

SEIR EPIDEMIC MODEL WITH TWO TIME DELAYS

TAYEBEH WAEZIZADEH

DEPARTMENT OF PURE MATHEMATICS, FACULTY OF MATHEMATICS AND
COMPUTER & MAHANI MATHEMATICAL RESEARCH CENTER, SHAHID BAHONAR
UNIVERSITY OF KERMAN 76169-1411, KERMAN, IRAN
E-MAIL: WAEZIZADEH@UK.AC.IR

ABSTRACT. In this paper we introduce the *SEIR* epidemic model with latent and infectious time delays, which are denoted by ω and τ respectively. As follows we consider two different cases, $\omega = 0$ and $\omega \neq 0 \neq \tau$. Stability near disease-free and endemic equilibrium points in different cases are investigated.

AMS Classification: 92B05, 12H20, 37G20

Keywords: Epidemic; Differential Equation; Equilibrium Point; Asymptotically Stable; Modeling

(Received: 18 January 2016, Accepted: 14 July 2016)

1. INTRODUCTION

The mathematical modeling for epidemic disease has been started since 1927, when Kermak and McKendrick constructed a system of ODE. Their model was a "compartment model" [4, 13] (For more details see, [8, 3, 16, 7]). The most basic and important questions to ask for epidemic disease models are global asymptotic behaviors [21, 19].

After infectious individuals lived through an infection period, he/she recovers completely or dies because of disease and transfer to the removed class, so in the mathematical model the number of the removed of the infectious should be considered.

Hence we have a "time delay" parameter which is "the length of infection period". And it is important to consider it in the model.

Time delay is introduced into epidemic disease models, which greatly enriches biologic background. The global asymptotic stability of the well-known SIR epidemic model with a time delay was investigated in [1, 2, 18]. In [14, 9], SIR and SEIR models with discrete delays were analyzed. They proved the global stability of disease-free equilibrium point. Mc Cluskey in [15] gave proofs for the global stability of the endemic equilibrium for SIR and SEIR epidemic models with both finite and infinite time delay. The authors of [20] have been considered a disease transmission model of SEIR type with exponential demographic structure. In it the latent period is assumed to be constant. They are considered the global attractiveness of the disease-free equilibrium and existence of endemic equilibrium.

On the other hand, for some diseases (for example, tuberculosis, influenza, measles and AIDS) on sufficient contact with an infective, a susceptible individual becomes exposed, that is, infected but not yet infective. This individual remains in the exposed class a certain latent period before becoming infective.

For many infectious diseases, it is important to consider influences of delays that represents, latent period. This is because not only the lengths of delays may long. For example, the latent period for HIV could reach 10 years, but also different dynamic behavior of mathematical models can be induced by time delays. Thus, it is realistic to introduce a latent delay into the system.

Now most of the research literature on SEIR epidemic models are established by ODE [8, 5, 6] delayed ODE [19, 10, 17].

Therefore, in present paper we consider SEIR epidemic model with two time delays, the length of infection period and latent period. As follow, we study their dynamic behaviors.

The organization of this paper is as follows. In the next section, we introduce the SEIR epidemic model with time delays. In section 3, the dynamic behavior of the SEIR epidemic model with a time delay is investigated and the sufficient conditions of asymptotically stability at equilibrium points are obtained. The stability of disease-free equilibrium point with two time delay is investigated in section 4. Lastly, we investigate the dynamic behavior near the endemic equilibrium point.

2. MATHEMATICAL MODEL

Suppose the population size is the constant, K . The population is divided into four compartments. The first is susceptible individuals, who are not infected but if they have sufficient contact with infected people, they will be infective. The second compartment is the exposed class, which is the class of infected individuals who are not infective yet. The third group is the infective class and the last compartment is the removed class which is the class of removed individuals because of disease. This compartments show with S , E , I and R respectively [12, 11].

In this model, an individual who enters to the class I at time t should have been infected at time $t - \omega$ and who removes at time t should have been infective at time $t - \omega - \tau$ where two constants ω and τ are time delays.

The birth rate in population is $B = \mu K$ and the natural death rate is μ . β is the transmission rate of infection and α is the recovery rate. All parameters are positive and between zero and one. The mathematical model is [13]

$$\begin{aligned}\frac{dS}{dt} &= \mu[K - S(t)] - \beta S(t)I(t); \\ \frac{dE}{dt} &= \beta S(t)I(t) - \beta S(t - \omega)I(t - \omega)e^{-\mu\omega} - \mu E(t); \\ \frac{dI}{dt} &= \beta S(t - \omega)I(t - \omega)e^{-\mu\omega} - \beta S(t - \omega - \tau)I(t - \omega - \tau)e^{-\mu(\omega + \tau)}e^{-\alpha\tau} - (\mu + \alpha)I(t); \\ \frac{dR}{dt} &= \beta S(t - \omega - \tau)I(t - \omega - \tau)e^{-\mu(\omega + \tau)}e^{-\alpha\tau} - \mu R(t);\end{aligned}$$

