. Then*
(i) h (Orbit of x under f) = Orbit of $(h(x))$ under g.
(ii) If x is a periodic point of f with period k, then $h(x)$ is a periodic point of g with period k.
(iii) If the periodic points of f are dense in X, then the periodic points of g are dense in Y.
(iv) If f is transitive, so is g.
(v) If f is chaotic, so is g.

Sometimes, if a function f is difficult to analyze, we can try to study some simple conjugate of f. At one end we have equicontinuous functions, and at the other end we have chaotic functions. Between these extreme classes there are many other distinct types of dynamical behaviour.

6. A Devaney-chaotic Dynamical system using p-adic addition with a carry

Consider the sets $\mathbb{Z}_p \subset \mathbb{Q}_p \subset A^{\mathbb{Z}}$ where $A = \{0, 1, 2, 3, 4 \cdots , \ p-1\}$ and p is a prime number. The first question is whether we can extend the non-Archimedean topology of $\mathbb{Q}_p$ to $A^{\mathbb{Z}}$. The norm cannot be extended because $\|x\|_p$ is p^m where m is the largest integer such that m^{th} coordinate of x in the canonical representation of x is nonzero. There is no such m for elements outside $\mathbb{Q}_p$. For the same reason, the metric also cannot be extended in a natural way.
We can at most extend the topology to $A^{\mathbb{Z}}$, by defining open balls centred at x in a similar fashion. Let x_i denote the i^{th} coordinate of x, for any element of $A^{\mathbb{Z}}$. Define $\mathbf{B}_{p^r}(x) = \{ \ y \in A^{\mathbb{Z}} \mid x_i = y_i \ \forall \ i > r\}$. Then $\mathbb{Q}_p$ is an open subset of $A^{\mathbb{Z}}$ in this topology. In fact, $\mathbb{Q}_p = \bigcup_{i=-\infty}^{i=+\infty} \mathbf{B}_{p^i}(0)$. $\mathbb{Q}_p$ is complete implies that it is closed in $A^{\mathbb{Z}}$. Thus $\mathbb{Q}_p$ is both open and closed. Hence any element outside $\mathbb{Q}_p$ is in a different component, and any continuous function on $\mathbb{Q}_p$ cannot be extended

naturally and uniquely to $A^{\mathbb{Z}}$.

Next consider $\mathbb{Q}_p$ as a subset of $A^{\mathbb{Z}}$. We denote the non-Archimedean topology of $\mathbb{Q}_p$ by T_1. Let A have discrete topology. Let T_2 denote the product topology on $A^{\mathbb{Z}}$ and its restrictions to $\mathbb{Z}_p$ and $\mathbb{Q}_p$.

We know that T_2 is induced by the metric $d(x,y) = 2^{-r}$ where $r = \min\{i \geq 0 \mid x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}$. We can as well replace 2 by p, and define the distance as p^{-r}. Note that it will induce the same topology.

Proposition 6.1. (i) *$T_1 = T_2$ on $\mathbb{Z}_p$.*

(ii) *T_1 is finer than T_2 on $\mathbb{Q}_p$.*

(iii) *$\mathbb{Q}_p$ is dense in $A^{\mathbb{Z}}$ (with respect to T_2)*

(iv) *$\mathbb{Z}_p$ is a closed subset of $A^{\mathbb{Z}}$ in T_2*

Proof. (i) If $x = \cdots x_{-2}\ x_{-1}\ \overbrace{x_0}^{0^{th}}\ 0\ 0 \cdots$ then $\mathrm{B}_{p^{-r}}(x)$, that is the ball of radius p^{-r}, for a positive integer r, centered at x is the same in both the topologies of $\mathbb{Z}_p$. It consists of elements of the type

$x = \cdots *\ *\ *\ x_{-r} \cdots x_{-2}\ x_{-1}\ \overbrace{x_0}^{0^{th}}\ 0\ 0 \cdots$.

(ii) Consider open balls of radius p^{-r} in both topologies. It is enough to consider $r > 1$. Let $\mathrm{B}^1_{p^{-r}}(x)$ be the ball of radius p^{-r} centered at x in T_1, and $\mathrm{B}^2_{p^{-r}}(x)$ the corresponding one in T_2 for $\mathbb{Q}_p$.

