

ON OSCULATING, NORMAL AND RECTIFYING BI-NULL CURVES IN $\mathbb{R}_3^6$

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ABSTRACT. In this paper we give the necessary and sufficient conditions for bi-null curves in $\mathbb{R}_3^6$ to be osculating, normal or rectifying curves in terms of their curvature functions.

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1. Introduction

In the Euclidean 3-space $\mathbb{R}^3$ there exist three classes of characterisirc curves, so called rectifying, normal and osculating curves. Rectifying curves are introduced by B. Y. Chen in [1] as a space curve whose position vector always lies in its rectifying plane. Here, the rectifying plane is spanned by the tangent vector $T(s)$ and the binormal vector $B(s)$. Thus, the position vector α of a rectifying curve in $\mathbb{R}^3$ satisfies the equation $\alpha(s) = \lambda(s)T(s) + \mu(s)B(s)$ for some differentiable

functions $\lambda(s)$ and $\mu(s)$. In Minkowski 3-space $\mathbb{R}_1^3$, rectifying curves have similar geometric properties as in $\mathbb{R}^3$. Some characterizations of spacelike, timelike and null rectifying curves, lying fully in $\mathbb{R}_1^3$, are given in [11]. Moreover, the definition of a rectifying curve is generalized to 4-dimensional Euclidean and Minkowski spaces in [5, 6, 10].

In analogy with the Euclidean case, a normal curve in $\mathbb{R}_1^3$ is defined in [3] as a curve whose position vector always lies in its normal plane. Some characterizations of spacelike, timelike and null normal curves, lying fully in $\mathbb{R}_1^3$, are given in [3, 4, 2]. In addition, normal curves in Minkowski space-time $\mathbb{R}_1^4$ are defined in [7] as a curve whose position vector always lies in its normal space $T^\perp$, which represents the orthogonal complement of the tangent vector field of the curve.

Similarly, osculating curves in $\mathbb{R}_1^4$ are defined in [8] as a curve whose position vector always lies in its osculating space, which represents the orthogonal complement of the first binormal or second binormal vectors field of the curve. Timelike, spacelike and null osculating curves are studied in [8] and [9].

On the other hand, the second named author (in [12]) gave the notion of bi-null curves in semi-Euclidean spaces $\mathbb{R}_2^n$ of index 2, together with the unique Frenet frame and the curvatures. He also discussed some properties of bi-null curves in terms of the curvatures. Recently we study bi-null curves in $\mathbb{R}_2^5$ and $\mathbb{R}_3^6$ ([13, 15]). Also we studied osculating, normal and rectifying bi-null curves in $\mathbb{R}_2^5$ ([14]).

In this paper, we characterize osculating bi-null Cartan curves in $\mathbb{R}_3^6$. We consider four types of osculating bi-null Cartan curves in $\mathbb{R}_3^6$ which we call as "first kind of osculating curve", "second kind of osculating curve", "third kind of osculating curve" and "fourth kind of osculating curve". Then we give the necessary and sufficient conditions for bi-null Cartan curves in $\mathbb{R}_3^6$ to be osculating curves in terms of their curvature functions for all types of osculating curves. In addition, we study normal bi-null Cartan curves and rectifying bi-null Cartan curves in $\mathbb{R}_3^6$.

2. Preliminaries

In this section, following [15], we recall the Frenet equations for bi-null Cartan curves in $\mathbb{R}_3^6$. Let $\mathbb{R}_3^6$ be the 6-dimensional semi-Euclidean space of index 3 with standard coordinate system $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 - dx_5^2 - dx_6^2.$$

We denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}_3^6$. Recall that a vector $v \in \mathbb{R}_3^6 \setminus \{0\}$ can be spacelike if $\langle v, v \rangle > 0$, timelike if $\langle v, v \rangle < 0$ and null (lightlike) if $\langle v, v \rangle = 0$. In particular, the vector $v = 0$ is said to be spacelike. The norm of a vector v is given by $\|v\| = \sqrt{|\langle v, v \rangle|}$. Two vectors v and w are said to be orthogonal, if $\langle v, w \rangle = 0$.

