

ON SYMMETRIES OF GENERALIZED ROBERTSON-WALKER SPACETIMES AND APPLICATIONS

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ABSTRACT. The purpose of the present article is to study and characterize several types of symmetries of generalized Robertson-Walker spacetimes. Conformal vector fields, curvature and Ricci collineations are studied. Many implications for existence of these symmetries on generalized Robertson-Walker spacetimes are obtained. Finally, Ricci solitons on generalized Robertson-Walker spacetimes admitting conformal vector fields are investigated.

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1. AN INTRODUCTION

Robertson-Walker spacetimes have been extensively studied in both mathematics and physics for a long time [6, 12, 14, 21, 22]. This family of spacetimes is a

very important family of cosmological models in general relativity[8]. A generalized $(n + 1)$ –dimensional Robertson-Walker (GRW) spacetime is a warped product manifold $I \times_f M$ where M is an n –dimensional Riemannian manifold without any additional assumptions on its fiber. The family of generalized Robertson-Walker spacetimes widely extends the classical Robertson-Walker spacetimes $I \times_f S_\kappa$ where S_κ is a 3–dimensional Riemannian manifold with constant curvature κ .

The study of spacetime symmetries is of great interest in both mathematics and physics. The existence of some symmetries in a spacetime is helpful in solving Einstein field equation and in providing further insight to conservative laws of dynamical systems(see [13] one of the best references for 4–dimensional spacetime symmetries). Although considerable research has been devoted to the study of symmetries of classical 4–dimensional spacetimes[13, 17, 19, 25], rather less attention has been paid to the study of symmetries of generalized n –dimensional spacetimes.

The increasing interest in generalized Robertson-Walker spacetimes has heightened the need for the study of their symmetries. However, as far as we know, there is no study on generalized Robertson-Walker spacetimes investigating neither conformal vector fields nor different types of collineations up to this paper in which we intend to fill this gap by providing many answers of the following questions: Under what conditions is a vector field on $I \times_f M$ a conformal vector field or a certain collineation? What does the fiber M inherit from a generalized Robertson-Walker spacetime $I \times_f M$ admitting a conformal vector field or a collineation? The main purpose of the current article is to study and explore both conformal vector fields and collineations on generalized Robertson-Walker spacetimes. In 2006, Grigori Perelman used Ricci soliton to solve the long standing Poincaré conjecture posed in 1904. Since then an enormous amount of works has continued to study Ricci soliton. Accordingly, we study Ricci solitons on generalized Robertson-Walker spacetimes admitting a potential conformal vector field. We get some partial answers of the question: How the geometry of generalized Robertson-Walker spacetimes admitting Ricci soliton structure is affected?

This paper is structured as follows. The next section presents some connection and curvature related formulas of generalized Robertson-Walker spacetimes

that are needed. Section 3 presents a study of conformal vector fields on generalized Robertson-Walker spacetimes. Ricci solitons on generalized Robertson-Walker spacetimes admitting a potential conformal vector field are considered. Finally, in Section 4, we investigated curvature and Ricci collineations on generalized Robertson-Walker spacetimes.

2. PRELIMINARIES

First, we want to fix some definitions and concepts. The warped product $M_1 \times_f M_2$ of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) is the product manifold $M_1 \times M_2$ equipped with the metric tensor

$$(2.1) \quad g = \pi_1^*(g_1) \oplus (f \circ \pi_1)^2 \pi_2^*(g_2)$$

with a smooth function $f : M_1 \rightarrow (0, \infty)$ where $\pi_i : M_1 \times_f M_2 \rightarrow M_i$ is the natural projection map of the Cartesian product $M_1 \times M_2$ onto M_i , $i = 1, 2$ and $*$ denotes the pull-back operator on tensors. The factors (M_1, g_1) and (M_2, g_2) are usually called the base manifold and fiber manifold respectively while f is called as the warping function[7, 20]. In particular, if $f = 1$, then $M_1 \times_1 M_2 = M_1 \times M_2$ is the usual Cartesian product manifold. It is clear that the submanifold $M_1 \times \{q\}$ is isometric to M_1 for every $q \in M_2$. Moreover, $\{p\} \times M_2$ is homothetic to M_2 for every $p \in M_1$. Throughout this article we refer to these factor submanifolds as M_1 and M_2 . The lift of a vector field $X_i \in \mathfrak{X}(M_i)$ to $\mathfrak{X}(M_1 \times M_2)$ is the unique vector field in $\mathfrak{X}(M_1 \times M_2)$ that is π_i -related to X_i in $\mathfrak{X}(M_i)$ and π_j -related to zero vector field in $\mathfrak{X}(M_j)$, $i \neq j$. Thus a vector field X_i in $\mathfrak{X}(M_i)$ is identified with the horizontal or the vertical vector field on $M_1 \times M_2$ that is π_i -related to X_i and π_j -related to zero vector field in $\mathfrak{X}(M_j)$. We use the same notation for a vector field and for its lift to the product manifold. Also, a function ω_i on M_i will be identified with $\omega_i \circ \pi_i$. Note that we use the same notation for the vector field $\nabla^i \omega_i$ and for its lift to $\mathfrak{X}(M_1 \times M_2)$.

Generalized Robertson-Walker spacetimes are well-known examples of warped product spaces. A generalized Robertson-Walker spacetime is often pictured in the form of warped product $\bar{M} = I \times_f M$ with fiber (M, g) any n -dimensional Riemannian manifold and base an open connected subinterval $(I, -dt^2)$ of the real

line $\mathbb{R}$ endowed with the metric

$$(2.2) \quad \bar{g} = -dt^2 \oplus f^2 g,$$

where dt^2 is the Euclidean metric on I .

The family of generalized Robertson-Walker spacetimes $\bar{M} = I \times_f M$ widely extends the classical Robertson-Walker spacetimes $I \times_f S_k$ where $M = S_k$ is a 3-dimensional Riemannian manifold of constant sectional curvature k . The warping function $f(t)$ is sometimes called the scale factor. This factor tells us how big is the space-like slice at sometime t .

For example the $(n+1)$ -dimensional spherically symmetric Friedmann-Robertson-Walker metric is given by

$$ds^2 = -dt^2 + f^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_{n-1}^2 \right),$$

where the spherical sector is given by $d\Omega_{n-1}^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \dots + \sin^2 \theta_{n-2} d\theta_{n-1}^2$ [15]. The Einstein field equation for $(n+1)$ -dimensional spacetime is given by

$$\text{Ric} - \frac{r}{2}g = k_n T,$$

where k_n is the multidimensional gravitational constant[15].