Let $x = \cdots x_{-2}\ x_{-1}\ \overbrace{x_0}^{0^{th}}\ x_1\ x_2\ \cdots x_m\ 0\ 0 \cdots$ be an element of $\mathbb{Q}_p$. Then $\mathrm{B}^1_{p^{-r}}(x)$ consists of elements of the type

$\cdots * * *\ x_{-r}\ x_{-r+1}\ \cdots x_{-2}\ x_{-1}\ \overbrace{x_0}^{0^{th}}\ x_1\ x_2\ \cdots x_m\ 0\ 0 \cdots$, whereas $\mathrm{B}^2_{p^{-r}}(x)$ consists of elements of the type

$\cdots\ *\ *\ x_{-r}\ x_{-r+1}\ \cdots x_{-2}\ x_{-1}\ \overbrace{x_0}^{0^{th}}\ x_1\ x_2\ \cdots x_r\ *\ *\ \cdots$.

Therefore $\mathrm{B}^1_{p^{-r}}(x) \subset \mathrm{B}^2_{p^{-r}}(x)$, and so T_1 is finer than T_2 on $\mathbb{Q}_p$.

(iii) Let

$a = \cdots\ x_{-r-1}\ x_{-r}\ x_{-r+1}\ \cdots x_{-2}\ x_{-1}\ \overbrace{x_0}^{0^{th}}\ x_1\ x_2\ \cdots x_r\ \cdots$ be an element of $A^{\mathbb{Z}}$.

Define

$$a_n = \cdots\ x_{-r-1}\ x_{-r}\ x_{-r+1}\ \cdots x_{-2}\ x_{-1}\ \overbrace{x_0}^{0^{th}}\ x_1\ x_2\ \cdots\ x_n\ 0\ 0\ 0\ \cdots \tag{4}$$

Then $d(a_m, a_n) \leq p^{-k}$ where k is min $\{m, n\}$, hence $\{a_n\}$ is a Cauchy sequence in T_2, which clearly converges to a in T_2. It is not Cauchy in T_1, unless $a \in \mathbb{Q}_p$.

(iv) We prove that the complement of $\mathbb{Z}_p$ is open. Take any element

$$x = \cdots\ x_{-2}\ x_{-1}\ \overbrace{x_0}^{0^{th}}\ x_1\ x_2\ \cdots\ \underbrace{x_m}_{\neq 0}\ x_{m+1}\ *\ *\ \cdots$$

of $A^{\mathbb{Z}} \setminus \mathbb{Z}_p$, where $x_m \neq 0$ for some $m > 0$. Take $\delta = p^{-m}$. Then $\mathrm{B}^2_\delta(x) \subset A^{\mathbb{Z}} \setminus \mathbb{Z}_p$, because for all y in $\mathrm{B}^2_\delta(x)$, the m^{th} coordinate $y_m = x_m \neq 0$, and so $y \notin \mathbb{Z}_p$.

□

Our aim is to find what functions $\mathbb{Z}_p \to \mathbb{Z}_p$ and $\mathbb{Q}_p \to \mathbb{Q}_p$ can be extended to functions $A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, to give rise to new dynamical systems, particulary chaotic.
In order to extend a function f on $\mathbb{Q}_p$ to a continuous function on $A^{\mathbb{Z}}$, f should be uniformly continuous in T_2, as $A^{\mathbb{Z}}$ is compact. We can see that any of the algebraic functions like polynomials, or even multiplication by a constant is not uniformly continuous. It is because when x and c are multiplied, whose norms are say, p^m and p^j, the product has norm p^{m+j}, and the digits upto $m+j$ depend both on x and c. The"carry"travels from right to left; so even if x and y agree on a large range of coordinates around the 0^{th} postion, say $-j$ to j, xc and yc need not agree around the 0^{th} position. In other words, we cannot choose a j such that $d(x, y) < p^{-j}$ implies $d(xc, yc) < \varepsilon$, for small ε.

Consider adding a constant to any element of $\mathbb{Q}_p$. This gives a uniformly continuous function on $\mathbb{Q}_p$.