We say that a curve $\gamma(t)$ in $\mathbb{R}_3^6$ is a bi-null curve if $\text{span}\{\gamma'(t), \gamma''(t)\}$ is an isotropic 2-plane for all t . That is,

$$\langle \gamma'(t), \gamma'(t) \rangle = \langle \gamma'(t), \gamma''(t) \rangle = \langle \gamma''(t), \gamma''(t) \rangle = 0,$$

and $\{\gamma'(t), \gamma''(t)\}$ are linearly independent. The condition is independent of the choice of the parameter.

We say that a bi-null curve $\gamma(t)$ in $\mathbb{R}_3^6$ is parameterized by the bi-null arc if $\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \rangle = 1$. If a bi-null curve $\gamma(t)$ in $\mathbb{R}_3^6$ satisfies $\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \rangle \neq 0$, then by the anti-isometry of $\mathbb{R}_3^6$, we may assume that $\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \rangle > 0$, and we can see that

$$u(t) = \int_{t_0}^t \left\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \right\rangle^{1/6} dt$$

becomes the bi-null arc parameter.

Let us say that a bi-null curve $\gamma(t)$ in $\mathbb{R}_3^6$ with $\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \rangle > 0$ is a bi-null Cartan curve if $\{\gamma'(t), \gamma''(t), \gamma^{(3)}(t), \gamma^{(4)}(t), \gamma^{(5)}(t)\}$ are linearly independent for any t .

For a bi-null Cartan curve $\gamma(t)$ in $\mathbb{R}_3^6$ with bi-null arc parameter t , there exists a unique Frenet frame $\{L_1, L_2, N_1, N_2, W_1, W_2\}$ such that

$$\begin{aligned} \gamma' &= L_1, & L'_1 &= L_2, & L'_2 &= W_1, \\ W'_1 &= -k_0 L_2 - N_2, \\ (2.1) \quad N'_2 &= -k_1 L_1 - N_1 + k_0 W_1, \\ N'_1 &= k_1 L_2 + k_2 W_2, \\ W'_2 &= k_2 L_1 \end{aligned}$$

where N_1, N_2 are null, $\langle L_1, N_1 \rangle = \langle L_2, N_2 \rangle = 1$, $\{L_1, N_1\}$, $\{L_2, N_2\}$ and $\{W_1, W_2\}$ are mutually orthogonal, $\{W_1, W_2\}$ are orthonormal of signature $(+, -)$, and $\{L_1, L_2, N_1, N_2, W_1, W_2\}$ is positively oriented.

We say that the pseudo-orthonormal frame $\{L_1, L_2, N_1, N_2, W_1, W_2\}$ is the Cartan frame, and the functions $\{k_0, k_1, k_2\}$ are the Cartan curvatures of γ .

The following theorem can be shown similarly to the classical fundamental existence theorem for curves in $\mathbb{R}^3$.

Theorem 1. *Let $k_0(t)$, $k_1(t)$ and $k_2(t)$ be differentiable functions on $(t_0 - \varepsilon, t_0 + \varepsilon)$. Let p_0 be a point in $\mathbb{R}_3^6$, and $\{L_1^0, L_2^0, N_1^0, N_2^0, W_1^0, W_2^0\}$ be a pseudo-orthonormal basis of $\mathbb{R}_3^6$. Then there exists a unique bi-null Cartan curve $\gamma(t)$ in $\mathbb{R}_3^6$ with $\gamma(t_0) = p_0$, bi-null arc parameter t and curvatures k_0, k_1, k_2 whose Cartan frame $\{L_1, L_2, N_1, N_2, W_1, W_2\}$ satisfies*

$$\begin{aligned} L_1(t_0) &= L_1^0, & L_2(t_0) &= L_2^0, & N_1(t_0) &= N_1^0, \\ N_2(t_0) &= N_2^0, & W_1(t_0) &= W_1^0, & W_2(t_0) &= W_2^0. \end{aligned}$$

3. Osculating bi-null Cartan curves

In this section, we consider the conditions for a bi-null Cartan curve in $\mathbb{R}_3^6$ to be an osculating curve.

Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}_3^6$ with bi-null arc t . Then the bi-null Cartan curve γ is called the first, second, third or fourth kind of osculating bi-null Cartan curve if its position vector (with respect to some chosen origin) always lies in the orthogonal space $N_1^\perp = \text{span}\{L_2, N_1, N_2, W_1, W_2\}$, $N_2^\perp = \text{span}\{L_1, N_1, N_2, W_1, W_2\}$, $W_1^\perp = \text{span}\{L_1, L_2, N_1, N_2, W_2\}$ or $W_2^\perp = \text{span}\{L_1, L_2, N_1, N_2, W_1\}$ respectively. As a result, the position vector of the first, second, third or fourth kind of osculating curve satisfies respectively the following equations

$$(3.1) \quad \gamma(t) = \mu_1(t) L_2(t) + \mu_2(t) N_1(t) + \mu_3(t) N_2(t) + \mu_4(t) W_1(t) + \mu_5(t) W_2(t),$$

$$(3.2) \quad \gamma(t) = \mu_1(t) L_1(t) + \mu_2(t) N_1(t) + \mu_3(t) N_2(t) + \mu_4(t) W_1(t) + \mu_5(t) W_2(t),$$

$$(3.3) \quad \gamma(t) = \mu_1(t) L_1(t) + \mu_2(t) L_2(t) + \mu_3(t) N_1(t) + \mu_4(t) N_2(t) + \mu_5(t) W_2(t),$$

$$(3.4) \quad \gamma(t) = \mu_1(t) L_1(t) + \mu_2(t) L_2(t) + \mu_3(t) N_1(t) + \mu_4(t) N_2(t) + \mu_5(t) W_1(t)$$

for some differentiable functions $\mu_i(t)$ for $i \in \{1, 2, 3, 4, 5\}$.

If $k_2 = 0$ for the first, second and third kinds, then $\gamma(t)$ is included in a hyperplane of index 2, which is the case in [14]. So we assume that $k_2 \neq 0$ in these cases.

Let us consider the first kind of osculating bi-null Cartan curve in $\mathbb{R}_3^6$. Then, the position vector $\gamma(t)$ satisfies (3.1). Differentiating (3.1) with respect to t , we get

$$\begin{aligned} L_1 &= (-\mu_3 k_1 + \mu_5 k_2) L_1 + (\mu'_1 + \mu_2 k_1 - \mu_4 k_0) L_2 + (\mu'_2 - \mu_3) N_1 + (\mu'_3 - \mu_4) N_2 \\ &\quad + (\mu'_4 + \mu_1 + k_0 \mu_3) W_1 + (\mu_2 k_2 + \mu'_5) W_2, \end{aligned}$$

which implies that

$$\begin{aligned} (3.5) \quad \mu'_1 + \mu_2 k_1 - \mu_4 k_0 &= 0, \quad \mu'_4 + \mu_1 + k_0 \mu_3 = 0, \quad \mu'_2 - \mu_3 = 0, \\ \mu'_3 - \mu_4 &= 0, \quad \mu_2 k_2 + \mu'_5 = 0, \quad -\mu_3 k_1 + \mu_5 k_2 = 1. \end{aligned}$$

Solving the system of these equations, we have

$$\mu_1 = -\mu'_2 k_0 - \mu_2^{(3)}, \quad \mu_3 = \mu'_2, \quad \mu_4 = \mu''_2, \quad \mu_5 = \frac{1}{k_2} (1 + \mu'_2 k_1),$$

$$\mu_2 k_1 - 2\mu''_2 k_0 - \mu'_2 k'_0 - \mu_2^{(4)} = 0$$

and

$$\mu_2 k_2 + \left(\frac{1}{k_2} (1 + \mu'_2 k_1) \right)' = 0.$$

Using above equations, we get the following theorem:

