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ON INEXTENSIBLE FLOWS OF CURVES ACCORDING TO ALTERNATIVE MOVING FRAME

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ABSTRACT. In this paper, we investigate inextensible flows of curves according to alternative moving frame in Euclidean space $\mathbb{E}^3$. Using the Frenet frame of the given curve, we present partial differential equations. The concepts with the inextensible flows are analyzed by using alternative moving frame.

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1. Introduction

Flows of particles is an important active research field in the differential geometry studies and it is heavily studied for a long time and it is still under consideration. Also, it introduces the miniature, architect, simulation of kinematic motion or design of highways and mechanic tools. Uzunoğlu et. al. have discussed a new frame called it as a alternative moving frame, [17].

The flow of a curve is said to be inextensible if the arclength is preserved. Physically, inextensible curve flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example can be described by inextensible curve flows. Such motions arise quite naturally in a wide range of physical applications. There are many approaches to the elastica. For example, Körpınar have constructed a new method for inextensible flows of timelike curves in a conformally flat, quasi conformally flat and conformally symmetric 4-dimensional LPSasakian manifold. With this new representation it has been derived the necessary and sufficient condition for the given curve to be the inextensible flow. By using curvature tensor field, he gave some characterizations for curvatures of a timelike curve in a conformally flat, quasi conformally flat and conformally symmetric 4-dimensional LP-Sasakian manifold, [8].

On the other hand, The study of flow theory has received considerable attention in mathematical physics. For example, Körpınar et al have have been extensively discussed time-evolution equations that the intrinsic quantities of curves to construct binormal spherical curves as surfaces in terms of inextensible flows in E^3. They gave necessary and sufficient condition for a binormal spherical curves as surfaces to be a Bonnet surface, [9].

In this paper, we investigate inextensible flows of curves according to alternative moving frame in Euclidean space $\mathbb{E}^3$. Using the Frenet frame of the given curve, we present partial differential equations. The concepts with the inextensible flows are analyzed by using alternative moving frame.

2. Preliminaries

Let $\alpha : I \to \mathbb{E}^3$ be an arbitrary curve in $\mathbb{E}^3$. Recall that the curve α is said a unit speed curve (or parameterized by arclength functions) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$,where $\langle , \rangle$ denotes the standard inner product of R^3 given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. In particular, the norm of an arbitrary vector $a \in R^n$ is given by $\|a\|^2 = \langle a, a \rangle$.

In Euclidean 3-space, we know that each unit speed curve has at least four continuous derivatives, one can associate three orthogonal unit vector fields $\mathbf{T}, \mathbf{N}, \mathbf{B}$ are the tangent, the principal normal and the binormal vector fields, respectively.

Firstly, assume that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along α. Then, the Frenet frame satisfies the following Frenet-Serret equations:

$$\begin{aligned} \mathbf{T}'(s) &= \kappa(s)\mathbf{N}(s), \\ \mathbf{N}'(s) &= -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s), \\ \mathbf{B}'(s) &= -\tau(s)\mathbf{N}(s), \end{aligned} \tag{2.1}$$

where κ is the curvature of α and τ its torsion.

Secondly, denote by $\{\mathbf{N}, \mathbf{C}, \mathbf{N} \times \mathbf{C} = \mathbf{W}\}$ the alternative moving frame along the curve α in Euclidean 3-space. Putting

$$\mathbf{N}, \mathbf{C} = \frac{\mathbf{N}'}{\|\mathbf{N}'\|}, \mathbf{W} = \frac{\tau\mathbf{T} + \kappa\mathbf{B}}{\sqrt{\kappa^2 + \tau^2}}$$

are the unit principal normal vector, the derivative of principal normal vector and the Darboux vector, respectively. For the derivatives of the alternative moving frame, we have

$$\begin{aligned} \mathbf{N}'(s) &= f(s)\mathbf{C}(s), \\ \mathbf{C}'(s) &= -f(s)\mathbf{N}(s) + g(s)\mathbf{W}(s), \\ \mathbf{W}'(s) &= -g(s)\mathbf{C}(s), \end{aligned} \tag{2.1}$$

where $f = \kappa\sqrt{1 + H^2}$ and $g = \sigma f$ are curvatures of the curve α in terms of alternative moving frame, [17].