The following results are special cases of similar results on warped product manifolds[7, 20, 23, 24]. Let $\bar{M} = I \times_f M$ be a generalized Robertson-Walker spacetime equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2 g$. Then the Levi-Civita connection $\bar{D}$ on $\bar{M}$ is

$$(2.3) \quad \begin{aligned} \bar{D}_{\partial_t} \partial_t &= 0 & \bar{D}_{\partial_t} X &= \bar{D}_X \partial_t = \dot{f} X \\ \bar{D}_X Y &= D_X Y + f \dot{f} g(X, Y) \partial_t \end{aligned}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$ where D is the Levi-Civita connection on M and dots indicate differentiation with respect to t . The curvature tensor $\bar{R}$ of $\bar{M}$ is given by

$$(2.4) \quad \begin{aligned} \bar{R}(\partial_t, \partial_t) \partial_t &= \bar{R}(\partial_t, \partial_t) X = \bar{R}(X, Y) \partial_t = 0 \\ \bar{R}(X, \partial_t) \partial_t &= -\ddot{f} X & \bar{R}(\partial_t, X) Y &= f \ddot{f} g(X, Y) \partial_t \\ \bar{R}(X, Y) Z &= R(X, Y) Z + \dot{f}^2 [g(X, Z) Y - g(Y, Z) X], \end{aligned}$$

where R is curvature tensor of M . Finally, the Ricci curvature tensor $\bar{\text{Ric}}$ on $\bar{M}$ is

$$(2.5) \quad \begin{aligned} \bar{\text{Ric}}(\partial_t, \partial_t) &= \frac{-n\ddot{f}}{f} & \bar{\text{Ric}}(X, \partial_t) &= 0 \\ \bar{\text{Ric}}(X, Y) &= \text{Ric}(X, Y) - f^\diamond g(X, Y), \end{aligned}$$

where $f^\diamond = -f\ddot{f} - (n-1)\dot{f}^2$. $\bar{M}$ is Einstein with $\bar{\text{Ric}} = \left(\frac{\bar{r}}{n+1}\right)\bar{g}$ if and only if M is Einstein with $\text{Ric} = \left(\frac{r}{n}\right)g$ and the f satisfies

$$(2.6a) \quad \ddot{f} = \frac{\bar{r}}{n(n+1)}f$$

$$(2.6b) \quad \frac{r}{n} = (n-1)\left(f\ddot{f} - \dot{f}^2\right).$$

3. CONFORMAL VECTOR FIELDS ON GRW SPACETIMES AND APPLICATIONS

In this section, we investigate several types of symmetries of generalized Robertson-Walker spacetimes. Necessary and sufficient conditions are derived for a generalized Robertson-Walker spacetime to admit a conformal vector field. Then Ricci soliton on GRW admitting a conformal vector field is considered.

Now, we will recall the definitions of conformal vector fields on an arbitrary pseudo-Riemannian manifold. Let (M, g, D) be a pseudo-Riemannian manifold with metric g where D is the Levi-Civita connection on M . A vector field $\zeta \in \mathfrak{X}(M)$ is called a Killing vector field if $\mathcal{L}_\zeta g = 0$. By using symmetry of $(\mathcal{L}_\zeta g)(X, Y)$, it is easy to show that ζ is a Killing vector field if and only if $g(D_X \zeta, X) = 0$ for any vector field $X \in \mathfrak{X}(M)$. A pseudo-Riemannian n -dimensional manifold has at most $n(n+1)/2$ independent Killing vector fields. The symmetry generated by Killing vector fields is called isometry. A pseudo-Riemannian manifold which admits a maximum such symmetry has a constant sectional curvature. A vector field ζ is called a conformal vector field if $\mathcal{L}_\zeta g = \rho g$ for some smooth function $\rho : M \rightarrow \mathbb{R}$. ζ is called homothetic if ρ is constant and Killing if $\rho = 0$. Also, ζ is called a concircular vector field if $D_X \zeta = \rho X$ for any vector field $X \in \mathfrak{X}(M)$ [9]. Let $\zeta \in \mathfrak{X}(M)$ be a concircular vector field on M , then $\mathcal{L}_\zeta g(X, Y) = 2\rho g(X, Y)$ i.e. ζ is a conformal vector field with conformal factor 2ρ . A concircular vector field is a parallel vector field if $\rho = 0$. Moreover, for a constant factor ρ , we have $R(X, Y)\zeta = 0$.

3.1. Conformal vector fields. Let $\bar{M} = I \times_f M$ be a generalized Robertson-Walker spacetime equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2g$. Suppose that $h\partial_t, x\partial_t, y\partial_t \in \mathfrak{X}(I)$ and $\zeta, X, Y \in \mathfrak{X}(M)$, then

$$(3.7) \quad (\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g})(\bar{X}, \bar{Y}) = -2\dot{h}xy + f^2(\mathcal{L}_{\zeta}g)(X, Y) + 2hf\dot{f}g(X, Y),$$

where $\bar{\zeta} = h\partial_t + \zeta$, $\bar{X} = x\partial_t + X$ and $\bar{Y} = y\partial_t + Y$. An important consequence of this proposition is the following.