Proposition 6.2. *Let*

$c = \cdots\ c_{-1}\ \overbrace{c_0}^{0^{th}}\ c_1\ \cdots\ c_{n-1}\ c_n\ 0\ 0\ 0 \cdots$ *be any fixed element of* $\mathbb{Q}_p$. *Let* $\varphi : \mathbb{Q}_p \to \mathbb{Q}_p$ *be given by* $\varphi(x) = x + c$. *This can be extended to a function on* $A^{\mathbb{Z}}$ *that is continuous in the product topology.*

Proof. For any

$x = \cdots\ x_{-2} x_{-1}\ \overbrace{x_0}^{0^{th}}\ x_1\ x_2\ \cdots\ x_m\ 0\ 0\ 0\ \cdots$, take $j > |n|$. Adding the constant c affects only the n^{th} coordinates and the coordinates to the left. If x and y agree from $-j^{th}$ to j^{th} coordinate, so do $\varphi(x)$ and $\varphi(y)$. This n depends only on the constant c and not on x. If $d(x,y) < p^{-j}$, then $d(\varphi(x),\varphi(y)) < p^{-j}$. Therefore adding a constant gives a uniformly continuous function on $\mathbb{Q}_p$. Therefore, if $\{a_n\}$ is a sequence in $\mathbb{Q}_p$ converging to x in $A^{\mathbb{Z}}$, then $\varphi(a_n)$ converges to a unique point $\varphi(x)$ in $A^{\mathbb{Z}}$. Therefore $\varphi : x \mapsto x + a$ extends to a continuous function on $A^{\mathbb{Z}}$. $\square$

Note that $(A^{\mathbb{Z}}, \varphi)$ is similar to the adding machine in [5]. Combining φ with powers σ^k of the shift map, we get maps that are chaotic in the sense of Devaney. We denote $\sigma^k(x)$ also by $p^k x$. Fix an element a of $\mathbb{Q}_p$ and define a function f on $\mathbb{Q}_p$ by

$$f : x \mapsto a + p^k x \tag{5}$$

This is chaotic according to the definition by Devaney, as we can see that it is conjugate to a power of the shift map.

Proposition 6.3. *Let k be any nonzero integer, and let a be a p-adic number. Let f be the function $A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by $f : x \mapsto a + p^k x$. Then f is conjugate to σ^k.*

Proof. let $b = \frac{a}{1-p^k}$. Define $\varphi : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by $\varphi(x) = x - b$ Then

$$\begin{aligned} \varphi \circ f \circ \varphi^{-1}(x) &= \varphi \circ f(x+b) \\ &= \varphi(p^k x + p^k b + a) = p^k x + p^k b + a - b \\ &= p^k x + b(p^k - 1) + a = p^k(x) = \sigma^k(x). \end{aligned} \tag{6}$$

$\square$

As 0 is the only fixed point of σ^k, it follows that b is the only fixed point of f. For f resitricted to $\mathbb{Q}_p$ and positive k it is an attractive fixed point. For negative k it is a repelling fixed point.

Proposition 6.4. *Let $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $f(x) = p^k x + a$, for a constant a in $\mathbb{Q}_p$, and positive k. For all x in $\mathbb{Q}_p$, $f^n(x)$ converges to $b = \frac{a}{1-p^k}$.*

Proof. For any positive integer n, and any x in $\mathbb{Q}_p$,

$$\begin{aligned} f^n(x) &= xp^{nk} + a(1 + p^k + p^{2k} + \cdots + p^{(n-1)k}) \\ &= xp^{nk} + a\frac{1-p^{nk}}{(1-p^k)} \\ &= (x - b)p^{nk} + b \end{aligned} \tag{7}$$

For positive k, p^{nk} tends to zero as n tends to infinity, in the topology T_1 of $\mathbb{Q}_p$ and so in the coarser topology T_2 too. For any x in $\mathbb{Q}_p$, xp^{nk} also tends to zero in this topology, and so in the coarser topology T_2 too. Therefore $f^n(x)$ tends to b for any x in $\mathbb{Q}_p$. $\square$

(Note : For negative k, p^{nk} does not tend to zero when n tends to infinity, and so $f^n(x)$ cannot tend to b for $x \neq b$.)
The periodic points of f are $\overline{w} + b$, where w is a word of length $|nk|$ in $A^{\mathbb{Z}}$. For any transitive point y of σ^k, $y + b$ is a transitive point of f.

7. An expansive map with positive entropy

When the above function f is applied to x, the coordinates at the far right are affected only by the shift map. We look for a function that affects all the coordinates. For computational purposes, it is good if we can start calculating the digits of $f(x)$ at the centre, that is around the 0^{th} coordinate and proceed iteratively in both directions. We extend the above addition to an addition in $A^{\mathbb{Z}}$. It would be better if can add two elements of $A^{\mathbb{Z}}$, in a way different from the usual coordinate-wise addition mod p, i.e., making use of the the "carry ".