3. Inextensible Flows according to Alternative Moving Frame

Let $\alpha(u, t)$ is a one parameter family of smooth curves in $\mathbb{E}^3$.

Any flow of α can be represented according to alternative moving frame as

$$\frac{\partial \alpha}{\partial t} = \beta_1\mathbf{N} + \beta_2\mathbf{C} + \beta_3\mathbf{W}, \tag{3.1}$$

where $\beta_1, \beta_2, \beta_3 \in C^\infty(\mathbb{E}^3)$.

Letting the arclength variation be

$$s(u, t) = \int_0^u v du.$$

Also, inextensible flow of curve can be expressed by the condition

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u \frac{\partial v}{\partial t}du=0.$$

Definition 3.1. *The flow* $\frac{\partial \alpha}{\partial t}$ *in* $\mathbb{E}^3$ *are said to be inextensible if*

$$\frac{\partial}{\partial t}\left|\frac{\partial \alpha}{\partial u}\right|=0.$$

Lemma 3.2. *Let* $\frac{\partial \alpha}{\partial t}$ *be a smooth flow of the curve* α *according to the alternative moving frame. The flow is inextensible if and only if*

$$\frac{\partial v}{\partial t}=\frac{\tau}{\varpi}(\frac{\partial \beta_3}{\partial u}+\beta_2 gv)-\frac{\kappa}{\varpi}(\frac{\partial \beta_2}{\partial u}+\beta_1 fv-\beta_3 gv) \tag{3.2}$$

where $\varpi=\sqrt{\kappa^2+\tau^2}$ *and* $\beta_1,\beta_2,\beta_3\in C^\infty(\mathbb{E}^3)$.

Proof. By using Definition 3.1, we obtain

$$\frac{\partial}{\partial u}\frac{\partial \alpha}{\partial t}=(\frac{\partial \beta_1}{\partial u}-\beta_2 fv)\mathbf{N}+(\frac{\partial \beta_2}{\partial u}+\beta_1 fv-\beta_3 gv)\mathbf{C}+(\frac{\partial \beta_3}{\partial u}+\beta_2 gv)\mathbf{W}.$$

From the alternative moving frame, we have

$$\begin{aligned}\mathbf{B} &= \frac{\kappa}{\varpi}\mathbf{W}+\frac{\tau}{\varpi}\mathbf{C},\\ \mathbf{T} &= \frac{\tau}{\varpi}\mathbf{W}-\frac{\kappa}{\varpi}\mathbf{C},\end{aligned}$$

where $\varpi=\sqrt{\kappa^2+\tau^2}$.

Suppose that $\frac{\partial \alpha}{\partial t}$ be a smooth flow of the curve α . Using definition of α, we have

$$\frac{\partial v}{\partial t}=<\frac{\tau}{\varpi}\mathbf{W}-\frac{\kappa}{\varpi}\mathbf{C},(\frac{\partial \beta_1}{\partial u}-\beta_2 fv)\mathbf{N}+(\frac{\partial \beta_2}{\partial u}+\beta_1 fv-\beta_3 gv)\mathbf{C}+(\frac{\partial \beta_3}{\partial u}+\beta_2 gv)\mathbf{W}>.$$

Therefore, by using (3.1), an algebraic calculus gives that

$$\frac{\partial v}{\partial t}=\frac{\tau}{\varpi}(\frac{\partial \beta_3}{\partial u}+\beta_2 gv)-\frac{\kappa}{\varpi}(\frac{\partial \beta_2}{\partial u}+\beta_1 fv-\beta_3 gv)$$

Making necessary calculations from above equation, we have (3.2), which proves the lemma.