Theorem 1. *A vector field $\bar{\zeta} = h\partial_t + \zeta$ on $\bar{M} = I \times_f M$ is conformal if and only if ζ is a conformal vector field on M with factor $\rho = 2\left(\dot{h} - \frac{hf}{f}\right)$. Moreover, the conformal factor of $\bar{\zeta}$ is $2\dot{h}$.*

Proof. Let $\bar{\zeta} = h\partial_t + \zeta$ be a conformal vector field on $\bar{M} = I \times_f M$ with factor $\bar{\rho}$, then

$$-\bar{\rho}xy + \bar{\rho}f^2g(X, Y) = -2\dot{h}xy + f^2(\mathcal{L}_{\zeta}g)(X, Y) + 2hf\dot{f}g(X, Y)$$

for any vector fields $\bar{X} = x\partial_t + X$ and $\bar{Y} = y\partial_t + Y$. This equation yields $\bar{\rho} = 2\dot{h}$ and hence

$$(\mathcal{L}_{\zeta}g)(X, Y) = \left(\bar{\rho} - \frac{2hf}{f}\right)g(X, Y),$$

i.e. ζ is a conformal vector field on M with conformal factor $\rho = 2\dot{h} - \frac{2hf}{f}$. The converse is direct. $\square$

Corollary 1. *Let $\bar{M} = I \times_f M$ be a generalized Robertson-Walker spacetime equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2g$. Then, $\bar{\zeta} = a\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ is a Matter collineation on $\bar{M}$ if $\zeta \in \mathfrak{X}(M)$ is a Killing vector field on M .*

It is well-known that if X and Y are conformal vector fields on $\bar{M}$ with $X = \mu Y$ for some smooth function μ , then μ is constant. Thus the above theorem represents a good characterization to both time-like and space-like conformal vector fields on $\bar{M}$. This result reveals that the dimension of the conformal algebra $C(\bar{M})$ of $\bar{M}$ is at least $r + 1$ where r is the dimension of $C(M)$. Suppose that M is maximally symmetric, then

$$\frac{n^2 + 3n + 4}{2} \leq \dim C(\bar{M}) \leq \frac{(n + 2)(n + 3)}{2}.$$

Let $\bar{\zeta} = h\partial_t + \zeta$ be a Killing vector field on a generalized Robertson-Walker spacetime $\bar{M}$ and let $r = (1/2)\bar{g}(\bar{\zeta}, \bar{\zeta})$. Then

$$\bar{g}(\bar{\nabla}r, \bar{X}) = -\bar{g}(\bar{D}_{\bar{\zeta}}\bar{\zeta}, \bar{X})$$

for any vector field $\bar{X} \in \mathfrak{X}(\bar{M})$ i.e. $\bar{\nabla}r = -\bar{D}_{\bar{\zeta}}\bar{\zeta}$. Thus the Hessian definition with some computations yield

$$\bar{H}^r(\bar{X}, \bar{X}) = -\bar{R}(\bar{\zeta}, \bar{X}, \bar{\zeta}, \bar{X}) + \bar{g}(\bar{D}_{\bar{X}}\bar{\zeta}, \bar{D}_{\bar{X}}\bar{\zeta}),$$

where $\bar{R}(\bar{X}, \bar{Y}, \bar{X}, \bar{Y}) = \bar{g}(\bar{R}(\bar{X}, \bar{Y})\bar{X}, \bar{Y})$. Taking the trace of both sides imply

$$\bar{\Delta}r = -\bar{\text{Ric}}(\bar{\zeta}, \bar{\zeta}) + \bar{g}(\bar{D}_{\bar{\zeta}}\bar{\zeta}, \bar{D}_{\bar{\zeta}}\bar{\zeta}).$$

Assume that $\dot{f} = \dot{h} = 0$. Then

$$\bar{\Delta}r = -\text{Ric}(\zeta, \zeta) + f^2g(D\zeta, D\zeta).$$

Theorem 2. *Let $\bar{\zeta} = h\partial_t + \zeta$ be a Killing vector field on a generalized Robertson-Walker spacetime $\bar{M}$ and let $r = \frac{1}{2}\bar{g}(\bar{\zeta}, \bar{\zeta})$. Then*

$$\bar{\Delta}r = -\bar{\text{Ric}}(\bar{\zeta}, \bar{\zeta}) + \bar{g}(\bar{D}_{\bar{\zeta}}\bar{\zeta}, \bar{D}_{\bar{\zeta}}\bar{\zeta}).$$

Moreover, if $\dot{f} = \dot{h} = 0$, then

$$\bar{\Delta}r = -\text{Ric}(\zeta, \zeta) + f^2g(D\zeta, D\zeta).$$

The metric $\bar{g} = -dt^2 \oplus f^2g$ on $\bar{M}$ can be expressed as a conformal metric to a product one on $I \times M$. The metric $\bar{g}$ may be rewritten as follows

$$\bar{g} = f^2 \left(-\frac{1}{f^2}dt^2 + g \right) = f^2\tilde{g},$$

where $\tilde{g} = -d\tau^2 + g$ and $d\tau^2 = \frac{1}{f^2}dt^2$. Now, we consider the consequences of replacing $\bar{g}$ on $\bar{M}$ by $\tilde{g} = -d\tau^2 + g$. Similar discussions on 4-dimensional spacetimes and on doubly warped spacetimes are provided in [13, Chapter 11] and [10] respectively. It is clear that a conformal vector field ζ on (M, g) with conformal factor ρ is a conformal vector field on $(M, \bar{g})$ where conformal factor $\rho = \hat{\rho} + 2\zeta(\ln f)$. Likewise, a conformal vector field $\bar{\zeta} = h\partial_t + \zeta$ on $(\bar{M}, \bar{g})$ with conformal factor $\tilde{\rho}$ is a conformal vector field on $(\bar{M}, \tilde{g})$ with factor $\bar{\rho} = \tilde{\rho} + 2\zeta(\ln f)$. Using Theorem 1 in [1] together with this discussion we get the following.

Theorem 3. *Let $\bar{M} = I \times_f M$ be a generalized Robertson-Walker spacetime $(\bar{M}, \bar{g})$ and let $\tilde{g} = -d\tau^2 + \hat{g}$ and $d\tau^2 = \frac{1}{f^2}dt^2$. Then,*

- (1) a Killing vector field ζ on (M, g) is a Killing vector field on $(\bar{M}, \bar{g})$,
- (2) $(\bar{M}, \bar{g})$ admits a homothetic vector field if (M, g) admits a homothetic vector field,
- (3) each conformal vector field on $(\bar{M}, \bar{g})$ is a conformal vector field on (M, g) .

Theorem 4. Let $\bar{\zeta} = af\partial_t + \zeta$ be a vector field on a generalized Robertson-Walker spacetime $(\bar{M}, \bar{g})$. Assume that ζ is a Killing vector field on (M, g) . Then $\bar{\zeta}$ is a conformal vector field on $(\bar{M}, \bar{g})$.

Theorem 5. Let $\zeta \in \mathfrak{X}(M)$ be homothetic vector fields on (M, g) with factor c . Assume that $\zeta(f) = 0$. Then $\bar{\zeta} = c(at + b)\partial_t + 2a\zeta$ is a homothetic vector field on $(\bar{M}, \bar{g})$ with factor $2ac$.