Theorem 2. *Let γ be a bi-null Cartan curve in $\mathbb{R}_3^6$ with bi-null arc parameter t and curvatures k_0, k_1 and $k_2 \neq 0$. Then γ is congruent to a first kind of osculating curve if and only if there exists a function μ_2 satisfying the following simultaneous equations:*

$$(3.6) \quad \mu_2 k_1 - 2\mu''_2 k_0 - \mu'_2 k'_0 - \mu_2^{(4)} = 0$$

and

$$(3.7) \quad \mu_2 k_2 + \left(\frac{1}{k_2} (1 + \mu'_2 k_1) \right)' = 0.$$

Proof. Suppose that γ is a first kind of osculating bi-null Cartan curve in $\mathbb{R}_3^6$. Then it is clear from above calculations that the relations (3.6) and (3.7) hold.

Conversely, assume that the equations (3.6) and (3.7) hold. Let us consider a vector $X \in \mathbb{R}_3^6$ as follows

$$X = \gamma + \left(\mu'_2 k_0 + \mu_2^{(3)} \right) L_2 - \mu_2 N_1 - \mu'_2 N_2 - \mu''_2 W_1 - \frac{1}{k_2} (1 + \mu'_2 k_1) W_2.$$

Then we obtain $X' = 0$ which implies that γ is congruent to a first kind of osculating curve. $\square$

From Theorems 1 and 2, we get the following theorem.

Theorem 3. *Let $k_0(t)$, $k_1(t)$ and $k_2(t) \neq 0$ be differentiable functions. Suppose that there exists a differentiable function $\mu_2(t)$ such that*

$$(3.8) \quad \mu_2 k_1 - 2\mu_2'' k_0 - \mu_2' k_0' - \mu_2^{(4)} = 0$$

and

$$(3.9) \quad \mu_2 k_2 + \left(\frac{1}{k_2} (1 + \mu_2' k_1) \right)' = 0.$$

Then there exists a first kind of osculating bi-null Cartan curve $\gamma(t)$ in $\mathbb{R}_3^6$ with bi-null arc parameter t and curvatures k_0, k_1, k_2 .

Example 1. *The followings satisfy the equations (3.8) and (3.9).*

- (i) $\mu_2 = 0, k_0 = t^2/2, k_1 = 1, k_2 = 1.$
- (ii) $\mu_2 = t, k_0 = -t^4/8, k_1 = -t^2/2, k_2 = 1.$

Let us consider the second kind of osculating bi-null Cartan curve in $\mathbb{R}_3^6$. Then, the position vector $\gamma(t)$ satisfies (3.2). Differentiating (3.2) with respect to t , we get

$$\begin{aligned} L_1 &= (\mu_1' - \mu_3 k_1 + \mu_5 k_2) L_1 + (\mu_1 + \mu_2 k_1 - \mu_4 k_0) L_2 + (\mu_2' - \mu_3) N_1 \\ &\quad + (\mu_3' - \mu_4) N_2 + (\mu_4' + \mu_3 k_0) W_1 + (\mu_5' + \mu_2 k_2) W_2, \end{aligned}$$

which implies that

$$\begin{aligned} \mu_1' - \mu_3 k_1 + \mu_5 k_2 &= 1, \quad \mu_1 + \mu_2 k_1 - \mu_4 k_0 = 0, \quad \mu_2' - \mu_3 = 0 \\ \mu_3' - \mu_4 &= 0, \quad \mu_4' + \mu_3 k_0 = 0, \quad \mu_5' + \mu_2 k_2 = 0. \end{aligned}$$