Let $a = \cdots\ a_{-2}\ a_{-1}\ \overbrace{a_0}^{0^{th}}\ a_1\ a_2\ \cdots$ and $b = \cdots\ b_{-2}\ b_{-1}\ \overbrace{b_0}^{0^{th}}\ b_1\ b_2\ \cdots$ be any two elements of $A^{\mathbb{Z}}$. We define a new kind of "addition "as follows.
$a + b = c = \ldots\ c_{-2}\ c_{-1}\ \overbrace{c_0}^{0^{th}}\ c_1\ c_2 \cdots$ where
$\cdots\ c_{-2}\, c_{-1}\, c_0$ is the usual sum of the p-adic integers $\cdots\ a_{-2}\, a_{-1}\, a_0$ and $\cdots\ b_{-2}\, b_{-1}\, b_0$, with carries transferred to the left, and $\cdots\ c_2\ c_1$ is the usual sum of the p-adic integers $\cdots\ a_2\ a_1$ and $\cdots\ b_2\ b_1$. In other words, the given elements are split after the 0^{th} position, the two parts are considered as separate p-adic integers and added in the usual way. For the left part, addition proceeds from right to left, and for the right part it proceeds from left to right. Actually this operation makes $A^{\mathbb{Z}}$ into a topological group. The additive identity of the group is the zero sequence. The additive inverse of an element a is defined in a similar fashion as that of p-adic integers.
Let a_{-m} be the first non-zero digit of a on the left side (starting at the 0^{th} position), and let a_n be the first non-zero digit of a on the right side (starting at the 1^{st} position).

constant c and not on x. If $d(x,y) < p^{-j}$, then $d(\varphi(x), \varphi(y)) < p^{-j}$. Therefore adding a constant gives a uniformly continuous function on $\mathbb{Q}_p$. Therefore, if $\{a_n\}$ is a sequence in $\mathbb{Q}_p$ converging to x in $A^{\mathbb{Z}}$, then $\varphi(a_n)$ converges to a unique point $\varphi(x)$ in $A^{\mathbb{Z}}$. Therefore $\varphi : x \mapsto x + a$ extends to a continuous function on $A^{\mathbb{Z}}$. □

Note that $(A^{\mathbb{Z}}, \varphi)$ is similar to the adding machine in [5]. Combining φ with powers σ^k of the shift map, we get maps that are chaotic in the sense of Devaney. We denote $\sigma^k(x)$ also by $p^k x$. Fix an element a of $\mathbb{Q}_p$ and define a function f on $\mathbb{Q}_p$ by

$$f : x \mapsto a + p^k x \tag{5}$$

This is chaotic according to the definition by Devaney, as we can see that it is conjugate to a power of the shift map.

Proposition 6.3. *Let k be any nonzero integer, and let a be a p-adic number. Let f be the function $A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by $f : x \mapsto a + p^k x$. Then f is conjugate to σ^k.*

Proof. let $b = \frac{a}{1-p^k}$. Define $\varphi : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by $\varphi(x) = x - b$ Then

$$\begin{aligned} \varphi \circ f \circ \varphi^{-1}(x) &= \varphi \circ f(x+b) \\ &= \varphi(p^k x + p^k b + a) = p^k x + p^k b + a - b \\ &= p^k x + b(p^k - 1) + a = p^k(x) = \sigma^k(x). \end{aligned} \tag{6}$$

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As 0 is the only fixed point of σ^k, it follows that b is the only fixed point of f. For f resitricted to $\mathbb{Q}_p$ and positive k it is an attractive fixed point. For negative k it is a repelling fixed point.

Proposition 6.4. *Let $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $f(x) = p^k x + a$, for a constant a in $\mathbb{Q}_p$, and positive k. For all x in $\mathbb{Q}_p$, $f^n(x)$ converges to $b = \frac{a}{1-p^k}$.*

Proof. For any positive integer n, and any x in $\mathbb{Q}_p$,

$$\begin{aligned} f^n(x) &= xp^{nk} + a(1 + p^k + p^{2k} + \cdots + p^{(n-1)k}) \\ &= xp^{nk} + a\frac{1-p^{nk}}{(1-p^k)} \\ &= (x - b)p^{nk} + b \end{aligned} \tag{7}$$

and σ^{nk}, and it keeps moving to the left with successive applications of f and σ^k. Therefore

$$d(f^n(x), f^n(y)) = d(\sigma^{nk}(x), \sigma^{nk}(y)) \text{ for all } n \geq 0. \tag{8}$$

Moreover, this distance decreases as n increases.