Lemma 3.3. *Let* $\frac{\partial \alpha}{\partial t}$ *be a smooth flow of the curve* α *according to the alternative moving frame. The flow is inextensible if and only if*

$$\tau(\frac{\partial \beta_3}{\partial u}+\beta_2 gv)=\kappa(\frac{\partial \beta_2}{\partial u}+\beta_1 fv-\beta_3 gv), \tag{3.3}$$

where $\beta_1,\beta_2,\beta_3\in C^\infty(\mathbb{E}^3)$.

Proof. Using lemma 3.2, we easily have (3.3).

Theorem 3.4. *Let $\frac{\partial \alpha}{\partial t}$ be a smooth inextensible flow of the curve α according to alternative moving frame. Then, the flow of $\boldsymbol{N}$ is given by*

$$\begin{aligned}\frac{\partial \mathbf{N}}{\partial t} &= \frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s} - \beta_2 f) - (\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g)f - \frac{\partial \kappa}{\partial t}]\mathbf{N} \\ &+ \frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g) + (\frac{\partial \beta_1}{\partial s} - \beta_2 f)f - (\frac{\partial \beta_3}{\partial s} + \beta_2 g)g]\mathbf{C} \\ &+ \frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s} + \beta_2 g) + (\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g)g]\mathbf{W}.\end{aligned}$$

Theorem 3.5. *Let $\frac{\partial \alpha}{\partial t}$ be a smooth inextensible flow of the curve α according to alternative moving frame. Then, the flow of $\boldsymbol{C}$ is given by*

$$\begin{aligned}\frac{\partial \mathbf{C}}{\partial t} &= \frac{1}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s} - \beta_2 f) - (\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g)f - \frac{\partial \kappa}{\partial t}]] \\ &- f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g) + (\frac{\partial \beta_1}{\partial s} - \beta_2 f)f - (\frac{\partial \beta_3}{\partial s} + \beta_2 g)g]]]\mathbf{N} \\ &+ \frac{1}{f}[-\frac{\partial f}{\partial t} + \frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g) + (\frac{\partial \beta_1}{\partial s} - \beta_2 f)f - (\frac{\partial \beta_3}{\partial s} + \beta_2 g)g]] \\ &+ [\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s} - \beta_2 f) - (\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g)f - \frac{\partial \kappa}{\partial t}]]f - [\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s} + \beta_2 g) \\ &+ (\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g)g]]g]\mathbf{C} + \frac{1}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s} + \beta_2 g) + (\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g)g]] \\ &+ [\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g) + (\frac{\partial \beta_1}{\partial s} - \beta_2 f)f - (\frac{\partial \beta_3}{\partial s} + \beta_2 g)g]]g]\mathbf{W}.\end{aligned}$$

Theorem 3.6. *Let $\frac{\partial \alpha}{\partial t}$ be a smooth inextensible flow of the curve α according to alternative moving frame. Then, the flow of $\boldsymbol{W}$ is given by*

$$\begin{aligned}\frac{\partial \mathbf{W}}{\partial t} &= \frac{\varpi}{\tau}[(\frac{\partial \beta_1}{\partial s} - \beta_2 f) + \frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s} - \beta_2 f) - (\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g)f \\ &- \frac{\partial \kappa}{\partial t}]] - f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g) + (\frac{\partial \beta_1}{\partial s} - \beta_2 f)f - (\frac{\partial \beta_3}{\partial s} + \beta_2 g)g]]]]\mathbf{N} \\ &+ \frac{\varpi}{\tau}[(\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g) + \frac{1}{f}\frac{\kappa}{\varpi}[-\frac{\partial f}{\partial t} + \frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g) \\ &+ (\frac{\partial \beta_1}{\partial s} - \beta_2 f)f - (\frac{\partial \beta_3}{\partial s} + \beta_2 g)g]] + [\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s} - \beta_2 f) - (\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g)f \\ &- \frac{\partial \kappa}{\partial t}]]f - [\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s} + \beta_2 g) + (\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g)g]]g] + \frac{\partial}{\partial t}[\frac{\kappa}{\varpi}]]\mathbf{C} \\ &+ \frac{\varpi}{\tau}[(\frac{\partial \beta_3}{\partial s} + \beta_2 g) + \frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s} + \beta_2 g) + (\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g)g]] \\ &+ [\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s} + \beta_1 f - \beta_3 g) + (\frac{\partial \beta_1}{\partial s} - \beta_2 f)f - (\frac{\partial \beta_3}{\partial s} + \beta_2 g)g]]g] - \frac{\partial}{\partial t}[\frac{\tau}{\varpi}]]\mathbf{W}.\end{aligned}$$