R does not appear in the other equations and it has not an important role in epidemic so, the fourth equation is omitted in dynamics. The phase space is

$$T_0 = \{(S, E, I) : 0 \leq S + E + I \leq K\}.$$

The model has two equilibrium points. $Q_0 = (K, 0, 0)$ is disease free equilibrium point and endemic equilibrium point is $Q^* = (S^*, E^*, I^*)$ where

$$S^* = \frac{\mu + \alpha}{\beta A}, I^* = \frac{\mu K A}{\mu + \alpha} - \mu(\mu + \alpha) \text{ and } E^* = \frac{1}{\mu}(1 - e^{-\mu\omega})\beta I^* \frac{K(\mu + \alpha)}{\mu + \alpha - \beta(\mu + \alpha)^2 + \beta K A}$$

and $A = e^{-\mu\omega}(1 - e^{-\tau(\mu + \alpha)})$. The endemic equilibrium point exists if $K A \geq (\mu + \alpha)^2$ and $\tau > 0$.

3. DYNAMICAL BEHAVIOR OF THE MODEL WITH A TIME DELAY

In this section, dynamical behavior of the model when $\omega = 0$ is investigated. By these assumptions, the model is reduced to the following model

$$\frac{dS}{dt} = \mu[K - S(t)] - \beta S(t)I(t);$$

$$\begin{aligned}\frac{dI}{dt} &= \beta S(t)I(t) - \beta S(t - \tau)I(t - \tau)e^{-\tau(\mu + \alpha)} - (\mu + \alpha)I(t); \\ \frac{dR}{dt} &= \beta S(t - \tau)I(t - \tau)e^{-\tau(\mu + \alpha)} - \mu R(t);\end{aligned}$$

The basic reproduction number is

$$R_0 = \frac{\beta K(1 - e^{-\tau(\mu + \alpha)})}{\mu + \alpha}.$$

The third equation is omitted in calculations, because removed individuals does not have an effective relationship in epidemic and R doesn't appear in the other equations. The disease-free equilibrium point is $Q_0 = (S_0, I_0)$, where $S_0 = K$ and $I_0 = 0$.

The characteristic matrix at Q_0 is

$$(3.1) \quad \begin{pmatrix} \mu + \lambda & \beta K \\ 0 & \beta KC + \mu + \alpha + \lambda \end{pmatrix}$$

where $C = e^{-\tau(\mu + \alpha + \lambda)} - 1$. The characteristic equation is

$$(\mu + \lambda)(\beta KC + \mu + \alpha + \lambda) = 0.$$

One of eigenvalues is $-\mu$ and the others satisfy the following equation

$$(3.2) \quad \lambda + a_1 e^{-\tau\lambda} + a_2 = 0$$

where

$$a_1 = \beta K e^{-\tau(\mu + \alpha)}$$

and

$$a_2 = \mu + \alpha - \beta K$$

Let $\lambda = \lambda_1 + i\lambda_2$. Separating the real and imaginary parts of characteristic equation, we obtain

$$\lambda_1 + a_2 + a_1 e^{-\tau\lambda_1} \cos\lambda_2\tau = 0$$

$$\lambda_2 - a_1 e^{-\tau\lambda_1} \sin\lambda_2\tau = 0$$

Without loss of generality, consider $\lambda_2 > 0$, so the following theorem is established.

Theorem 3.1. *If $\mu + \alpha > \beta K(1 + e^{-\tau(\mu + \alpha)})$, then Q_0 is asymptotically stable.*

Proof. By the above relations, since $\lambda_2 > 0$, then $\sin\lambda_2\tau > 0$. Therefore $2k\pi \leq \lambda_2\tau \leq 2k\pi + \pi$ for all $k \in \mathbb{Z}$.

If $2k\pi \leq \lambda_2\tau \leq 2k\pi + \frac{\pi}{2}$, $\cos\lambda_2\tau > 0$ and since $a_2 > 0$ then $\lambda_1 < 0$ for all $\tau > 0$.

Hence Q_0 is asymptotically stable.

In the other case, if $2k\pi + \frac{\pi}{2} \leq \lambda_2\tau \leq (2k+1)\pi$ and $\lambda_1 > 0$, then by the theorem assumption $\mu + \alpha > \beta K(1 + e^{-\tau(\mu+\alpha)}) > \beta K(1 + e^{-\tau(\mu+\alpha+\lambda_1)})$.

Hence, $\mu + \alpha - \beta K > \beta K e^{-\tau(\mu+\alpha+\lambda_1)}$ and $a_2 > a_1 e^{-\tau\lambda_1} > -a_1 e^{-\tau\lambda_1} \cos\lambda_2\tau$.

Therefore, $a_2 + a_1 e^{-\tau\lambda_1} \cos\lambda_2\tau > 0$, and this is a contradiction. $\square$

Remark 3.2. If Q_0 is asymptotically stable, then $R_0 < 1$.