Solving the system of these equations, we have

$$\begin{aligned} \mu_1 &= k_0 \mu_2'' - k_1 \mu_2, \quad \mu_3 = \mu_2', \quad \mu_4 = \mu_2'', \\ \mu_5 &= \frac{1}{k_2} \left(1 + k_1' \mu_2 + 2k_1 \mu_2' - k_0' \mu_2'' - k_0 \mu_2^{(3)} \right) \end{aligned}$$

and

$$\begin{aligned} \mu_2^{(3)} + k_0 \mu_2' &= 0, \\ \left(\frac{1}{k_2} \left(1 + k_1' \mu_2 + 2k_1 \mu_2' - k_0' \mu_2'' - k_0 \mu_2^{(3)} \right) \right)' + k_2 \mu_2 &= 0. \end{aligned}$$

Using above equations, we get the following theorem:

Theorem 4. *Let γ be a bi-null Cartan curve in $\mathbb{R}_3^6$ with bi-null arc parameter t and curvatures k_0, k_1 and $k_2 \neq 0$. Then γ is congruent to a second kind of osculating curve if and only if there exists a function μ_2 satisfying the following simultaneous equations:*

$$(3.10) \quad \mu_2^{(3)} + k_0\mu_2' = 0, \quad \left(\frac{1}{k_2} \left(1 + k_1'\mu_2 + 2k_1\mu_2' - k_0'\mu_2'' - k_0\mu_2^{(3)} \right) \right)' + k_2\mu_2 = 0.$$

Proof. Suppose that γ is a second kind of osculating bi-null Cartan curve in $\mathbb{R}_3^6$. Then it is clear from above calculations that the relation (3.10) holds.

Conversely, assume that the equations (3.10) hold. Let us consider a vector $X \in \mathbb{R}_3^6$ as follows

$$\begin{aligned} X &= \gamma(t) - (k_0\mu_2'' - k_1\mu_2) L_1 - \mu_2 N_1 - \mu_2' N_2 - \mu_2'' W_1 \\ &\quad - \frac{1}{k_2} \left(1 + k_1'\mu_2 + 2k_1\mu_2' - k_0'\mu_2'' - k_0\mu_2^{(3)} \right) W_2. \end{aligned}$$

Then we obtain $X' = 0$ which implies that γ is congruent to a second kind of osculating curve. $\square$

From the above Theorem and Theorem 1, we have

Theorem 5. *Let $k_0(t), k_1(t)$ and $k_2(t) \neq 0$ be differentiable functions. Suppose that there exists a differentiable function $\mu_2(t)$ such that*

$$(3.11) \quad \mu_2^{(3)} + k_0\mu_2' = 0, \quad \left(\frac{1}{k_2} \left(1 + k_1'\mu_2 + 2k_1\mu_2' - k_0'\mu_2'' - k_0\mu_2^{(3)} \right) \right)' + k_2\mu_2 = 0.$$

Then there exists a second kind of osculating bi-null Cartan curve $\gamma(t)$ in $\mathbb{R}_3^6$ with bi-null arc parameter t and curvatures k_0, k_1, k_2 .

Example 2. *The followings satisfy the equation (3.11).*

- (i) $\mu_2 = t, k_0 = 0, k_1 = -t^2/8, k_2 = 1.$
- (ii) $\mu_2 = t^2, k_0 = 0, k_1 = -t^2/18, k_2 = 1.$

Let us consider the third kind of osculating bi-null Cartan curve in $\mathbb{R}_3^6$. Then, the position vector $\gamma(t)$ satisfies (3.3). Differentiating (3.3) with respect to t , we get

$$\begin{aligned} L_1 &= (\mu_1' - k_1\mu_4 + \mu_5k_2) L_1 + (\mu_1 + \mu_2' + k_1\mu_3) L_2 + (\mu_3' - \mu_4) N_1 + \mu_4' N_2 \\ &\quad + (\mu_2 + k_0\mu_4) W_1 + (\mu_3k_2 + \mu_5') W_2, \end{aligned}$$

which implies that

$$(3.12) \quad \begin{aligned} \mu'_1 - k_1\mu_4 + \mu_5k_2 &= 1, & \mu_1 + \mu'_2 + k_1\mu_3 &= 0, & \mu'_3 - \mu_4 &= 0, \\ \mu_3k_2 + \mu'_5 &= 0, & \mu_2 + k_0\mu_4 &= 0, & \mu'_4 &= 0. \end{aligned}$$