Theorem3.7.

$$[\frac{\partial}{\partial s}[\frac{1}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}-\beta_2 f)-(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial\kappa}{\partial t}]]-f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}$$
$$+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]]]-f[\frac{1}{f}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}$$
$$+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}-\beta_2 f)$$
$$-(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial\kappa}{\partial t}]]f-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_3}{\partial s}+\beta_2 g)$$
$$+(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]]]=$$
$$[-\frac{\partial f}{\partial t}+\frac{g\varpi}{\tau}[(\frac{\partial\beta_1}{\partial s}-\beta_2 f)+\frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}-\beta_2 f)-(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)f$$
$$-\frac{\partial\kappa}{\partial t}]]-f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]]]$$
$$-\frac{f}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}-\beta_2 f)-(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial\kappa}{\partial t}]].$$

Proof. Let α be a unit speed curve with the alternative moving frame. From the definition of this frame, we know that

$$\frac{\partial}{\partial s}\frac{\partial\mathbf{C}}{\partial t} = [\frac{\partial}{\partial s}[\frac{1}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}-\beta_2 f)-(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial\kappa}{\partial t}]]$$
$$-f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]]]$$
$$-f[\frac{1}{f}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)+$$
$$(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]$$
$$+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}-\beta_2 f)-(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial\kappa}{\partial t}]]f-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_3}{\partial s}+\beta_2 g)$$
$$+(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]]]\mathbf{N}+[\frac{\partial}{\partial s}[\frac{1}{f}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)$$
$$+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}-\beta_2 f)-(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)f$$
$$-\frac{\partial\kappa}{\partial t}]]f-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_3}{\partial s}+\beta_2 g)+(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]]+f[\frac{1}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}$$
$$-\beta_2 f)-(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial\kappa}{\partial t}]]-f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}$$
$$-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]]]-g[\frac{1}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_3}{\partial s}+\beta_2 g)+(\frac{\partial\beta_2}{\partial s}+\beta_1 f$$
$$-\beta_3 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]g]]]\mathbf{C}$$

$$+[\frac{\partial}{\partial s}[\frac{1}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_3}{\partial s}+\beta_2 g)+(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}$$
$$+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]g]]+g[\frac{1}{f}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}$$
$$+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}-\beta_2 f)-(\frac{\partial\beta_2}{\partial s}$$
$$+\beta_1 f-\beta_3 g)f-\frac{\partial\kappa}{\partial t}]]f-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_3}{\partial s}+\beta_2 g)+(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]]]\mathbf{W}.$$

Then using the following equality:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathbf{C}=-\frac{\partial f}{\partial t}\mathbf{N}+\frac{\partial g}{\partial t}\mathbf{W}-f\frac{\partial}{\partial t}\mathbf{N}+g\frac{\partial}{\partial t}\mathbf{W}$$