The study of Killing vector fields of constant length is remarkable in that they correspond to isometries of constant displacement. Consequently, these vector fields are in relation with Clifford-Wolf translation in Riemannian manifolds[4]. In the following, Killing vector fields of constant length on a standard static spacetime $\bar{M} = I_f \times M$ are considered. The following result is an analogue of a similar result in Riemannian manifolds.

Theorem 6. A Killing vector field $\bar{\zeta} = h\partial_t + \zeta$ on a generalized Robertson-Walker spacetime $\bar{M} = I \times_f M$ equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2g$ has a constant length if and only if ζ satisfies

$$(3.8) \quad D_\zeta \zeta + \frac{2hf}{f} \dot{\zeta} = 0 \text{ and } h\dot{h} + f\dot{f}g(\zeta, \zeta) = 0.$$

Proof. Suppose that $\bar{\zeta}$ is a Killing vector field. Then

$$\bar{g}(\bar{D}_{\bar{\zeta}}\bar{\zeta}, \bar{Y}) = -\bar{g}(\bar{\zeta}, \bar{D}_{\bar{Y}}\bar{\zeta}).$$

But Equations (2.3) and (3.8) yield $\bar{D}_{\bar{\zeta}}\bar{\zeta} = 0$ and so

$$\bar{g}(\bar{\zeta}, \bar{D}_{\bar{Y}}\bar{\zeta}) = 0,$$

i.e. $\bar{g}(\bar{\zeta}, \bar{\zeta})$ is constant and so $\bar{\zeta}$ has a constant length. Conversely, if $\bar{\zeta}$ has constant length, then

$$\bar{g}(\bar{\zeta}, \bar{D}_{\bar{Y}}\bar{\zeta}) = -\bar{g}(\bar{D}_{\bar{\zeta}}\bar{\zeta}, \bar{Y}) = 0$$

for any vector field $\bar{Y} \in \mathfrak{X}(\bar{M})$ i.e. $\bar{D}_{\bar{\zeta}}\bar{\zeta} = 0$. Now, we can use Equation (2.3) to get the result. $\square$

Corollary 2. *Let $\bar{\zeta} = h\partial_t + \zeta$ be a Killing vector field of constant length on a generalized Robertson-Walker spacetime $\bar{M} = I \times_f M$ equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2g$. Then the flow lines of ζ are geodesics on M if and only if f is constant or $h = 0$.*

Theorem 7. *Let $\bar{\zeta} = h\partial_t + \zeta$ be a Killing vector field on a generalized Robertson-Walker spacetime $\bar{M} = I \times_f M$ equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2g$ and $\alpha(s)$, $s \in \mathbb{R}$, be a geodesic on $(\bar{M}, \bar{g})$ with tangent vector field $\bar{X} = x\partial_t + X$. Assume that $\dot{f} = 0$. Then h is constant and $\zeta \in \mathfrak{X}(M)$ is a Jacobi vector field along the integral curves of X .*

Theorem 8. *Let $\bar{\zeta} = h\partial_t + \zeta$ be a conformal vector field along a curve $\alpha(s)$ with unit tangent vector $\bar{V} = v\partial_t + V$ on a generalized Robertson-Walker spacetime $\bar{M} = I \times_f M$ equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2g$. Then the conformal factor ρ of $\bar{\zeta}$ is given by*

$$\rho = 2 \left[-v^2\dot{h} + hf\dot{f}\|V\|^2 + f^2g(D_V\zeta, V) \right].$$

Proof. Let $\bar{\zeta}$ be a conformal vector field with conformal factor ρ . Then the conformal factor ρ is given by

$$\rho = 2\bar{g}(\bar{D}_{\bar{V}}\bar{\zeta}, \bar{V}).$$

Thus

$$\begin{aligned} \rho &= 2\bar{g}(\bar{D}_{\bar{V}}\bar{\zeta}, \bar{V}) \\ &= 2 \left[-v^2\dot{h} + hf\dot{f}\|V\|^2 + f^2g(D_V\zeta, V) \right]. \end{aligned}$$

□

In the sequential, we study the structure of concircular vector fields on generalized Robertson-Walker spacetimes.

Theorem 9. *Let $\bar{\zeta} = h\partial_t + \zeta$ be a vector field on a generalized Robertson-Walker spacetime $\bar{M} = I \times_f M$ equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2g$. Then $\bar{\zeta}$ is a concircular vector field on $\bar{M}$ if and only if one of the following conditions holds:*

- (1) $h = af$ and $\zeta = 0$, or
- (2) ζ is a concircular vector field on M with factor $\rho = \dot{h}$ and f is constant.

Proof. Let $\bar{X} = x\partial_t + X$ be any vector field on $\bar{M}$. Then for any scalar function ρ we get that

$$\bar{D}_{\bar{X}}\bar{\zeta} - \rho\bar{X} = \left(x\dot{h} + f\dot{f}g(X, \zeta) - x\rho\right)\partial_t + \frac{x\dot{f}}{f}\zeta + \frac{h\dot{f}}{f}X + D_X\zeta - \rho X.$$

Suppose that $\bar{\zeta}$ is concircular on $\bar{M}$, then

$$\begin{aligned} x\dot{h} + f\dot{f}g(X, \zeta) - x\rho &= 0 \\ \frac{x\dot{f}}{f}\zeta + \frac{h\dot{f}}{f}X + D_X\zeta - \rho X &= 0. \end{aligned}$$

If $\dot{f}$ does not vanish, then $g(X, \zeta) = 0$ for all X i.e. $\zeta = 0$. Thus

$$\begin{aligned} x(\dot{h} - \rho) &= 0 \\ \left(\frac{h\dot{f}}{f} - \rho\right)X &= 0 \end{aligned}$$

and therefore $h = a\dot{f}$ for some constant a . However, if $\dot{f} = 0$, then we get that

$$\begin{aligned} x(\dot{h} - \rho) &= 0 \\ D_X\zeta - \rho X &= 0, \end{aligned}$$

i.e. ζ is a concircular vector field on M with factor $\rho = \dot{h}$.