The endemic equilibrium point is $Q^* = (S^*, I^*)$, where $S^* = \frac{\mu+\alpha}{\beta A}$ and $I^* = \frac{\mu K A}{\mu+\alpha} - \mu(\mu + \alpha)$ which $A = 1 - e^{-\tau(\mu+\alpha)}$. The equilibrium point exists if $K A > (\mu + \alpha)$.

The characteristic equation at the endemic equilibrium point Q^* is

$$(3.3) \quad \lambda^2 + a_1\lambda + a_2 + e^{-\tau\lambda}(a_3\lambda + a_4) = 0$$

where

$$a_1 = 2\mu + \alpha + \beta(I^* - S^*);$$

$$a_2 = \beta^2 S^* I^* (1 - e^{-\tau(\mu+\alpha)}) + (\mu + \beta I^*)(\mu + \alpha - \beta S^*);$$

$$a_3 = \beta S^* e^{-\tau(\mu+\alpha)};$$

$$a_4 = \beta S^* e^{-\tau(\mu+\alpha)}(\mu + \beta I^*);$$

If the real part of an eigenvalue be zero then $\lambda = iy$ and the characteristic equation is $-y^2 + ia_1y + a_2 + e^{-i\tau y}(a_3iy + a_4) = 0$.

Separating the real and imaginary parts of the characteristic equation, we obtain

$$y^2 - a_2 - a_4 \cos\tau y + a_3 y \sin\tau y = 0$$

$$a_1 y + a_3 y \cos\tau y + a_4 \sin\tau y = 0$$

Hence, there is τ_0 such that $\sin\tau_0 y = \frac{y(a_3(a_2 - y^2) - a_1 a_4)}{a_4^2 + a_3^2 y^2}$. Set

$$\tau = \frac{1}{y} \left(\text{Arcsin} \frac{y(a_3(a_2 - y^2) - a_1 a_4)}{a_4^2 + a_3^2 y^2} + \pi \right).$$

Therefore, $\sin\tau y > 0$.

Lemma 3.3. Let $\beta(\mu + \alpha) < \frac{1}{2}$ and $\mu(K\beta A - \mu - \alpha) > \frac{\mu+\alpha}{A}$, then $a_1 > a_2$.

Proof. By theorem assumptions, we have $\beta^2 S^* I^* A = \mu\beta(K - S^*)A = \mu(K\beta A - \mu - \alpha) > \frac{\mu+\alpha}{A}$. On the other hand, $\beta(\mu + \alpha) < \frac{1}{2}$, then

$$\beta^2 S^* I^* A + (\mu + \alpha)\left(\frac{1}{A} - 1\right) < \beta^2 S^* I^* A + (\mu + \alpha)\frac{1}{A} < 2\beta^2 S^* I^* A < \frac{\beta\mu K A}{\mu+\alpha}.$$

Therefore, $a_1 - a_2 > 2\mu + \alpha + \beta(I^* - S^*) - \beta^2 S^* I^* A = \mu(1 - \beta(\mu + \alpha)) + \mu + \alpha + \frac{\beta\mu K A}{\mu+\alpha} - \frac{\mu+\alpha}{A} - \beta^2 S^* I^* A > 0$. $\square$

Theorem 3.4. *If $\beta(\mu + \alpha) < \frac{1}{2}$, $\mu(K\beta A - \mu - \alpha) > \frac{\mu + \alpha}{A}$ and $\tau = \frac{1}{y}(\text{Arcsin} \frac{y(a_3(a_2 - y^2) - a_1 a_4)}{a_4^2 + a_3^2 y^2} + \pi)$ where y is imaginary part of eigenvalue of characteristic, then Q^* is asymptotically stable.*

Proof. Let $\lambda = x + iy$ be an eigenvalue of the characteristic equation 3.3. Without loss of generality we consider $y > 0$. Separating real and imaginary parts of characteristic equation. We obtain

$$x^2 - y^2 + a_1 x + a_2 + e^{-\tau x}((a_3 x + a_4) \cos \tau y - a_3 y \sin \tau y) = 0;$$

$$2xy - a_1 y + e^{-\tau x}(a_3 y \cos \tau y + (a_3 x + a_4) \sin \tau y) = 0;$$

We have

$$(1 + (\frac{a_3 x + a_4}{a_3 y})^2) \sin \tau y = \frac{x^2 - y^2 + a_1 x + a_2}{a_3 y e^{-\tau x}} - \frac{(a_3 x + a_4)(2xy + a_1 y)}{a_3^2 y^2 e^{-\tau x}}.$$

Since $\sin \tau y > 0$, so

$$x^2 a_3 y - y^3 a_3 + a_1 a_3 x y + a_2 a_3 y - (a_3 x + a_4)(2xy - a_1 y) > 0.$$

By some calculation, we have

$$a_3 x^2 + 2x a_4 + y^2 a_3 - a_2 a_3 + a_1 a_4 < 0.$$

Clearly $a_4 > a_3$. If $a_2 < 0$, there is no nonnegative x which satisfies above equation.