we have

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathbf{C} \;=\; & [-\frac{\partial f}{\partial t}+\frac{g\varpi}{\tau}[(\frac{\partial\beta_1}{\partial s}-\beta_2 f)+\frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}-\beta_2 f)-(\frac{\partial\beta_2}{\partial s}\\
&+\beta_1 f-\beta_3 g)f-\frac{\partial\kappa}{\partial t}]]-f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}\\
&+\beta_2 g)g]]]]-\frac{f}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}-\beta_2 f)-(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial\kappa}{\partial t}]]\mathbf{N}+[\frac{g\varpi}{\tau}[(\frac{\partial\beta_2}{\partial s}\\
&+\beta_1 f-\beta_3 g)+\frac{1}{f}\frac{\kappa}{\varpi}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f\\
&-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}-\beta_2 f)-(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial\kappa}{\partial t}]]f\\
&-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_3}{\partial s}+\beta_2 g)\\
&+(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]+\frac{\partial}{\partial t}[\frac{\kappa}{\varpi}]]-\frac{f}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}\\
&-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]\mathbf{C}+[\frac{\partial g}{\partial t}+\frac{g\varpi}{\tau}[(\frac{\partial\beta_3}{\partial s}+\beta_2 g)\\
&+\frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_3}{\partial s}+\beta_2 g)\\
&+(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}\\
&+\beta_2 g)g]]g]-\frac{\partial}{\partial t}[\frac{\tau}{\varpi}]]-\frac{f}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_3}{\partial s}+\beta_2 g)+(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]\mathbf{W}.
\end{aligned}$$

By using above mixed differentiation and coefficient of N, we have corollary. This completes the proof.

Corollary 3.8.

$$
\begin{aligned}
& [\frac{\partial}{\partial s}[\frac{1}{f}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]] \\
& +[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]]f-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s} \\
& +\beta_1 f-\beta_3 g)g]]g]]+f[\frac{1}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]] \\
& -f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]]] \\
& -g[\frac{1}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s} \\
& +\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]g]]] \\
= & [\frac{g\varpi}{\tau}[(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+\frac{1}{f}\frac{\kappa}{\varpi}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g) \\
& +(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s}+\beta_1 f \\
& -\beta_3 g)f-\frac{\partial \kappa}{\partial t}]]f-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]+\frac{\partial}{\partial t}[\frac{\kappa}{\varpi}]] \\
& -\frac{f}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]].
\end{aligned}
$$

Proof. It is obvious by using Theorem 3.7.

Corollary 3.9.

$$
\begin{aligned}
& [\frac{\partial}{\partial s}[\frac{1}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s} \\
& +\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]g]]+g[\frac{1}{f}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s} \\
& +\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s} \\
& +\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]]f-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]]] \\
= & [\frac{\partial g}{\partial t}+\frac{g\varpi}{\tau}[(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+\frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]] \\
& +[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]g]-\frac{\partial}{\partial t}[\frac{\tau}{\varpi}]] \\
& -\frac{f}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]].
\end{aligned}
$$

Proof. It is obvious by using Theorem 3.7.

Theorem 3.10.

$$\begin{aligned}
& [\frac{\partial}{\partial s}[\frac{\varpi}{\tau}[(\frac{\partial \beta_1}{\partial s}-\beta_2 f)+\frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f) \\
& -(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]] \\
& -f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]]]]- \\
& f[\frac{\varpi}{\tau}[(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g) \\
& +\frac{1}{f}\frac{\kappa}{\varpi}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f \\
& -(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]]f \\
& -[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]+\frac{\partial}{\partial t}[\frac{\kappa}{\varpi}]]]] \\
= \; & -\frac{g}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]]-f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s} \\
& +\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]].
\end{aligned}$$

Proof. Let α be a unit speed curve with the alternative moving frame. Differentiating $\frac{\partial \mathbf{W}}{\partial t}$ with respect to s, we have