Conversely, we have

$$\bar{D}_{\bar{X}}\bar{\zeta} = \left[x\dot{h} + f\dot{f}g(X, \zeta)\right]\partial_t + D_X\zeta + \left(x\frac{\dot{f}}{f}\right)\zeta + \left(h\frac{\dot{f}}{f}\right)X.$$

Finally, any one of the above conditions implies that $\bar{D}_{\bar{X}}\bar{\zeta} = \dot{h}\bar{X}$ and consequently $\bar{\zeta}$ is concircular on $\bar{M}$. $\square$

Theorem 10. *Let $\bar{\zeta} = h\partial_t + \zeta$ be a concircular vector field on a generalized Robertson-Walker spacetime $\bar{M} = I \times_f M$ equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2g$. Then*

$$\text{Ric}(\zeta, \zeta) = 0 \quad \text{and} \quad \kappa(X, \zeta) = 0$$

for any vector field $X \in \mathfrak{X}(M)$, where $\kappa(X, \zeta)$ is the sectional curvature of the plane spanned by X and ζ .

Proof. Using the above theorem we get that ζ is zero or concircular with factor $\dot{h}$. If $\zeta = 0$, then $\text{Ric}(\zeta, \zeta) = 0$ and $\kappa(X, \zeta) = 0$. Now suppose that ζ is concircular on M i.e.

$$D_X \zeta = \dot{h} X$$

for any vector field $X \in \mathfrak{X}(M)$. Let $e \in \mathfrak{X}(M)$ be a unit vector field, then

$$\begin{aligned} R(\zeta, e, \zeta, e) &= g(-D_\zeta D_e \zeta + D_e D_\zeta \zeta + D_{[\zeta, e]} \zeta, e) \\ &= \dot{h} g(-D_\zeta e + D_e \zeta + [\zeta, e], e) \\ &= 0. \end{aligned}$$

Thus $\text{Ric}(\zeta, \zeta) = 0$ and $\kappa(X, \zeta) = 0$. □

3.2. Ricci Soliton with a potential conformal vector field. A smooth vector field ζ on a Riemannian manifold (M, g) is said to define a Ricci soliton (M, g, ζ, λ) if it satisfies the soliton equation

$$(3.9) \quad \frac{1}{2} \mathcal{L}_\zeta g + \text{Ric} = \lambda g,$$

where $\mathcal{L}_\zeta$ is the Lie-derivative with respect to ζ , Ric is the Ricci tensor and λ is a constant. If $\zeta = \text{grad} u$, for a smooth function u on M , the Ricci soliton $(M, g, \zeta, \lambda) = (M, g, u, \lambda)$ is called a gradient Ricci soliton and the function u is called the potential function. The study of Ricci solitons was first introduced by Hamilton as fixed or stationary points of the Ricci flow in the space of the metrics on M modulo diffeomorphism and scaling. Gradient Ricci solitons are natural generalizations of Einstein manifolds [2, 3, 5, 11, 16, 18]. For $n \geq 3$, ζ is Killing if and only if (M, g) is Einstein. Let us take the Lie derivative of both sides of Equation (3.9) in direction of ζ , then we have

$$\frac{1}{2} \mathcal{L}_\zeta \mathcal{L}_\zeta g + \mathcal{L}_\zeta \text{Ric} = \lambda \mathcal{L}_\zeta g.$$

A vector field ζ is called 2-Killing if $\mathcal{L}_\zeta \mathcal{L}_\zeta g = 0$ [23]. Every Killing vector field is 2-Killing and the converse is not generally true. Assume that (M, g, ζ, λ) is a Ricci soliton and ζ is a 2-Killing vector field. Then ζ is a Killing vector field if it is a Ricci collineation.

Now, we consider a Ricci soliton structure on a generalized Robertson-Walker spacetime.

Theorem 11. *Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton where $\bar{M} = I \times_f M$ is a generalized Robertson-Walker spacetime and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$. Then*

- (1) $n\ddot{f} = -(\dot{h} - \lambda)f$, and
- (2) $(M, g, f^2\zeta, \lambda f^2 + f^\diamond - hf\dot{f})$ is a Ricci soliton whenever $\lambda f^2 + f^\diamond - hf\dot{f}$ is constant.

Proof. Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton, then

$$\frac{1}{2}(\mathcal{L}_{\bar{\zeta}}\bar{g})(\bar{X}, \bar{Y}) + \bar{\text{Ric}}(\bar{X}, \bar{Y}) = \lambda\bar{g}(\bar{X}, \bar{Y}),$$

where $\bar{X} = x\partial_t + X$ and $\bar{Y} = y\partial_t + Y$ are vector fields on $\bar{M}$. Let $X = Y = 0$, then

$$\left(\dot{h} + \frac{n}{f}\ddot{f}\right)g_I(x\partial_t, y\partial_t) = \lambda g_I(x\partial_t, y\partial_t).$$

Thus, $n\ddot{f} = -(\dot{h} - \lambda)f$. Now, let $x = y = 0$, then

$$\frac{1}{2}\left(f^2(\mathcal{L}_{\zeta}g)(X, Y) + 2hf\dot{f}g(X, Y)\right) + \text{Ric}(X, Y) - f^\diamond g(X, Y) = \lambda f^2 g(X, Y)$$

where $f^\diamond = -f\ddot{f} - (n-1)\dot{f}^2$. Simply, we may rewrite it as follows,

$$\frac{1}{2}f^2(\mathcal{L}_{\zeta}g)(X, Y) + \text{Ric}(X, Y) = (f^\diamond - hf\dot{f} + \lambda f^2)g(X, Y).$$

Thus $(M, g, f^2\zeta, \mu)$ is a Ricci soliton, where $\mu = \lambda f^2 + f^\diamond - hf\dot{f}$. □

Theorem 12. *Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton where $\bar{M} = I \times_f M$ is a generalized Robertson-Walker spacetime and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ be a conformal vector field on $\bar{M}$. Then (M, g) is an Einstein manifold with factor*

$$\mu = (n-1)(f\ddot{f} - \dot{f}^2).$$

Moreover, the conformal factor is $\lambda - \frac{n\ddot{f}}{f}$.