Hence, the proof is complete.

If $a_2 > 0$, using lemma and the theorem assumptions we have $a_1 > a_2$. Therefore, $x < 0$ and Q^* is asymptotically stable. $\square$

4. The stability of disease-free equilibrium point with two time delays

In this section, is investigated the stability of disease-free equilibrium point in different conditions where $\tau \neq 0$ and $\omega \neq 0$. The basic reproduction number is

$$(4.1) \quad R_0 = \frac{\beta K(1 - e^{-(\mu + \alpha)\tau})}{\mu + \alpha} + \frac{\beta K(1 - e^{-\mu\omega})}{\mu}$$

The linearizing system at Q_0 yields the following characteristic equation

$$\det \begin{pmatrix} -\mu - \lambda & 0 & -\beta K \\ 0 & -\mu - \lambda & \beta K(1 - e^{-\omega(\lambda + \mu)}) \\ 0 & 0 & \beta K e^{-\omega(\lambda + \mu)} - \beta K C - (\mu + \alpha) - \lambda \end{pmatrix} = 0$$

where $C = e^{-(\omega + \tau)(\lambda + \mu)} e^{-\alpha\tau}$. Two of roots are $-\mu$ and about the other eigenvalues the following theorems is established.

Theorem 4.1. *If $R_0 < 1$ and $\tau < \omega(e^{\tau(\mu + \alpha)} - 1)$, the characteristic equation has no any positive real solution.*

Proof. $h(\lambda) = \beta K e^{-\omega(\lambda+\mu)}(1 - e^{-\tau(\lambda+\mu+\alpha)}) - (\mu + \alpha) - \lambda$ is the characteristic equation.

Then $h(0) = \beta K e^{-\omega\mu}(1 - e^{-\tau(\mu+\alpha)}) - (\mu + \alpha)$ and $R_0 < 1$, then $h(0) < (\mu + \alpha)(e^{-\mu\omega} - 1) < 0$ and

$$\begin{aligned} h'(\lambda) &= -\beta\omega K e^{-\omega(\lambda+\mu)}(1 - e^{-\tau(\lambda+\mu+\alpha)}) + \tau\beta K e^{-\omega(\mu+\lambda)}e^{-\tau(\lambda+\mu+\alpha)} - 1 \\ &= \beta K e^{-\omega(\lambda+\mu)}((\tau + \omega)e^{-\tau(\lambda+\mu+\alpha)} - \omega) - 1. \end{aligned}$$

Since $\tau < \omega(e^{\tau(\mu+\alpha)} - 1)$ then $\omega(1 - e^{-\tau(\mu+\alpha)}) > \tau e^{-\tau(\mu+\alpha)}$ and $(\omega + \tau)e^{-\tau(\lambda+\mu+\alpha)} < (\omega + \tau)e^{-\tau(\mu+\alpha)} < \omega$. Hence, $h'(\lambda) < 0$ and $h(\lambda) < 0$ for all $\lambda > 0$. Thus, if $h(\lambda)$ has a real root it is negative, so all real eigenvalues are negative. $\square$

Theorem 4.2. *If $R_0 < 1$ and $\omega > \frac{Ln2}{\mu}$, $h(\lambda)$ has no complex eigenvalue with zero real part.*

Proof. If $h(\lambda)$ has a complex root with zero real part, so $h(\lambda) = 0$ and $\lambda = \pm iy$ where y is a positive real number, $h(iy) = \beta K e^{-\omega(iy+\mu)}(1 - e^{-\tau(iy+\mu+\alpha)}) - (\mu + \alpha) - iy = 0$. Separating the real and imaginary parts of $h(iy)$, we obtain

$$Reh(iy) = \beta K e^{-\mu\omega}(\cos\omega y - e^{-(\mu+\alpha)\tau}\cos(\omega + \tau)y) - (\mu + \alpha) = 0$$

$$Imh(iy) = \beta K e^{-\mu\omega}(\sin\omega y - e^{-(\mu+\alpha)\tau}\sin(\omega + \tau)y) - y = 0$$

Solving the equation system, the following equation is established

$$\beta^2 K^2 e^{-2\mu\omega}(1 + e^{-2(\mu+\alpha)\tau} - 2e^{-(\mu+\alpha)\tau}\cos\tau y) = y^2 + (\mu + \alpha)^2.$$

Also by the theorem assumptions, $e^{-2\mu\omega} < \frac{1}{4}$ and $\beta K < \mu + \alpha$. Hence, $y^2 + (\mu + \alpha)^2 < 4\beta^2 K^2 e^{-2\mu\omega} < 4(\mu + \alpha)^2 e^{-2\mu\omega}$, so $y^2 < (\mu + \alpha)^2(4e^{-2\mu\omega} - 1) < 0$ and this is a contradiction. Hence all the complex roots of the characteristic equation have nonzero real parts. $\square$