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \mathbf{W}}{\partial t} = \; & [\frac{\partial}{\partial s}[\frac{\varpi}{\tau}[(\frac{\partial \beta_1}{\partial s}-\beta_2 f)+\frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s} \\
& +\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]]-f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s} \\
& -\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]]]]-f[\frac{\varpi}{\tau}[(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+\frac{1}{f}\frac{\kappa}{\varpi}[-\frac{\partial f}{\partial t} \\
& +\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]] \\
& +[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]]f-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s} \\
& +\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]+\frac{\partial}{\partial t}[\frac{\kappa}{\varpi}]]]]\mathbf{N}+[\frac{\partial}{\partial s}[\frac{\varpi}{\tau}[(\frac{\partial \beta_2}{\partial s}+\beta_1 f \\
& -\beta_3 g)+\frac{1}{f}\frac{\kappa}{\varpi}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f \\
& -(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]]f \\
& -[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]+\frac{\partial}{\partial t}[\frac{\kappa}{\varpi}]]]+f[\frac{\varpi}{\tau}[(\frac{\partial \beta_1}{\partial s}
\end{aligned}$$

$$
\begin{aligned}
&-\beta_2 f)+\frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}-\beta_2 f)-(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial\kappa}{\partial t}]]\\
&-f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]]]]\\
&-g[\frac{\varpi}{\tau}[(\frac{\partial\beta_3}{\partial s}+\beta_2 g)+\frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_3}{\partial s}+\beta_2 g)+(\frac{\partial\beta_2}{\partial s}+\beta_1 f\\
&-\beta_3 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]g]\\
&-\frac{\partial}{\partial t}[\frac{\tau}{\varpi}]]]]\mathbf{C}+[\frac{\partial}{\partial s}[\frac{\varpi}{\tau}[(\frac{\partial\beta_3}{\partial s}+\beta_2 g)+\frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_3}{\partial s}+\beta_2 g)+(\frac{\partial\beta_2}{\partial s}\\
&+\beta_1 f-\beta_3 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)\\
&+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]g]\\
&-\frac{\partial}{\partial t}[\frac{\tau}{\varpi}]]]+g[\frac{\varpi}{\tau}[(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)+\frac{1}{f}\frac{\kappa}{\varpi}[-\frac{\partial f}{\partial t}\\
&+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)\\
&+(\frac{\partial\beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial\beta_3}{\partial s}+\beta_2 g)g]]\\
&+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_1}{\partial s}-\beta_2 f)-(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)f\\
&-\frac{\partial\kappa}{\partial t}]]f-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial\beta_3}{\partial s}+\beta_2 g)+(\frac{\partial\beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]+\frac{\partial}{\partial t}[\frac{\kappa}{\varpi}]]]]\mathbf{W}.
\end{aligned}
$$

On the other hand, differentiating $\dfrac{\partial\mathbf{W}}{\partial s}$ with respect to t, gives

$$
\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathbf{W}=-\frac{\partial}{\partial t}g\mathbf{C}-g\frac{\partial}{\partial t}\mathbf{C}.
$$

Since

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathbf{W} &= -\frac{g}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]] \\
&-f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]]\mathbf{N} \\
&+[-\frac{\partial g}{\partial t}-\frac{g}{f}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g) \\
&+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]] \\
&+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]]f \\
&-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]]\mathbf{C} \\
&-\frac{g}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]] \\
&+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]g]\mathbf{W}.
\end{aligned}
$$

This completes the proof.

Corollary 3.11.

$$
\begin{aligned}
&[\frac{\partial}{\partial s}[\frac{\varpi}{\tau}[(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+\frac{1}{f}\frac{\kappa}{\varpi}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g) \\
&+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s}+\beta_1 f \\
&-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]]f-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]+\frac{\partial}{\partial t}[\frac{\kappa}{\varpi}]]] \\
&+f[\frac{\varpi}{\tau}[(\frac{\partial \beta_1}{\partial s}-\beta_2 f)+\frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]] \\
&-f[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]]]]-g[\frac{\varpi}{\tau}[(\frac{\partial \beta_3}{\partial s} \\
&+\beta_2 g)+\frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s} \\
&+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]g]-\frac{\partial}{\partial t}[\frac{\tau}{\varpi}]]]] \\
=\ &[-\frac{\partial g}{\partial t}-\frac{g}{f}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g) \\
&+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]] \\
&+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]]f-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g) \\
&+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]].
\end{aligned}
$$

Proof. It is obvious by using Theorem 3.10.