Case(ii) Let $x_i = y_i$ for $0 \leq i < mk + l$ and $x_{mk+l} \neq y_{mk+l}$ for some $m \geq 0$ and some l with $0 \leq l < k$. Clearly $j \leq mk + l$. Then for $n \leq m$, $(f^n(x))_i = (f^n(y))_i$ and $(\sigma^{nk}(x))_i = (\sigma^{nk}(y))_i$ for $0 \leq i < (m-n)k + l$ and $(f^n(x))_{(m-n)k+l} \neq (f^n(y))_{(m-n)k+l}$ and $(\sigma^{nk}(x))_{(m-n)k+l} \neq (\sigma^{nk}(y))_{(m-n)k+l}$. Now $(m-n)k + l \geq 0$, and upto this n, any difference in the coordinates of x and y from the right side of 0^{th} position is not yet transferred to the left. At the left side, if at all there is a difference in the coordinates of x and y, it moves further to the left with successive applications of f and σ^k. Therefore we can say that

$$d(f^n(x), f^n(y)) = d(\sigma^{nk}(x), \sigma^{nk}(y)) \ for \ \ 0 \leq n \leq m. \tag{9}$$

For $n = m+1$, $(\sigma^{nk}(x))_{-k+l} \neq (\sigma^{nk}(y))_{-k+l}$. Therefore $d(\sigma^{nk}(x), \sigma^{nk}(y)) \geq p^{l-k}$. For $f^n(x)$ and $f^n(y)$ there may be a difference in the coordinates at any of the positions $l-k,\ l-k+1, \cdots, -1, 0$, or to the right side of 0^{th} position. In any case we can say that

$$d(f^n(x), f^n(y)) \geq p^{-k} \text{ and } d(\sigma^{nk}(x), \sigma^{nk}(y)) \geq p^{-k}. \tag{10}$$

We will prove that f is expansive, using the fact that σ^k is expansive. Take $\epsilon = p^{-k}$. For $x \neq y$, we have to find $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) \geq \epsilon$. If $d(x, y) \geq \epsilon$, we are through.

Let $d(x, y) < \epsilon$. Suppose that there is some positive j such that $d(\sigma^{jk}(x), \sigma^{jk}(y)) \geq \epsilon$. It means that there was a difference in the coordinates of x and y at a position after k, which was brought to a position $\leq k$, after j applications of σ^k. So we are in Case(ii). We can find a positive n satisfying (10).

Next suppose that $d(\sigma^{jk}(x), \sigma^{jk}(y)) < \epsilon$ for all positive j. Then we are in Case(i), and $x_i = y_i$ for all $i \geq 0$. We have to find a positive n such that $d(f^{-n}(x), f^{-n}(y)) \geq \epsilon$.

$f^{-1}(x) = \sigma^{-k}(x - a)$. Let $j = mk + l$, with $m \geq 1$, and $0 \leq l < k$, be the smallest positive integer such that $x_{-j} \neq y_{-j}$. It follows that $(x-a)_i = (y-a)_i$ for all $i > -j$, and $(x-a)_{-j} \neq (y-a)_{-j}$. Therefore $(\sigma^{-k}(x-a))_i = (\sigma^{-k}(y-a))_i$ for $-j+k < i$ and $(\sigma^{-k}(x-a))_{-j+k} \neq (\sigma^{-k}(y-a))_{-j+k}$. Therefore $d(\sigma^{-k}(x-a), \sigma^{-k}(y-a)) = p^{-j+k}$. Thus by one application of f^{-1} the distance is increased by a factor of p^k. Successive applications of f^{-1}, m times, gives $d(f^{-m}(x), f^{-m}(y) \geq \epsilon$. Thus f is expansive, hence is conjugate to a subshift.

Now we find the topological entropy of f. We prove that $B_f(x,r,n) = B_{\sigma^k}(x,r,n)$ (see (4)), for sufficiently large n and sufficiently small r. Let $r < p^{-2k}$ and $n > 2$. Suppose that $y \in B_f(x,r,n)$ for some x. Then $d(f^i(x), f^i(y)) \leq r < p^{-2k}$, for $0 \leq i \leq n-1$. If x and y are as in Case(i), clearly $d(\sigma^{ik}(x), \sigma^{ik}(y)) \leq r$ for $0 \leq i \leq n-1$, and $y \in B_{\sigma^k}(x,r,n)$.