From (3.12), we get

$$\begin{aligned} \mu_1 &= c_4k'_0 - k_1(c_4t + c_3), & \mu_2 &= -c_4k_0, & \mu_3 &= c_4t + c_3, & \mu_4 &= c_4, \\ \mu_5 &= \frac{1}{k_2} (1 - c_4k''_0 + k'_1(c_4t + c_3) + 2c_4k_1), \end{aligned}$$

and

$$\left(\frac{1}{k_2} (1 - c_4k''_0 + k'_1(c_4t + c_3) + 2c_4k_1) \right)' + k_2(c_4t + c_3) = 0$$

where $c_3, c_4 \in \mathbb{R}$.

From the above, we have

Theorem 6. *Let γ be a bi-null Cartan curve in $\mathbb{R}_3^6$ with bi-null arc parameter t and curvatures k_0, k_1 and $k_2 \neq 0$. Then γ is congruent to a third kind of osculating curve if and only if*

$$(3.13) \quad \left(\frac{1}{k_2} (1 - c_4k''_0 + k'_1(c_4t + c_3) + 2c_4k_1) \right)' + k_2(c_4t + c_3) = 0$$

where $c_3, c_4 \in \mathbb{R}$.

Proof. Suppose that γ is a third kind of osculating bi-null Cartan curve in $\mathbb{R}_3^6$. Then it is clear from above calculations that the relation (3.13) holds.

Conversely, assume that the equation (3.13) holds. Let us consider a vector $X \in \mathbb{R}_3^6$ as follows

$$\begin{aligned} X &= \gamma(t) - (c_4k'_0 - k_1(c_4t + c_3))L_1 + c_4k_0L_2 - (c_4t + c_3)N_1 - c_4N_2 \\ &\quad - \frac{1}{k_2} (1 - c_4k''_0 + k'_1(c_4t + c_3) + 2c_4k_1)W_2. \end{aligned}$$

Then we obtain $X' = 0$ which implies that γ is congruent to a third kind of osculating curve. $\square$

From the above Theorem and Theorem 1, we have

Theorem 7. *Let $k_0(t)$, $k_1(t)$ and $k_2(t) \neq 0$ be differentiable functions. Suppose that there exists real numbers c_3 and c_4 which satisfy the following*

$$(3.14) \quad \left(\frac{1}{k_2} (1 - c_4k''_0 + k'_1(c_4t + c_3) + 2c_4k_1) \right)' + k_2(c_4t + c_3) = 0.$$

Then there exists a third kind of osculating bi-null Cartan curve $\gamma(t)$ in $\mathbb{R}_3^6$ with bi-null arc parameter t and curvatures k_0, k_1, k_2 .

Example 3. The followings satisfy the equation (3.14).

(i) $c_3 = 1, c_4 = 0, k_0 = t, k_1 = -t^2/2, k_2 = 1$.

(ii) $c_3 = 0, c_4 = 1, k_0 = t, k_1 = -t^2/8, k_2 = 1$.

Let us consider the fourth kind of osculating bi-null Cartan curve in $\mathbb{R}_3^6$. Then, the position vector $\gamma(t)$ satisfies (3.4). Differentiating (3.4) with respect to t , we get

$$\begin{aligned} L_1 &= (\mu'_1 - k_1\mu_4) L_1 + (\mu_1 + \mu'_2 + k_1\mu_3 - \mu_5k_0) L_2 + (\mu'_3 - \mu_4) N_1 \\ &\quad + (\mu'_4 - \mu_5) N_2 + (\mu_2 + k_0\mu_4 + \mu'_5) W_1 + \mu_3k_2W_2, \end{aligned}$$

which implies that

$$\begin{aligned} (3.15) \quad \mu_1 + \mu'_2 + k_1\mu_3 - \mu_5k_0 &= 0, \quad \mu'_3 - \mu_4 = 0, \quad \mu_3k_2 = 0, \\ \mu_2 + k_0\mu_4 + \mu'_5 &= 0, \quad \mu'_1 - k_1\mu_4 = 1, \quad \mu'_4 - \mu_5 = 0. \end{aligned}$$

