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UPPER AND LOWER SOLUTIONS METHOD FOR HIGHER ORDER FRACTIONAL INITIAL VALUE PROBLEMS

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Abstract. This paper concerns the existence of solution to an initial fractional problem of arbitrary order. The main tools for this study are the lower ad upper solutions method and Schauder's fixed point Theorem.

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1. Introduction

Fractional calculus which is a generalization of the classical differentiation and integration, is proved to be a powerful tool to explore memory properties of materials. Due to the intensive development of the theory of fractional calculus, the study of nonlinear differential equations of fractional order attracted more attention and many papers and monographs are devoted to this subject, see [1-12]. The results on existence and uniqueness for fractional boundary value problem as well as initial value problems are proved by means of different methods such, the method of lower

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ad upper solutions, degree theory of coincidence of Mawhin, fixed point theorems, the monotone iterative technique...[1-8,12].

The purpose of this paper is to prove the existence of solutions for the following initial value problem (P):

$$(P)\begin{cases} D_{0+}^{q}u(t)=g(t)f\left(u(t),D_{0+}^{q-1}u(t)\right),0\leq t\leq 1,\\ u(0)=0,D_{0+}^{q-i}u(0)=0,i=1,...n-1, \end{cases}$$

where D_{0+}^{q} denotes Riemann-Liouville fractional derivative of order $q\geq 2$, n is the smallest integer greater than q, the given functions f and g are such that $g\in C\left([0,1],\mathbb{R}^{+}\right)$ and $f\in C\left(\mathbb{R}^{2},\mathbb{R}\right)$. Our investigations are based on the Schauder fixed point theorem and the method of lower and upper solutions. We obtain some results on the localization of the solution of the problem in question. Moreover, we give the explicit expressions of the lower and upper solutions. This method has been the main tools in the study of different problems on both fractional and ordinary differential equations, we refer to [1-3,6,7,9,12].

Let us recall some basic definitions and known results from [8,10,11].

The Riemann-Liouville fractional integral of order $q>0$ of a function g is defined by

$$I_{0+}^{q}g(t)=\frac{1}{\Gamma(q)}\int_{0}^{t}\frac{g(s)}{(t-s)^{1-q}}ds,$$

while its Riemann fractional derivative of order q is given by

$$D_{0+}^{q}g(t)=\left(\frac{d}{dt}\right)^{n}I_{0+}^{n-q}g(t).$$

We have the following properties on fractional calculus. Let $p,q\geq 0$, $f\in L_{1}[0,1]$, we have:

1- $I_{0+}^{q}D_{0+}^{q}f(t)=f(t)+\sum_{i=1}^{n}c_{n}t^{q-j}$, for almost everywhere on $[0,1]$.

2- If $p>0$, $m\in\mathbb{N}$, then $D^{m}D_{0+}^{p}f(t)=D_{0+}^{p+m}f(t)$.

3- If $p>0$ and $q\geq 0$, then $D_{0+}^{q}t^{p-1}(x)=\frac{\Gamma(p)}{\Gamma(p-q)}x^{p-q-1}$, $D_{0+}^{q}t^{q-j}(x)=0$, $j=1,2,...,n$.

Let

$$AC^{n}[0,1]=\left\{u\in C^{n-1}[0,1],u^{(n-1)}\ absolutely\ continuous\ function\ on\ [0,1]\right\}.$$

By lower and upper solutions of problem (P), we mean respectively, two functions α, $\beta\in AC^{n}[0,1]$ satisfying:

a) $D_{0^+}^{q}\alpha(t) \leq g(t) f\left(\alpha(t), D_{0^+}^{q-1}\alpha(t)\right)$, for $0 < t < 1$,
$\alpha(0) \leq 0$, $D_{0^+}^{q-i}\alpha(0) \leq 0, i = 2, ..., n-1$.
b) $D_{0^+}^{q}\beta(t) \geq g(t) f\left(\beta(t), D_{0^+}^{q-1}\beta(t)\right)$, for $0 < t < 1$,
$\beta(0) = 0$, $D_{0^+}^{q-i}\beta(0) \geq 0, i = 0, 1, ...n-1$.

2. Existence of solution

Denote by (P1) the corresponding boundary value problem of order $(q-1)$, that is

$$(P1)\begin{cases} D_{0^+}^{q-1}u(t) = v(t), 0 \leq t \leq 1 \\ u(0) = 0, D_{0^+}^{q-1-i}u(0) = 0, i = 1, ..., n-2. \end{cases}$$

By using the initials conditions, we obtain the solution of problem (P1)

$$u(t) = I_{0^+}^{q-1}v(t) = \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2} v(s)\, ds.$$

Define the operator T on E by

$$Tv(t) = \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2} v(s)\, ds = I_{0^+}^{q-1}v(t), t \in [0,1].$$

where $E = C([0,1], \mathbb{R})$ is equipped with the uniform norm $||u|| = \max_{t\in[0,1]} |u(t)|$. Since $u(t) = Tv(t)$ and $D_{0^+}^{q-1}u(0) = 0$, then problem (P) is equivalent to the following first order initial value problem denoted by (P2)

$$(P2)\begin{cases} v'(t) = g(t) f(Tv(t), v(t)), 0 \leq t \leq 1 \\ v(0) = 0 \end{cases}$$

Next, we give somme Lemmas that will be used in the sequel.