If x and y are not as in Case(i), then take m and l as in Case(ii). Then $n - 1 \leq m$, because of (10), and from (9) it follows that $y \in B_{\sigma^k}(x,r,n)$. Therefore $B_f(x,r,n) \subseteq B_{\sigma^k}(x,r,n)$. By a similar argument $B_{\sigma^k}(x,r,n) \subseteq B_f(x,r,n)$. It follows that $S_d(f,r,n) = S_d(\sigma^k,r,n)$, and from (2) and (3) of section (3) both f and σ^k have same topological entropy, that is $k \, log \, p$. $\square$

8. A positively expansive map

Definition 8.1. *A dynamical system (X,f) is positively expansive if $\exists\ \epsilon > 0$, $\forall\ x \neq y \in X$, $\exists\ n \geq 0$ such that $d(f^n(x), f^n(y)) \geq \epsilon$.*

We try to modify the above map f to get a positively expansive map, with positive topological entropy.

For any $x = \cdots x_{-2}\ x_{-1}\ \overbrace{x_0}^{0^{th}}\ x_1\ x_2\ \cdots$ in $A^{\mathbb{Z}}$, define the "reflection" about 0^{th} coordinate, $r(x)$, as $\cdots x_2\ x_1\ \overbrace{x_0}^{0^{th}}\ x_{-1}\ x_{-2}\ \cdots$. That is $r(x)_i = x_{-i}$ for all $i \in \mathbb{Z}$. Let x and y be two distinct points in $A^{\mathbb{Z}}$, separated by a distance p^{-j}. To get a positively expansive function f, we have to increase the distance between them at each application of f. If there is a difference in the j^{th} coordinates, for $j > k > 0$, application of σ^k brings the difference to $(j-k)^{th}$ position, and so the distance increases by a factor of p^k. But if there is no difference in the positively numbered coordinates, application of σ^k will not increase the distances. So we use the function r to transfer the difference in coordinates at the left side to the right side. We use

a combination of σ, r and the addition. Now define
$f(x) = r(\sigma^k(x) + x)$. It is enough to consider positive values of k.

Proposition 8.2. *The function $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by $f(x) = r(\sigma^k(x) + x)$, where k is a positive integer, is continuous and positively expansive.*

Proof. Obviously, σ^k is a homeomorphism and r is an isometry. Therefore it is enough to verify that the addition map given by $(x, y) \mapsto x + y$ is continuous from $A^{\mathbb{Z}} \times A^{\mathbb{Z}}$ to $A^{\mathbb{Z}}$. Then the function f is (uniformly) continuous on $A^{\mathbb{Z}}$.
For any(x, y) in $A^{\mathbb{Z}} \times A^{\mathbb{Z}}$, consider the p^{-j} neighbourhood of $x + y$ in $A^{\mathbb{Z}}$. Consider any x' in $A^{\mathbb{Z}}$ with $d(x, x') < p^{-j}$, and any y' in $A^{\mathbb{Z}}$ with $d(y, y') < p^{-j}$. The coordinates of x are same as those of x' for $-j,\ -j+1\ ,\ -j+2,\ \cdots 0,\ \cdots\ ,\ j-1,\ j$. The same is true for y and y'. Therefore clearly the coordinates of $x + y$ and $x' + y'$ agree from $-j$ to j.
So, $d(x, x') < p^{-j}$ and $d(y, y') < p^{-j} \Rightarrow d(x+y, x'+y') < p^{-j}$. Thus the operation $+$ is continuous.
Next, we verify that f is positively expansive.

Let $x = \ \cdots\ x_{-j}\ x_{-j+1}\ \cdots\ x_{-2}\ x_{-1}\ \overbrace{x_0}^{0^{th}}\ x_1\ x_2\ \cdots x_{j-1}\ x_j\ \cdots$

$\sigma^k(x) = \ \cdots\ x_{k-j}\ x_{k-j+1}\ \cdots\ x_{k-2}\ x_{k-1}\ \overbrace{x_k}^{0^{th}}\ x_{k+1}\ x_{k+2}\ \cdots x_{k+j-1}\ x_{k+j}\ \cdots$

Choose an $\epsilon < p^{-k}$. If $x \neq y$, let $d(x, y) = p^{-j}$. If $j \leq k$, take $n = 0$.
Let $d(x, y) = p^{-j}$, with $j > k$.
Then $x_i = y_i$ for $-j + 1 \leq i \leq j - 1$, and either $x_j \neq y_j$ or $x_{-j} \neq y_{-j}$. We denote $\sigma^k(x) + x$ by x' and $\sigma^k(y) + y$ by y'.