Theorem 4.3. *If $\omega > \frac{Ln2}{\mu}$ and $\beta K < \mu + \alpha$, then the real part of complex eigenvalues of characteristic equation are negative.*

Proof. Let $\lambda = x + iy$ be a complex eigenvalue and $x > 0$. Without loss of generality, we assume that $y > 0$. Then the characteristic equation is

$$h(\lambda) = \beta K e^{-\mu\omega} e^{-\omega x}(\cos\omega y - i\sin\omega y)(1 - e^{-\tau(\mu+\alpha+x)}(\cos\tau y - i\sin\tau y)) - \mu - \alpha - x - iy.$$

The real and imaginary parts of $h(\lambda)$ are

$$Reh(\lambda) = \beta K e^{-\omega(\mu+x)}(\cos\omega y - e^{-\tau(\mu+\alpha+x)}\cos\omega y\cos\tau y) - \mu - \alpha - x +$$

$$\beta K e^{-\omega(\mu+x)}e^{-\tau(\mu+\alpha+x)}\sin\omega y\sin\tau y$$

$$= \beta K e^{-\omega(\mu+x)}(\cos\omega y - e^{-\tau(\mu+\alpha+x)}\cos(\omega + \tau)y) - \mu - \alpha - x = 0$$

$$\begin{aligned} Imh(\lambda) &= \beta K e^{-\omega(\mu+x)} (e^{-\tau(\mu+\alpha+x)} \cos \omega y \sin \tau y - \sin \omega y \\ &+ e^{-\tau(\mu+\alpha+x)} \sin \omega y \cos \tau y) - y = \beta K e^{-\omega(\mu+x)} (-\sin \omega y + e^{-\tau(\mu+\alpha+x)} \sin(\omega+\tau)y) - y = \\ &0. \end{aligned}$$

And

$$\begin{aligned} Re^2 h(\lambda) + Im^2 h(\lambda) &= \beta^2 K^2 e^{-2\omega(\mu+x)} (1 + e^{-2\tau(\mu+\alpha+x)} - 2e^{-\tau(\mu+\alpha+x)} \cos \tau y) \\ &= (\mu + \alpha + x)^2 + y^2. \end{aligned}$$

Using the theorem assumptions,

$$y^2 + (\mu + \alpha + x)^2 < 4\beta^2 K^2 e^{-2\omega(\mu+\alpha)} < 4(\mu + \alpha) e^{-2\omega(\mu+\alpha)} < 4(\mu + \alpha + x)^2 e^{-2\omega(\mu+\alpha)}.$$

Thus,

$$y^2 < (\mu + \alpha + x)^2 (4e^{-2\omega(\mu+\alpha)} - 1) < 0. \text{ This is a contradiction.} \quad \square$$

Corollary 4.4. *If $R_0 < 1$, $\omega > \frac{Ln2}{\mu}$, $\beta K < \mu + \alpha$ and $\tau < \omega(e^{\tau(\mu+\alpha)} - 1)$, the equilibrium point Q_0 is asymptotically stable.*

5. STABILITY AT Q^*

The linearization of system at the equilibrium point Q^* is

$$\begin{aligned} \frac{dx}{dt} &= -\mu x(t) - \beta S^* z(t) - \beta I^* x(t); \\ \frac{dy}{dt} &= \beta S^* z(t) + \beta I^* x(t) - e^{-\mu\omega} \beta S^* z(t - \omega) - \beta I^* x(t - \tau) e^{-\mu\omega} - \mu y(t); \\ \frac{dz}{dt} &= \beta S^* z(t - \omega) e^{-\mu\omega} + \beta x(t - \tau) I^* e^{-\mu\omega} - \beta S^* z(t - \omega - \tau) e^{-\mu(\omega+\tau)} e^{-\alpha\tau} \\ &- \beta x(t - \omega - \tau) I^* e^{-\mu(\omega+\tau)} e^{-\alpha\tau} - (\mu + \alpha) z(t). \end{aligned}$$

where $x(t - \tau) = e^{-\lambda\tau} x(t)$ and $z(t - \omega) = e^{-\lambda\omega} z(t)$.

The characteristic matrix is

$$\begin{pmatrix} -\mu - \lambda - \beta I^* & 0 & -\beta S^* \\ \beta I^* (1 - e^{-\mu\omega} e^{-\tau\lambda}) & -\mu - \lambda & \beta S^* (1 - e^{-\omega(\lambda+\mu)}) \\ \beta I^* (e^{-\mu\omega} e^{-\tau\lambda} - D) & 0 & \beta S^* (e^{-\omega(\lambda+\mu)} - D) - (\mu + \alpha) - \lambda \end{pmatrix}$$

where $D = e^{-(\mu+\lambda)(\omega+\tau)} e^{-\alpha\tau}$. The characteristic equation is

$$(5.1) \quad (-\mu - \lambda)(\lambda^2 + (a_1 + b_1)\lambda + a_2 + b_2) = 0.$$