Proof. Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton where $\bar{M} = I \times_f M$ is a generalized Robertson-Walker spacetime and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ be a conformal vector field on $\bar{M}$. Then

$$\bar{\text{Ric}}(\bar{X}, \bar{Y}) = (\lambda - \rho)\bar{g}(\bar{X}, \bar{Y}).$$

Let $X = Y = 0$, then by using Equations (2.5), we get that

$$(3.10) \quad \lambda - \rho = \frac{n\ddot{f}}{f}.$$

Now, we let $x = y = 0$, then

$$\begin{aligned}\bar{\text{Ric}}(X, Y) &= (\lambda - \rho) \bar{g}(X, Y) \\ \text{Ric}(X, Y) &= [(\lambda - \rho) f^2 + f^\circ] g(X, Y).\end{aligned}$$

By using equation (3.10), we get that

$$\text{Ric}(X, Y) = (n - 1) (f\ddot{f} - \dot{f}^2) g(X, Y).$$

Then (M, g) is an Einstein manifold with factor $\mu = (n - 1) (f\ddot{f} - \dot{f}^2)$. $\square$

Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton where $\bar{M} = I \times_f M$ is a generalized Robertson-Walker spacetime and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ be a conformal vector field on $\bar{M}$. Then (M, g) is Ricci flat if f is constant. If $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ is a Killing vector field on $\bar{M}$, then $\lambda = \frac{n\ddot{f}}{f}$.

Theorem 13. *Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton where $\bar{M} = I \times_f M$ is a generalized Robertson-Walker spacetime and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$. Assume that $H^f = 0$ and (M, g) is Einstein with factor $-(n - 1)\dot{f}^2$. Then $\bar{\zeta}$ is conformal with factor 2λ .*

Proof. Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton where $\bar{M} = I \times_f M$ is a generalized Robertson-Walker spacetime and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$. Then

$$\frac{1}{2} (\mathcal{L}_{\bar{\zeta}} \bar{g})(\bar{X}, \bar{Y}) + \bar{\text{Ric}}(\bar{X}, \bar{Y}) = \lambda \bar{g}(\bar{X}, \bar{Y})$$

for any vector fields $\bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M})$. Equation (2.5) implies that

$$\begin{aligned}\bar{\text{Ric}}(\bar{X}, \bar{Y}) &= \text{Ric}(X, Y) - f^\circ g(X, Y) \\ &= \text{Ric}(X, Y) + (n - 1) \dot{f}^2 g(X, Y) = 0.\end{aligned}$$

Thus

$$(\mathcal{L}_{\bar{\zeta}} \bar{g})(\bar{X}, \bar{Y}) = 2\lambda \bar{g}(\bar{X}, \bar{Y})$$

i.e. $\bar{\zeta}$ is a conformal vector field with factor 2λ . $\square$

Theorem 14. *Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton where $\bar{M} = I \times_f M$ is a generalized Robertson-Walker spacetime and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ be a concircular vector field on $\bar{M}$ with factor one. Then (M, g) is Ricci flat if $f\ddot{f} - \dot{f}^2$ is constant.*

Proof. A concircular vector field is conformal, and so the above theorem implies that

$$(3.11) \quad \text{Ric}(X, Y) = (n-1) \left(f\ddot{f} - \dot{f}^2 \right) g(X, Y)$$

for any vector fields $\bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M})$. Since $\bar{\zeta}$ is concircular, Theorems (9) and (10) yield

$$\text{Ric}(\zeta, \zeta) = 0.$$

Therefore, for a constant factor $\mu = (n+1)f\ddot{f} - (n-1)\dot{f}^2$ we have

$$\begin{aligned} \text{Ric}(X, Y) &= \mu g(X, Y) \\ \text{Ric}(\zeta, \zeta) &= 0 \\ \mu &= 0, \end{aligned}$$

i.e. (M, g) is Ricci flat. □

Theorem 15. *Let $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ be a vector field on a generalized Robertson-Walker spacetime $\bar{M} = I \times_f M$. Then $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ is a Ricci soliton if*

- (1) ζ is a conformal vector field with conformal factor 2ρ ,
- (2) (M, g) is Einstein with factor μ , and
- (3) $\left(\dot{h} - \rho \right) f^2 = \mu + h\dot{f}f - (n-1)f\ddot{f} + (n-1)\dot{f}^2$

Moreover, $\lambda = \dot{h} + \frac{n\ddot{f}}{f}$.

Proof. Let $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$, then

$$\frac{1}{2} (\bar{\mathcal{L}}_{\bar{\zeta}} \bar{g}) (\bar{X}, \bar{Y}) = -\dot{h}xy + \frac{1}{2} f^2 (\mathcal{L}_{\zeta} g) (X, Y) + hf\dot{f}g(X, Y)$$

and

$$\bar{\text{Ric}}(\bar{X}, \bar{Y}) = \text{Ric}(X, Y) - xy \frac{n\ddot{f}}{f} - f^{\diamond} g(X, Y).$$

Since ζ is a conformal vector field with conformal factor 2ρ ,

$$\frac{1}{2} (\mathcal{L}_{\zeta} g) (\bar{X}, \bar{Y}) = -\dot{h}xy + \left(\rho f^2 + hf\dot{f} \right) g(X, Y)$$

and

$$\bar{\text{Ric}}(\bar{X}, \bar{Y}) = -xy \frac{n\ddot{f}}{f} + (\mu - f^{\diamond}) g(X, Y),$$

where $f^{\diamond} = -f\ddot{f} - (n-1)\dot{f}^2$. Thus

$$\begin{aligned} & \frac{1}{2} (\mathcal{L}_{\bar{\zeta}} \bar{g}) (\bar{X}, \bar{Y}) + \bar{\text{Ric}} (\bar{X}, \bar{Y}) \\ = & -xy \left(\dot{h} + \frac{n\ddot{f}}{f} \right) + f^2 \left(\rho + \frac{h\dot{f}}{f} + \frac{\mu - f^\diamond}{f^2} \right) g(X, Y). \end{aligned}$$

The last condition implies that

$$\dot{h} + \frac{n\ddot{f}}{f} = \rho + \frac{h\dot{f}}{f} + \frac{\mu - f^\diamond}{f^2} = \lambda.$$