Lemma 1. If there exists a constant $A \geq 0$ such that $f(\theta, \sigma) \leq A$, for $0 \leq t \leq 1$, $0 \leq \theta \leq \frac{A\|g\|}{\Gamma(q)}$ and $0 \leq \sigma \leq A\|g\|$, then problem (P) has an upper solution.

Proof. Setting $\varphi(t) = A\int_0^t g(s)\, ds$, it yields

$$\begin{aligned} 0 &\leq T\varphi(t) = \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}\left(A\int_0^s g(r)\, dr\right) ds \\ &= \frac{A}{\Gamma(q)}\int_0^t (t-r)^{q-1} g(r)\, dr \leq \frac{A\|g\|}{\Gamma(q)}, \end{aligned}$$

thus $\varphi'(t) - g(t) f(T\varphi(t), \varphi(t)) \geq 0$ and $\varphi(0) \geq 0$. As a consequence we have $\beta(t) = T\varphi(t)$ is an upper solution of problem (P). □

Lemma 2. If there exists a constant $B \leq 0$, such that $f(\theta, \sigma) \geq B$, for $0 \leq t \leq 1$, $\frac{B\|g\|}{\Gamma(q)} \leq \theta \leq 0$ and $B\|g\| \leq \sigma \leq 0$, then problem (P) has a lower solution.

Proof. Setting $\psi(t) = B\int_0^t g(s)\,ds$, it yields

$$\begin{aligned} 0 &\geq T\psi(t) = \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}\left(B\int_0^s g(r)\,dr\right)ds \\ &= \frac{B}{\Gamma(q)}\int_0^t (t-r)^{q-1} g(r)\,dr \geq \frac{B\|g\|}{\Gamma(q)}, \end{aligned}$$

hence, $\psi'(t) - f(t, T\psi(t), \psi(t)) \leq 0$ and $\psi(0) \leq 0$, consequently, $\alpha(t) = T\psi(t)$ is a lower solution of problem (P). □

Lemma 3. Assume that hypotheses of Lemmas 1 and 2 are satisfied, then

$$\alpha(t) \leq \beta(t), D_{0^+}^{q-1}\beta(t) \geq D_{0^+}^{q-1}\alpha(t), 0 \leq t \leq 1.$$

Proof. Since $\beta(t) = T\varphi(t)$ and $\alpha(t) = T\psi(t)$ are upper and lower solutions of problem (P) respectively, then from

$$\begin{aligned} \beta(t) &= \frac{A}{\Gamma(q)}\int_0^t (t-r)^{q-1} g(r)\,dr \geq 0, \\ \alpha(t) &= \frac{B}{\Gamma(q)}\int_0^t (t-r)^{q-1} g(r)\,dr \leq 0, \end{aligned}$$

we get

$$D_{0^+}^{q-1}\beta(t) = \varphi(t) = A\int_0^t g(s)\,ds \geq B\int_0^t g(s)\,ds = \psi(t) = D_{0^+}^{q-1}\alpha(t),$$

that completes the proof. □

Let us introduce the operator $F : C[0,1] \to C[0,1]$ as

$$\begin{aligned} (Fv)(t) &= g(t) f(T(\min[\varphi, (\max(v, \psi))]), \min[\varphi, (\max(v, \psi))]), \\ 0 &\leq t \leq 1, \end{aligned}$$

and define the modified problem (P3) by

$$(P3)\begin{cases} v'(t) = (Fv)(t), 0 \leq t \leq 1 \\ v(0) = 0. \end{cases}$$

We have the following result

Lemma 4. Assume that v is a solution of problem (P3). Then $u = Tv$ is a solution of problem (P) satisfying

$$\begin{aligned} \alpha(t) &\leq u(t) \leq \beta(t), \\ D_{0^+}^{q-1}\alpha(t) &\leq D_{0^+}^{q-1}u(t) \leq D_{0^+}^{q-1}\beta(t), 0 \leq t \leq 1. \end{aligned}$$