Where

$$a_1 = 2\mu + \beta I^* + \alpha;$$

$$a_2 = (\mu + \alpha)(\mu + \beta I^*);$$

$$b_1 = \beta S^*(e^{-\omega\lambda}e^{-\tau\lambda}e^{-\mu(\omega+\tau)}e^{-\alpha\tau} - e^{-\omega(\lambda+\mu)});$$

and

$$b_2 = (-\mu - \beta I^*)\beta S^*e^{-\omega(\lambda+\mu)} + \mu D + \beta^2 S^* I^* D + \beta^2 S^* I^* e^{-\mu\omega}e^{-\tau\lambda} \\ - \beta^2 S^* I^* e^{-\lambda(\mu+\tau)}e^{-\mu(\omega+\tau)}e^{-\alpha\tau}.$$

It is obvious that, one of the eigenvalues is $-\mu$. To find others, set $H(\lambda) = \lambda^2 + a_1\lambda + a_2 + b_1\lambda + b_2$. We have

$$H(0) = (\mu + \alpha)(\mu + \beta I^*) + (-\mu - \beta I^*)\beta S^*e^{-\mu\omega}(1 - e^{-\tau(\mu+\alpha)}) + \beta^2 S^* I^* e^{-\mu\omega}(1 - e^{-\tau(\mu+\alpha)}) = (\mu + \alpha)(\mu + \beta I^*) + \mu\beta S^*e^{\mu\omega}e^{-\tau(\mu+\alpha)} > 0.$$

Hence, there is no zero eigenvalue. In the next theorem, we find conditions, which real eigenvalues are negative.

Theorem 5.1. *If $\omega > \tau > \frac{Ln2}{\mu+\alpha}$ and $\frac{\beta\mu KA}{\mu+\alpha} > \frac{\alpha}{\mu} + \beta(\mu + \alpha)$, then real eigenvalues are negative.*

Proof. Let λ be a positive real eigenvalue and $H(\lambda) = t(\lambda) + f(\lambda)$. Where $t(\lambda) = (2\mu + \alpha + \beta I^* - \beta S^*e^{-\omega(\lambda+\mu)})\lambda + \beta S^* D\lambda$

and

$$f(\lambda) = (\mu + \alpha)(\mu + \beta I^*) - (\mu + \beta I^*)\beta S^*e^{-\omega(\lambda+\mu)} + \mu\beta S^* D + \beta^2 S^* I^* e^{-\mu\omega}e^{-\tau\lambda}.$$

By theorem assumptions,

$$2\mu + \alpha + \beta I^* - \beta S^*e^{-\omega(\lambda+\mu)} \geq 2\mu + \alpha + \beta I^* - \beta S^*e^{-\omega\mu} = 2\mu + \alpha + \frac{\beta\mu KA}{\mu+\alpha} - \beta\mu(\mu + \alpha) - \frac{\mu+\alpha}{1-e^{-\tau(\mu+\alpha)}} > 2\mu + \alpha + \frac{\alpha}{\mu} + \beta(\mu + \alpha)(1 - \mu) - \frac{\mu+\alpha}{1-e^{-\tau(\mu+\alpha)}} > 2\mu + \alpha + \frac{\alpha}{\mu} - 2(\mu + \alpha) > 0.$$

Thus $t(\lambda) > 0$ for all $\lambda > 0$.

As follow $f(0) = (\mu + \alpha)(\mu + \beta I^*) - \mu\beta S^*e^{-\mu\omega}(1 - e^{-\tau(\mu+\alpha)})$. On the other hand, by the theorem assumption, $\frac{\beta\mu KA}{\mu+\alpha} > \frac{\alpha}{\mu} + \beta(\mu + \alpha)$, and $I^* = \frac{\mu KA}{\mu+\alpha} - \mu(\mu + \alpha)$, we have $\mu + \beta I^* = \mu(1 + \frac{\beta KA}{\mu+\alpha} - \beta(\mu + \alpha)) > \mu + \alpha > \mu(\mu + \alpha)$. Hence $f(0) > 0$.

Since $\tau > \frac{Ln2}{\mu+\alpha}$ and $\omega > \tau$, then $e^{-\tau(\mu+\alpha)} < \frac{1}{2}$ and $\omega > 2\omega e^{-\tau(\mu+\alpha)} > (\omega + \tau)e^{-\tau(\mu+\alpha)} > (\omega + \tau)e^{-\tau(\mu+\alpha+\lambda)}$.