Case(i) Suppose that $x_j \neq y_j$. Then $\sigma^k(x)_{j-k} \neq \sigma^k(y)_{j-k}$ and $\sigma^k(x)_i = \sigma^k(y)_i$ for $0 \leq i < j - k$, on the right side. On the left side, $\sigma^k(x)_i = \sigma^k(y)_i$ for $(-j + 1) \leq i \leq 0$. Therefore, $x'_i = y'_i$ for $-j + 1 \leq i < j - k$, and $x'_{j-k} \neq y'_{j-k}$. When r is applied, $(r(x'))_i = (r(y'))_i$ for $-j + k < i \leq j - 1$ and $(r(x'))_{-j+k} \neq (r(y'))_{-j+k}$. Therefore $d(f(x), f(y))$ is $p^{(-j+k)}$.

Case (ii) If $x_j = y_j$, then $x_{-j} \neq y_{-j}$. Then $\sigma^k(x)_i = \sigma^k(y)_i$ for $0 \leq i \leq j - k$, on the right side. On the left side, $\sigma^k(x)_i = \sigma^k(y)_i$ for $-j \leq i \leq 0$, and $x_{-j} \neq y_{-j}$. Therefore $x'_i = y'_i$ for $(-j + 1) \leq i \leq (j - k)$ and $x'_{-j} \neq y'_{-j}$. When r is applied, $(r(x'))_i = (r(y'))_i$ for $(-j + k) \leq i \leq (j - 1)$ and $(r(x'))_j \neq (r(y'))_j$. For $i < (-j + k)$ we cannot conclude anything about $(r(x'))_i$ and $(r(y'))_i$. Therefore $d(f(x), f(y))$ is either $> p^{(-j)}$, or

$d(f(x), f(y)) = p^{(-j)}$ and we are back in case(i). Another application of f will increase the distance.

It follows that application of f once or twice increases the distances by a factor of at least p^k. Repeating this process, we can bring the distance to a value $\geq p^{-k} > \epsilon$. Thus the function is positively expansive. $\square$

We use the following result from [3](p. 40, Proposition 21)

Proposition 8.3. *For a positively expansive system (X, f), topological entropy $h(X, f) \ > \sup\{\frac{p_n}{n}\}$, where p_n is the number of points with period n.*

It follows that $h(X, f)$ is greater than one, because there is a fixed point 0.

9. A map that increases small distances

Let X be a metrizable space. We say that a self map f of X **increases small distances** if there is a compatible metric d, and there exists an $\epsilon > 0$ such that $0 < d(x, y) < \epsilon$ implies $d(f(x), f(y)) > d(x, y)$. We can further modify the above map to a map that increases small distances, and give both upper and lower bounds for entropy. We make use of the following result from [1](p. 627, Proposition 5.1).

Proposition 9.1. *Let $f : X \to X$ be a map of a compactum X with metric d. Suppose there exist positive numbers $\varepsilon > 0$ and $1 < \lambda_2 \leq \lambda_1$ such that if $x,\ y \in X$ and $0 < d(x, y) \leq \varepsilon$, then $\lambda_2 d(x, y) \leq d(f(x), f(y)) \leq \lambda_1 d(x, y)$. Then the following inequalities hold.*

$$D_d(X) log\lambda_2 \ \leq\ h(f) \ \leq\ D_d(X) log\lambda_1 \tag{11}$$

In the above map f given by $f(x) = r(\sigma^k(x) + x)$, to increase small distances, one or two applications of f are needed. If we replace the "reflection " r by r_1, which reflects x about the $(-1)^{th}$ position, rather than the 0^{th} position, we get a map that not only increases small distances, but satisfies the conditions of Proposition 9.1.

If $x \ = \ \cdots\ x_{-3}\ x_{-2}\ x_{-1}\ \overbrace{x_0}^{0^{th}}\ x_1\ x_2\ x_3\ \cdots$

$r_1(x) \ = \ \cdots\ x_2\ x_1\ x_0\ x_{-1}\ \overbrace{x_{-2}}^{0^{th}}\ x_{-3}\ x_{-4}\ x_{-5}\ \cdots$

As $r_1(x) = \sigma^2(r(x))$, r_1 is a homeomorphism.