Proof. Denote by v a solution of problem (P3) and let us prove that $\psi(t) \leq v(t) \leq \varphi(t)$, for $t \in [0,1]$. If we suppose the contrary, then there exists $\tau_1 \in (0,1]$ such that $v(\tau_1) > \varphi(\tau_1)$. Put $\omega(t) = v(t) - \varphi(t)$, from the initial conditions $v(0) = \varphi(t) = 0$, it yields $\omega(0) = 0$. In view of the continuity of ω, we can find $\tau_2 \in [0, \tau_1)$ and $\tau_3 \in [\tau_1, 1]$ such that $\omega(\tau_2) = 0$ and $\omega(t) \geq 0$, $t \in [\tau_2, \tau_3]$. Furthermore, by differentiating and taking into account the definition of lower and upper solutions, we get $\omega'(t) = v'(t) - \varphi'(t) = g(t) f(T(\min[\varphi, (\max(v, \psi))]), \min[\varphi, (\max(v, \psi))]) - D_{0^+}^{q}\beta(t) \leq 0$, $t \in [\tau_2, \tau_3]$, then ω is decreasing on $[\tau_2, \tau_3]$. Since $\omega(\tau_2) = 0$, then $v(t) \leq \varphi(t)$, $t \in [\tau_2, \tau_3]$, that leads to a contradiction. Analogously, we show $\psi(t) \leq v(t)$, $t \in [0,1]$. Now, we see that $v'(t) = (Fv)(t) = g(t) f(Tv(t), v(t))$, hence v is a solution of (P2) and consequently $u = Tv$ is a solution of (P). To get the localization of the solution u, it suffices to apply the monotony of the operator T, that is $T\psi(t) \leq Tv(t) \leq T\varphi(t)$, $t \in [0,1]$ hence $\alpha(t) \leq u(t) \leq \beta(t)$ and $D_{0^+}^{q-1}\alpha(t) \leq D_{0^+}^{q-1}u(t) \leq D_{0^+}^{q-1}\beta(t)$, $t \in [0,1]$. □

Next we give the main result on the existence of solution for problem (P):

Theorem 5. If there exist two constants A and B such that $A \geq 0$, $B \leq 0$ and $A \geq |B|$ and assume that the following conditions are fulfilled

$(C1)$- $f(\theta, \sigma) \leq A$, for $0 \leq t \leq 1$, $0 \leq \theta \leq \frac{A\|g\|}{\Gamma(q)}$ and $0 \leq \sigma \leq A\|g\|$,

$(C2)$- $f(\theta, \sigma) \geq B$, for $0 \leq t \leq 1$, $\frac{B\|g\|}{\Gamma(q)} \leq \theta \leq 0$ and $B\|g\| \leq \sigma \leq 0$,

then there exists at least one solution u for problem (P) satisfying

$$\begin{aligned} \alpha(t) &\leq u(t) \leq \beta(t), \\ D_{0^+}^{q-1}\alpha(t) &\leq D_{0^+}^{q-1}u(t) \leq D_{0^+}^{q-1}\beta(t), 0 \leq t \leq 1. \end{aligned}$$

Proof. Let $R : C[0,1] \to C[0,1]$, be the operator

$$Rv(t) = \int_0^t (Fv)(s)\, ds, 0 \leq t \leq 1.$$

Let us remark that a fixed point v of R is a solution of (P3) and then it satisfies $\psi(t) \leq v(t) \leq \varphi(t)$, for $t \in [0,1]$. Set $\Omega = \{v \in C[0,1], \|v\| \leq A\|g\|\}$ and let

$v \in \Omega$. Remarking that φ and $\psi \in \Omega$ and taking condition (C1) into account, we obtain

$$|Rv(t)| = \left| \int_0^t (Fv)(s)\, ds \right|$$

$$\leq \int_0^t |g(s) f(T(\min[\varphi, (\max(v,\psi))])(s), \min[\varphi, (\max(v,\psi))(s)])|\, ds \leq A\|g\|,$$

that implies the uniform boundedness of $R(\Omega)$ and that $R(\Omega) \subset \Omega$. Let $0 \leq t_1 < t_2 \leq 1$, then we have

$$\begin{aligned} & |Rv(t_1) - Rv(t_2)| \\ \leq & \int_{t_1}^{t_2} |g(s) f(T(\min[\varphi, (\max(v,\psi))])(s), \min[\varphi, (\max(v,\psi))(s)])|\, ds \\ \leq & A\|g\|(t_2 - t_1), \end{aligned}$$

from which we conclude the equicontinuity of $R(\Omega)$. Thanks to Arzela-Ascoli Theorem, we deduce that R is completely continuous. From Schauder fixed point theorem we conclude that R has a fixed point $v \in \Omega$, and so $u = Tv$ is a solution of (P). By Lemma 4 we get $\alpha(t) \leq u(t) \leq \beta(t)$ and $D_{0^+}^{q-1}\alpha(t) \leq D_{0^+}^{q-1}u(t) \leq D_{0^+}^{q-1}\beta(t)$, $0 \leq t \leq 1$. This achieves the proof. □

Example 6. Let us choose the functions f and g in problem (P) as

$$\begin{aligned} f\left(u(t), D_{0^+}^{q-1}u(t)\right) &= \frac{\Gamma(q)}{4}u(t) + \frac{1}{4}D_{0^+}^{q-1}u(t), \\ g(t) &= 2t, \end{aligned}$$

then $f(\theta, \sigma) = \Gamma(q)\frac{\theta}{4} + \frac{\sigma}{4}$. For $A = 1$ and $B = -1$ we can see that all assumptions of Theorem 5 are satisfied. Thus there exists at least one solution u for problem (P) verifying $\frac{-2t^{q+1}}{\Gamma(q+2)} = \alpha(t) \leq u(t) \leq \beta(t) = \frac{2t^{q+1}}{\Gamma(q+2)}$.

If we choose $A = 1$ and $B = 0$, then problem (P) has at least one positive solution.

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