As a result

$$f'(\lambda) = (\mu + \beta I^*)\beta S^*\omega e^{-\omega(\lambda+\mu)} - \mu\beta S^*(\omega + \tau)D - \beta^2 S^* I^* e^{-\mu\omega}e^{-\tau\lambda} = \\ \beta^2 S^* I^* e^{-\mu\omega}(\omega e^{-\tau\lambda} - \tau e^{-\omega\lambda}) + \mu\beta S^*\omega e^{-\omega(\lambda+\mu)}(\omega - (\omega + \tau)e^{-\tau(\mu+\alpha+\lambda)}) > 0. \text{ Therefore } H(\lambda) > 0, \text{ for all } \lambda > 0 \text{ and the proof is complete. } \square$$

Theorem 5.2. *If $\frac{\mu K}{\beta} < \tau < \frac{Ln2}{\mu+\alpha}$ and $\mu < \beta e^{-\mu\omega} e^{-\tau(\mu+\alpha)}$, then Q^* is asymptotically stable.*

Proof. The following map is considered

$$L(S, I) = \begin{cases} S - S^* - Ln \frac{S}{S^*} + I - I^* & (S, I) \in \Gamma_1 \\ S^* - S + Ln \frac{S}{S^*} + I - I^* & (S, I) \in \Gamma_2 \\ S^* - S + Ln \frac{S}{S^*} + I^* - I & (S, I) \in \Gamma_3 \\ S - S^* - Ln \frac{S}{S^*} + I^* - I & (S, I) \in \Gamma_4 \end{cases}$$

Where,

$$\Gamma_1 = \{(S, I) : S \geq S^*, I \geq I^*\};$$

$$\Gamma_2 = \{(S, I) : S \leq S^*, I \geq I^*\};$$

$$\Gamma_3 = \{(S, I) : S \leq S^*, I < I^*\};$$

$$\Gamma_4 = \{(S, I) : S \geq S^*, I < I^*\};$$

We prove that this map is a Lyapunov function when $L(S^*, I^*) = 0$ and $L(S, I) > 0$ for all $(S, I) \in \cup_{i=1}^4 \Gamma_i - \{(S^*, I^*)\}$. Also $\dot{L} = \dot{S}(1 - \frac{1}{S}) + \dot{I}$ on Γ_1 . Since $S > S^*$, then $\dot{S} < 0$. Also by theorem assumptions

$$\begin{aligned} \dot{L} &< \dot{S} + \dot{I} = \mu(K - S(t)) - \beta S(t)I(t) + \beta S(t - \omega)I(t - \omega)e^{-\mu\omega} - \beta S(t - \omega - \tau)I(t - \omega - \tau)e^{-\mu(\omega+\tau)}e^{-\alpha\tau} - (\mu + \alpha)I(t) < \mu(K - S(t)) - \beta e^{-\mu\omega}(S(t)I(t) - S(t - \omega)I(t - \omega)) \\ &- \beta S(t - \omega - \tau)I(t - \omega - \tau)e^{-\mu(\omega+\tau)}e^{-\alpha\tau} - (\mu + \alpha)I(t) = \mu(K - S(t)) - \beta e^{-\mu\omega} \frac{d}{dt} \int_{t-\omega}^t S(u)I(u)du - \beta S(t - \omega - \tau)I(t - \omega - \tau)e^{-\mu(\omega+\tau)}e^{-\alpha\tau} - (\mu + \alpha)I(t). \end{aligned}$$

Using mean value theorem, there is $t^* \in (t - \omega, t)$ such that

$$\begin{aligned} \dot{L} &< \mu(K - S(t)) - \beta \omega e^{-\mu\omega} S(t^*)I(t^*) - \beta S(t - \omega - \tau)I(t - \omega - \tau)e^{-\mu(\omega+\tau)}e^{-\alpha\tau} - (\mu + \alpha)I(t) < \mu(K - S^*) - \beta K e^{-\mu\omega} e^{-\tau(\mu+\alpha)} - (\mu + \alpha)I^* < 0. \end{aligned}$$

Thus, $\dot{L} < 0$ on Γ_1 . If $(S, I) \in \Gamma_2$ Since $S < S^*$, clearly $\dot{S} > 0$. By theorem assumptions, $\dot{I} < 0$. Therefore, $\dot{L} = \dot{S}(\frac{1}{S} - 1) + \dot{I} < 0$ on Γ_2 .

If $(S, I) \in \Gamma_3$, since $S < S^*$, clearly $\dot{S} > \mu K - \mu S^* - \beta S^* I^* = 0$. By theorem assumption $\frac{\mu K}{\beta} < \tau < \frac{Ln2}{\mu+\alpha}$ we have

$$\dot{I} > \beta e^{\mu\omega} e^{-\tau(\mu+\alpha)} \frac{d}{dt} \int_{t-\omega-\tau}^{t-\omega} S(u)I(u)du - (\mu + \alpha)I(t).$$

Using mean value theorem, there is $t^* \in (t - \omega - \tau, t - \omega)$ which

$$\begin{aligned} \dot{I} &> \beta e^{-\mu\omega} e^{-\tau(\mu+\alpha)} \tau S(t^*)I(t^*) - (\mu + \alpha)I(t) \\ &> \beta e^{-\mu\omega} (1 - e^{-\tau(\mu+\alpha)}) \tau S(t^*)I(t^*) - (\mu + \alpha)I(t) \\ &= \beta A \tau S(t^*)I(t^*) - (\mu + \alpha)I(t) > \beta A \tau - (\mu + \alpha)I(t) > \beta A \tau - (\mu + \alpha)I^* > 0. \end{aligned}$$

Thus, $\dot{L} < 0$ on Γ_3 .