Assume that $k_2 \neq 0$. From (3.15), we get $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0$ which is a contradiction. Thus $k_2 = 0$. Then W_2 is constant and γ is included in a hyperplane of index 2.

Theorem 8. Let γ be a bi-null Cartan curve in $\mathbb{R}_3^6$ with bi-null arc parameter t and curvatures k_0, k_1, k_2 . If γ is congruent to a fourth kind of osculating curve, then it lies in some hyperplane of index 2.

4. Normal or rectifying bi-null Cartan curves

In this section, we consider the conditions for a bi-null Cartan curve in $\mathbb{R}_3^6$ to be a normal curve or a rectifying curve.

Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}_3^6$ with bi-null arc t . Then the bi-null Cartan curve γ is called a normal bi-null Cartan curve if its position vector (with respect to some chosen origin) always lies in its normal space defined as the orthogonal space $L_1^\perp$ of the tangent vector field L_1 . Since the normal space is a 5-dimensional subspace of $\mathbb{R}_3^6$, spanned by $\{L_1, L_2, N_2, W_1, W_2\}$, the position vector γ of a normal bi-null Cartan curve satisfies the following equation

$$(4.1) \quad \gamma(t) = \mu_1(t) L_1(t) + \mu_2(t) L_2(t) + \mu_3(t) N_2(t) + \mu_4(t) W_1(t) + \mu_5(t) W_2(t)$$

where $\mu_i(t)$ are some differentiable functions for $i \in \{1, 2, 3, 4, 5\}$.

Next, the bi-null Cartan curve γ is called a rectifying bi-null Cartan curve if its position vector (with respect to some chosen origin) always lies in its rectifying space defined as the orthogonal space $L_2^\perp$ of the principal normal vector field L_2 . Since the rectifying space is a 5-dimensional subspace of $\mathbb{R}_3^6$, spanned by $\{L_1, L_2, N_1, W_1, W_2\}$, the position vector γ of a rectifying bi-null Cartan curve satisfies the following equation

$$(4.2) \quad \gamma(t) = \mu_1(t) L_1(t) + \mu_2(t) L_2(t) + \mu_3(t) N_1(t) + \mu_4(t) W_1(t) + \mu_5(t) W_2(t)$$

where $\mu_i(t)$ are some differentiable functions for $i \in \{1, 2, 3, 4, 5\}$.

Let γ be a normal bi-null Cartan curve in $\mathbb{R}_3^6$ parametrized by bi-null arc parameter t and with curvatures k_0, k_1, k_2 . Then

$$(4.3) \quad \langle \gamma, L_1 \rangle = 0.$$

Differentiating (4.3) with respect to t , we find

$$\langle \gamma, L_2 \rangle = \langle \gamma, N_1 \rangle = \langle \gamma, N_2 \rangle = \langle \gamma, W_1 \rangle = 0.$$

From (4.1), we have

$$(4.4) \quad \gamma = \mu_5 W_2.$$

Differentiating (4.4) with respect to t , we obtain $\mu_5' = 0$ and $k_2 = 1/\mu_5$.

From the above, we get the following theorem.

Theorem 9. *Let γ be a bi-null Cartan curve in $\mathbb{R}_3^6$ with bi-null arc parameter t and curvatures k_0, k_1, k_2 . Then γ is a normal curve if and only if the curvature k_2 is a nonzero constant.*

Proof. Suppose that γ is a normal bi-null Cartan curve in $\mathbb{R}_3^6$ with bi-null arc parameter t and curvatures k_0, k_1, k_2 . From above calculations, k_2 is a nonzero constant.