Thus

$$\frac{1}{2} (\mathcal{L}_{\bar{\zeta}} \bar{g}) (\bar{X}, \bar{Y}) + \bar{\text{Ric}} (\bar{X}, \bar{Y}) = \lambda \bar{g} (\bar{X}, \bar{Y})$$

and the proof is complete. $\square$

4. CURVATURE AND RICCI COLLINEATIONS ON GRW

A Riemannian manifold M is said to admit a curvature collineation if the Lie derivative of the curvature tensor R vanishes in the direction of a vector field $\zeta \in \mathfrak{X}(M)$, that is $\mathcal{L}_\zeta R = 0$ where R is the Riemann curvature tensor. Likewise, M is said to admit a Ricci curvature collineation if there is a vector field $\zeta \in \mathfrak{X}(M)$ such that $\mathcal{L}_\zeta \text{Ric} = 0$ where Ric is the Ricci curvature tensor. It is clear that every Killing vector field is a curvature collineation and every curvature collineation is a Ricci curvature collineation. The converse is not generally true. Finally, a spacetime M is said to admit a matter collineation if there is a vector field $\zeta \in \mathfrak{X}(M)$ such that $\mathcal{L}_\zeta T = 0$ where T is the energy-momentum tensor. The Einstein's field equation (with cosmological constant) is given by

$$\text{Ric} - \frac{r}{2}g = \kappa T - \lambda g,$$

where r is the scalar curvature and λ is the cosmological constant[15]. Suppose that ζ is a Killing vector field, then $\mathcal{L}_\zeta T = 0$ i.e. ζ is a matter collineation field. Note that a matter collineation need not be a Killing vector field. Now, let us turn to curvature collineations. First, we present the following important proposition.

Proposition 1. *Let $\bar{\zeta} = h\partial_t + \zeta$ be a vector field on a generalized Robertson-Walker spacetime $\bar{M} = I \times_f M$ equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2g$. Then*

$$\begin{aligned}
(\bar{\mathcal{L}}_{h\partial_t} \bar{R})(\partial_t, \partial_t, \partial_t, \partial_t) &= (\bar{\mathcal{L}}_{h\partial_t} \bar{R})(X, \partial_t, \partial_t, \partial_t) = 0 \\
(\bar{\mathcal{L}}_{h\partial_t} \bar{R})(X, Y, \partial_t, \partial_t) &= (\bar{\mathcal{L}}_{h\partial_t} \bar{R})(X, Y, Z, \partial_t) = 0 \\
(\bar{\mathcal{L}}_{h\partial_t} \bar{R})(X, Y, Z, W) &= 0 \\
(\bar{\mathcal{L}}_{h\partial_t} \bar{R})(Y, \partial_t, \partial_t, W) &= -\left[h\dot{f}\ddot{f} + hf\ddot{f} + 2\dot{h}f\ddot{f}\right]g(Y, W) \\
(\bar{\mathcal{L}}_{\zeta} \bar{R})(\partial_t, \partial_t, \partial_t, \partial_t) &= (\bar{\mathcal{L}}_{\zeta} \bar{R})(X, \partial_t, \partial_t, \partial_t) = 0 \\
(\bar{\mathcal{L}}_{\zeta} \bar{R})(X, Y, \partial_t, \partial_t) &= (\bar{\mathcal{L}}_{\zeta} \bar{R})(X, Y, Z, \partial_t) = 0 \\
(\bar{\mathcal{L}}_{\zeta} \bar{R})(Y, \partial_t, \partial_t, W) &= -f\ddot{f}(\mathcal{L}_{\zeta}g)(Y, W)
\end{aligned}$$

$$\begin{aligned}
(\bar{\mathcal{L}}_{\zeta} \bar{R})(X, Y, Z, W) &= f^2(\mathcal{L}_{\zeta}R)(X, Y, Z, W) \\
&\quad + f^2\dot{f}^2[(\mathcal{L}_{\zeta}g)(X, Z)g(Y, W) + g(X, Z)(\mathcal{L}_{\zeta}g)(Y, W)] \\
&\quad - f^2\dot{f}^2[(\mathcal{L}_{\zeta}g)(Y, Z)g(X, W) + g(Y, Z)(\mathcal{L}_{\zeta}g)(X, W)],
\end{aligned}$$

where $X, Y, Z, W \in \mathfrak{X}(M)$.

The following theorem represents a characterization of curvature collineations on generalized Robertson-Walker spacetimes.

Theorem 16. *Let $\bar{M} = I \times_f M$ be a generalized Robertson-Walker spacetime equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2g$. Then,*

- (1) $\bar{\zeta} = h\partial_t$ is a curvature collineation on $\bar{M}$ if and only if $h\dot{f}\ddot{f} + hf\ddot{f} + 2\dot{h}f\ddot{f} = 0$.
- (2) $\bar{\zeta} = \zeta$ is a curvature collineation on $\bar{M}$ if ζ is a Killing vector field on M .

Theorem 17. *Let $\bar{\zeta} = h\partial_t + \zeta$ be a curvature collineation on a generalized Robertson-Walker spacetime $\bar{M} = I \times_f M$ equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2g$. Assume that $\Delta f \neq 0$. Then ζ is a Killing vector field on M .*

Now we will do the same job for Ricci collineations on generalized Robertson-Walker spacetimes

Proposition 2. *Let $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$, then*

$$\begin{aligned} (\bar{\mathcal{L}}_{\bar{\zeta}}\bar{\text{Ric}})(\bar{X}, \bar{Y}) &= -\frac{nxy}{f^2} \left(hf\ddot{f} - h\dot{f}\ddot{f} + 2\dot{h}f\ddot{f} \right) - h \left(f\ddot{f} + (2n-1)\dot{f}\ddot{f} \right) g(X, Y) \\ &\quad + (\mathcal{L}_{\zeta}\text{Ric})(X, Y) - f^{\diamond}(\mathcal{L}_{\zeta}g)(X, Y) \end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$.

The following results are immediate consequences of the above proposition.

Theorem 18. *Let $\bar{\zeta} = h\partial_t \in \mathfrak{X}(\bar{M})$ be a vector field on a generalized Robertson-Walker spacetime $\bar{M} = I \times_f M$. Assume that $H^f = 0$. Then, $\bar{\zeta}$ is a Ricci collineation on $\bar{M}$.*

Now, we consider the converse of the above result.

Theorem 19. *Let $\bar{\zeta} = h\partial_t \in \mathfrak{X}(\bar{M})$ be a Ricci collineation on a generalized Robertson-Walker spacetime $\bar{M} = I \times_f M$. Then one of the following conditions holds*

- (1) $H^f = 0$, or
- (2) $h = af^n$ for some constant a .