Clearly L is positive on Γ_4 and $\dot{L} < 0$ on Γ_4 .



CONCLUSION

In this paper, a mathematical model with two time delays is considered. The model is worked for infectious disease with long latent and infectious periods. As follow, the conditions of asymptotically stability for disease- free equilibrium point and endemic equilibrium point in two different cases are investigated.

REFERENCES

- [1] Edoardo Beretta, Tadayuki Hara, Wanbiao Ma, Yasuhiro Takeuchi, Global asymptotic stability of an SIR epidemic model with distributed time delay, *Nonlinear Analysis*, Vol. 47, 4107-4115 (2001).
- [2] Edoardo Beretta and Yasuhiro Takeuchi, Global stability of an SIR epidemic model with time delays, *Journal of Mathematical Biology*, Vol 33, 250-260 (1995).
- [3] Fred Brauer, *Compartmental models in epidemics*, Lecture notes in mathematics, Springer-Varlag Berlin Heidelberg, 2008.
- [4] Fred Brauer, Peter Van den Driessche, Jang-Mei Wu, *Mathematical Epidemiology*, Springer-Verlag, Berlin Heidelberg, 2008.
- [5] Jianyu Chen, An SIRS epidemic model, *Applied Mathematics-A Journal of Chinese Universities*, Vol. 19, 101-108 (2004).
- [6] Manuela Dordea, Nicolae Coman, Aspects of the Evolutionary Process in small Population, *StudiaUniversitatis Babes-Bolyai*, Vol. 2, 35-51 (2006).
- [7] Miklos Farkas, *Dynamical Models in Biology*, Academic Press, 2001.
- [8] Gang Huang, Yasuhiro Takeuch, Wanbiao Ma, Dongming Wei, Global Stability for Delay SIR and SEIR Epidemic Models with nonlinear Incidence Rate, *Bulletin of Mathematical Biology*, Vol. 72, 1192-1207 (2010).
- [9] Zhong Jin and Zhong Ma, The Stability of an SIR Epidemic Model with Time Delays, *Mathematical Biosciences And Engineering*, Vol. 3, 101-109 (2006).
- [10] Michael Y. Li, Xiaobo Liu, An SIR Epidemic Model with Time Delay and General Nonlinear Incidence Rate, *Abstract and Applied Analysis*, Vol. 2014, 1-7 (2014).
- [11] Tong Li, Qing Liu, Bo Li, The Analysis of an SIRS Epidemic Model with Discrete Delay on Scale-Free Network, *Applied Mathematics*, Vol. 6, 1939-1946 (2015).
- [12] Tong Li, Yunhua Xue, Global stability Analysis of Delayed SEIQR Epidemic Model with Quarantine and Latent, *Applied Mathematics*, Vol. 4, 109-117 (2013).
- [13] Zhong Ma, Jun Li, *Dynamical Modeling and Analysis of Epidemics*, World Scientific publishing Co. Pte. Ltd, 2009.
- [14] Wanbiao Ma, Mei Song and Yasuhiro Takeuchi, Global Stability of an SIR Epidemic Model with Time Delay, *Applied Mathematics Letters*, Vol. 17, 1141-1145 (2004).

- [15] Connell McCluskey, Global Stability of an SIR Epidemic Model with Delay and General Nonlinear Incidence, *Mathematical Biosciences and engineering*, Vol. 7, 837-850 (2010).
- [16] MohammadReza Molaei, A Mathematical Model for an Epidemic in an Open Society, *Studia Universitatis Babes-Bolyai*, 1, 61-65 (2010).
- [17] Yukihiro Nakata, Yoichi Enatsu, Yoshiaki Maroya, On The Global Stability of an SIRS Epidemic Model with Distributed Delays, *Discrete and Continuous Dynamical Systems*, 1119-1128 (2011).
- [18] Yasuhiro Takeuchi, Wanbiao Ma and Edoardo Berettac, Global asymptotic properties of a delay SIR epidemic model with finite incubation times, *Nonlinear Analysis*, Vol. 42, 931-947 (2000).
- [19] Weiqiang Wang, Global behavior of an SEIRS Epidemic Model with Time Delays, *Applied Mathematical Letters*, Vol. 15, 423-428 (2002).
- [20] Pan Yan, Sheng-Chi Liu, SEIR epidemic model with delay, *Australian Mathematical Society*, Vol. 48, 119-134 (2006).
- [21] Yi Yang, Limin Zou, Shigui Ruan, Global Dynamics of a Delayed Within-Host Viral Infection Model with both Viruos-to-Cell and Cell-to-Cell Transmissions, *Mathematical Biosciences*, 1-9 (in press).