Corollary 3.12.

$$
\begin{aligned}
&[\frac{\partial}{\partial s}[\frac{\varpi}{\tau}[(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+\frac{1}{f}\frac{\kappa}{\varpi}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]] \\
&+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]g]-\frac{\partial}{\partial t}[\frac{\tau}{\varpi}]]] \\
&+g[\frac{\varpi}{\tau}[(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+\frac{1}{f}\frac{\kappa}{\varpi}[-\frac{\partial f}{\partial t}+\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)+(\frac{\partial \beta_1}{\partial s} \\
&-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_1}{\partial s}-\beta_2 f)-(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)f-\frac{\partial \kappa}{\partial t}]]f \\
&-[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]g]+\frac{\partial}{\partial t}[\frac{\kappa}{\varpi}]]]] \\
= \quad &-\frac{g}{f}[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_3}{\partial s}+\beta_2 g)+(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g)g]]+[\frac{1}{\kappa}[\frac{\partial}{\partial s}(\frac{\partial \beta_2}{\partial s}+\beta_1 f-\beta_3 g) \\
&+(\frac{\partial \beta_1}{\partial s}-\beta_2 f)f-(\frac{\partial \beta_3}{\partial s}+\beta_2 g)g]]g].
\end{aligned}
$$

Proof. It is obvious by using Theorem 3.10.

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References

[1] V. Asil, Velocities of dual homothetic exponential motions in D^3. Iran. J. Sci. Technol. Trans. A: Sci. 31, 265–271 (2007).

[2] M. do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, Englewood Cliffs, 1976.

[3] I. Dimitric, Submanifolds of $\mathbb{E}^m$ *with harmonic mean curvature vector,* Bull. Inst. Math. Acad. Sinica 20, 53–65 (1992).

[4] J. Eells and L. Lemaire, A report on harmonic maps, Bull. London Math. Soc. 10, 1–68 (1978).

[5] J. Eells and J. H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86, 109-160 (1964).

[6] M. Gage, R.S. Hamilton, The heat equation shrinking convex plane curves, J. Differential Geom. 23, 69–96 (1986).

[7] M. Grayson, The heat equation shrinks embedded plane curves to round points, J. Differential Geom. 26, 285–314 (1987).

[8] T. Körpınar, A new method for inextensible flows of timelike curves in 4-dimensional LP-Sasakian manifolds, Asian-European Journal of Mathematics, 8 (4) (2015), DOI: 10.1142/S1793557115500734

[9] T. Körpınar, E. Turhan, Time Evolution Equations for Surfaces Generated via Binormal Spherical Image in Terms of Inextensible Flows in E^3, Journal of Dynamical Systems and Geometric Theories 12(2), 145-157 (2014)

[10] DY. Kwon, FC. Park, DP Chi, Inextensible flows of curves and developable surfaces, Appl. Math. Lett. 18, 1156-1162 (2005).

[11] T. Lamm, Biharmonic map heat flow into manifolds of nonpositive curvature, Calc. Var. 22, 421-445 (2005).

[12] D.J. Struik, Lectures on Classical Differential Geometry, Dover, New-York, 1988.

[13] E. Turhan and T. Körpınar, On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group Heis^3, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a, 641-648 (2010).

[14] E. Turhan and T. Körpınar, Parametric equations of general helices in the sol space, Bol. Soc. Paran. Mat., 31 (1), 99–104 (2013).

[15] M. Turgut, On the invariants of time-like dual curves, Hacettepe J. Math. Stat., 37, 129-133 (2008).

[16] D.J. Unger, Developable surfaces in elastoplastic fracture mechanics, Int. J. Fract. 50, 33–38 (1991).

[17] B. Uzunoğlu, İ. Gök, Y. Yaylı, A new approach on curves of constant precession, Applied Mathematics and Computation 275, 317-323 (2016)