Proposition 9.2. *Let $r_1 : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be given by $(r_1(x))_i = x_{-i-2}$. For $k > 2$, define $f(x) = r_1(\sigma^k(x) + x)$. Then for x and y with $0 < d(x,y) < p^{-k}$, $pd(x,y) \leq d(f(x),(y)) \leq p^k d(x,y)$.*

Proof. If $x = \cdots\ x_{-j}\ x_{-j+1}\ \cdots\ x_{-2}\ x_{-1}\ \overbrace{x_0}^{0^{th}}\ x_1\ x_2\ \cdots x_{j-1}\ x_j\ \cdots$

$\sigma^k(x) = \cdots\ x_{k-j}\ x_{k-j+1}\ \cdots\ x_{k-2}\ x_{k-1}\ \overbrace{x_k}^{0^{th}}\ x_{k+1}\ x_{k+2}\ \cdots x_{k+j-1}\ x_{k+j}\ \cdots$

Let $d(x,y) = p^{-j}$, with $j > k$.

Then $x_i = y_i$ for $-j+1 \leq i \leq j-1$, and either $x_j \neq y_j$ or $x_{-j} \neq y_{-j}$. We denote $\sigma^k(x) + x$ by x' and $\sigma^k(y) + y$ by y'.

Case(i) Suppose that $x_j \neq y_j$. Then $\sigma^k(x)_{j-k} \neq \sigma^k(y)_{j-k}$ and $\sigma^k(x)_i = \sigma^k(y)_i$ for $0 \leq i < j-k$, on the right side. On the left side, $\sigma^k(x)_i = \sigma^k(y)_i$ for $(-j+1) \leq i \leq 0$. Therefore, $x'_i = y'_i$ for $-j+1 \leq i < j-k$, and $x'_{j-k} \neq y'_{j-k}$. When r_1 is applied, $(r_1(x'))_i = (r_1(y'))_i$ for $-j+k-2 < i \leq j-3$ and $(r_1(x'))_{-j+k-2} \neq (r_1(y'))_{-j+k-2}$. Therefore $d(f(x),f(y))$ is at least $p^{(-j+k-2)}$.

Case(ii) If $x_j = y_j$, then $x_{-j} \neq y_{-j}$. So $\sigma^k(x)_i = \sigma^k(y)_i$ for $0 \leq i \leq (j-k)$, on the right side. On the left side, $\sigma^k(x)_i = \sigma^k(y)_i$ for $(-j+1) \leq i \leq 0$, and $x_{-j} \neq y_{-j}$. Therefore $x'_i = y'_i$ for $(-j+1) \leq i \leq (j-k)$ and $x'_{-j} \neq y'_{-j}$. When r_1 is applied, $(r_1(x'))_i = (r_1(y'))_i$ for $(-j+k-2) \leq i \leq (j-3)$ and $(r_1(x'))_{j-2} \neq (r_1(y'))_{j-2}$. Therefore $d(f(x),f(y))$ is at least $p^{(-j+2)}$.

In any case, we can conclude that $d(f(x),f(y)) \geq p^l d(x,y)$, where $l = min\{(k-2),2\} \geq 1$.

In both cases, we see that $d(f(x),f(y))$ cannot be more than $p^{(-j+k)}$. Hence $d(f(x),f(y)) \leq p^k d(x,y)$. □

For $X = A^{\mathbb{Z}}$, $B(x,p^{-m}) = \{y \in X | y_i = x_i\ for\ -m \leq\ i \leq m\}$. There are exactly p^{2m+1} balls of radius p^{-m}, and all of them are needed to cover $A^{\mathbb{Z}}$. Therefore

$$D_d(A^{\mathbb{Z}}) = \overline{\lim_{m\to\infty}} \frac{log(p^{2m+1})}{|log(p^{-m})|} = \overline{\lim_{m\to\infty}} \frac{(2m+1)log\ p}{mlog\ p} = 2$$

Now applying Proposition 9.1, with $\lambda_2 = p$ and $\lambda_1 = p^k$, we get

$$2\ log\ p \leq h(X,f) \leq 2k\ log\ p.$$

Remarks

- Only in section (6) we need p to be prime, to get a field $\mathbb{Q}_p$, so that $b = \frac{a}{1-p^k}$ makes sense.
- When p is prime, every rational number has a unique p-adic expansion, whose digits can be got by iterative algorithms.

Acknowledgements

We are very grateful to Prof. Hisao Kato of University of Tsukuba, Ibaraki, Japan for sending the paper [1].

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