Proof. Suppose that $\bar{\zeta}$ is a Ricci collineation on $\bar{M}$. Then

$$\begin{aligned} (\bar{\mathcal{L}}_{\bar{\zeta}}\bar{\text{Ric}})(x\partial_t, y\partial_t) &= -\frac{nxy}{f^2} \left(hf\ddot{f} - h\dot{f}\ddot{f} + 2\dot{h}f\ddot{f} \right) = 0 \\ (\bar{\mathcal{L}}_{\bar{\zeta}}\bar{\text{Ric}})(X, Y) &= -h \left(f\ddot{f} + (2n-1)\dot{f}\ddot{f} \right) g(X, Y) = 0. \end{aligned}$$

The second equation yields

$$f\ddot{f} = -(2n-1)\dot{f}\ddot{f}$$

and so the first equation implies

$$2\dot{f} \left[nh\dot{f} - \dot{h}f \right] = 0,$$

which proves the result. □

Theorem 20. *Let $\bar{\zeta} = \zeta \in \mathfrak{X}(\bar{M})$ be a vector field on a generalized Robertson-Walker spacetime $\bar{M} = I \times_f M$. Assume that $f^{\diamond} = 0$. Then, $\bar{\zeta}$ is a Ricci collineation on $\bar{M}$ if and only if ζ is a Ricci collineation on M .*

A vector field $\zeta \in \mathfrak{X}(M)$ is called a conformal Ricci collineation if

$$(\mathcal{L}_\zeta \text{Ric})(X, Y) = \rho g(X, Y)$$

for some smooth function ρ on M .

Theorem 21. *Let $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ be a Ricci collineation on a generalized Robertson-Walker spacetime $\bar{M} = I \times_f M$. Then, ζ is a conformal Ricci collineation on M if and only if ζ is a conformal vector field on M .*

REFERENCES

- [1] P.S. Apostolopoulos, J.G. Carot, *Conformal symmetries in warped manifolds*, Journal of Physics: Conference Series, vol. 8, pp. 28–33, (2005).
- [2] A. Barros, E. Ribeiro Jr., *Some characterizations for compact almost Ricci solitons*, Proc. Amer. Math. Soc. Vol. 140, 1033–1040, (2012).
- [3] A. Barros, José N. Gomes, E. Ribeiro Jr. *A note on rigidity of the almost Ricci soliton*, Archiv der Mathematik, Vol. 100, Number 5, 481–490, (2013).
- [4] V.N. Berestovskii, Yu.G. Nikonorov, *Killing vector fields of constant length on Riemannian manifolds*, Siberian Mathematical Journal, Vol. 49, Number 3, pp 395–407, (2008).
- [5] J. Bernstein, T. Mettler, *Two-Dimensional Gradient Ricci Solitons Revisited*, Int Math Res Not, Vol. 2015, Number 1, Pp. 78–98, (2015).
- [6] A.L. Besse, *Einstein Manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008.
- [7] R.L. Bishop, B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. Vol. 145, 1–49, (1969).
- [8] B.Y. Chen, S.W. Wei, *Differential Geometry of Submanifolds of Warped Product Manifolds $I \times_f S^{m-1}(k)$* , Journal of Geometry, Vol. 91, 21–42, (2008).
- [9] B.-Y. Chen, *A Simple Characterization Of Generalized Robertson-Walker Spacetimes*, General Relativity and Gravitation, Vol. 46, 1833–9, (2014).
- [10] H.K. El-Sayied, S. Shenawy, N. Syied, *Conformal killing vector fields on doubly warped product manifolds and applications*, Advances in Mathematical Physics, Vol. 2016, Article ID 6508309, 11 pages, (2016).
- [11] M. Fernández-López, Eduardo García-Río, *Rigidity of shrinking Ricci solitons*, Mathematische Zeitschrift, Vol. 269, Number 1–2, pp 461–466, (2011).
- [12] J.L. Flores, M. Sánchez, *Geodesic Connectedness and Conjugate Points in GRW spacetimes*, J. Geom. Phys., Vol. 36, Number 3–4, 285–314, (2000).
- [13] G.S. Hall, *Symmetries and Curvature Structure in General Relativity*, World Scientific Publishing Co. Ltd, London, 2004.
- [14] V. G. Ivancevic, T.T. Ivancevic, *Applied Differential Geometry: A Modern Introduction*, World Scientific Publishing Co. Ltd, London, 2007.

- [15] A.A. Garcia, S. Carlip, *n-dimensional generalizations of the Friedmann–Robertson–Walker cosmology*, Physics Letters B, Vol. 645, 101–107, (2007).
- [16] P. Petersen, W. Wylie, *Rigidity of gradient Ricci solitons*, Pacific Journal of Mathematics, Vol. 241, Number 2, 329–345, (2009).
- [17] W. Kuhnel, H. Rademacher, *Conformal vector fields on pseudo-Riemannian spaces*, Journal of Geometry and its Applications, Vol. 7, 237–250, (1997).
- [18] O. Munteanu, N. Sesum, *On Gradient Ricci Solitons*, Journal of Geometric Analysis, Vol. 23, Number 2, pp 539–561, (2013).
- [19] K. Nomizu, *On local and global existence of Killing vector fields*, Ann. of Math. Vol. 72, Number 1, 105–120, (1960).
- [20] B. O’Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press Limited, London, 1983.
- [21] M. Sánchez, *On the Geometry of Generalized Robertson-Walker Spacetimes: geodesics*, Gen. Relativ. Gravitation, Vol. 30, Number 6, 915–932, (1998).
- [22] M. Sánchez, *On the Geometry of Generalized Robertson-Walker Spacetimes: Curvature and Killing fields*, J. Geom. Phys., Vol. 31, Number 1, 1–15, (1999).
- [23] S. Shenawy, B. Unal, *2–Killing vector fields on warped product manifolds*, International Journal of Mathematics, Vol. 26, Number 8, 1550065(17 pages), (2015).
- [24] S. Shenawy, B. Unal, *The W_2 –curvature tensor on warped product manifolds and applications*, International Journal of Geometric methods in Modern Physics, Vol. 13, Number 7 1650099 (14 pages), (2016).
- [25] M. Steller, *Conformal vector fields on spacetimes*, Ann Glob Anal Geom Vol. 29, 293–317, (2006).