Conversely, let us assume that k_2 is a nonzero constant. Then we have

$$\frac{d}{dt} \left(\gamma - \frac{1}{k_2} W_2 \right) = 0.$$

So up to parallel translations of $\mathbb{R}_3^6$, it follows that γ is a rectifying curve. □

Let us assume that γ is a rectifying bi-null Cartan curve in $\mathbb{R}_3^6$ parameterized by bi-null arc parameter t and with curvatures k_0, k_1 and $k_2 \neq 0$. Then

$$(4.5) \quad \langle \gamma, L_2 \rangle = 0.$$

Differentiating (4.5) with respect to t , we find

$$\langle \gamma, L_2 \rangle = \langle \gamma, N_2 \rangle = \langle \gamma, W_1 \rangle = 0.$$

From (4.2), we have

$$(4.6) \quad \gamma = \mu_1 L_1 + \mu_3 N_1 + \mu_5 W_2.$$

Differentiating (4.6) with respect to t , we find

$$L_1 = (\mu'_1 + \mu_5 k_2) L_1 + (\mu_1 + \mu_3 k_1) L_2 + \mu'_3 N_1 + (\mu_3 k_2 + \mu'_5) W_2,$$

which leads to the following system of equations

$$(4.7) \quad \mu'_1 + \mu_5 k_2 = 1, \quad \mu_1 + \mu_3 k_1 = 0, \quad \mu_3 k_2 + \mu'_5 = 0, \quad \mu'_3 = 0.$$

Solving the equations (4.7), we obtain

$$(4.8) \quad \mu_3 = c_3, \quad \mu_1 = -c_3 k_1, \quad \mu_5 = \frac{1}{k_2} (1 + c_3 k'_1),$$

and

$$c_3 k_2^3 - k'_2 (1 + c_3 k'_1) + c_3 k_2 k''_1 = 0$$

where $c_3 \in \mathbb{R}$.

Using the above computations, we obtain the following theorem.

Theorem 10. *Let γ be a bi-null Cartan curve in $\mathbb{R}_3^6$ with bi-null arc parameter t and curvatures k_0, k_1 and $k_2 \neq 0$. Then γ is congruent to a rectifying curve if and only if*

$$(4.9) \quad c_3 k_2^3 - k'_2 (1 + c_3 k'_1) + c_3 k_2 k''_1 = 0$$

where $c_3 \in \mathbb{R}$.

Proof. Suppose that γ is a rectifying bi-null Cartan curve in $\mathbb{R}_3^6$ with bi-null arc parameter t and curvatures k_0, k_1 and $k_2 \neq 0$. Then it is clear from above calculations that the relation (4.9) holds.

Conversely, assume that the equation (4.9) holds. Let us consider a vector $X \in \mathbb{R}_3^6$ as follows

$$X = \gamma(t) + c_3 k_1 L_1 - c_3 N_1 - \left(\frac{1}{k_2} (1 + c_3 k_1') \right) W_2.$$

Then we obtain $X' = 0$ which implies that γ is congruent to a rectifying curve. $\square$

From the above Theorem and Theorem 1, we have

Theorem 11. *Let $k_0(t)$, $k_1(t)$ and $k_2(t) \neq 0$ be differentiable functions. Suppose that there exists a real number c_3 which satisfies the following*

$$(4.10) \quad c_3 k_2^3 - k_2' (1 + c_3 k_1') + c_3 k_2 k_1'' = 0.$$

Then there exists a rectifying bi-null Cartan curve $\gamma(t)$ in $\mathbb{R}_3^6$ with bi-null arc parameter t and curvatures k_0, k_1, k_2 .

Example 4. *The followings satisfy the equation (4.10).*

$$(i) \ c_3 = 0, \ k_0 = t, \ k_1 = t, \ k_2 = 1, \quad (ii) \ c_3 = 1, \ k_0 = t^2, \ k_1 = -t^2/2, \ k_2 = 1.$$

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