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PERTURBED ROBE'S RESTRICTED PROBLEM OF 2+2 BODIES WHEN THE PRIMARIES FORM A ROCHE ELLIPSOID-TRIAXIAL SYSTEM

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Abstract. The aim of this paper is to study the effect of perturbations in the Coriolis and centrifugal forces on the location and stability of the equilibrium solutions in the Robe's restricted problem of 2+2 bodies under the assumption that the hydrostatic equilibrium figure of the first primary is a Roche ellipsoid and the shape of the second primary is triaxial. The third and the fourth bodies (of mass m_3 and m_4 respectively) are small solid spheres of density ρ_3 and ρ_4 respectively inside the ellipsoid, with the assumption that the mass and the radius of the third and the fourth body are infinitesimal. We assume that m_2 is describing a circle around m_1. The masses m_3 and m_4 mutually attract each other, do not influence the motion of m_1 and m_2 but are influenced by them. We have taken into consideration all the three components of the pressure field in deriving the expression for the buoyancy force viz (i) due to the own gravitational field of the fluid (ii)that originating in the attraction of m_2 (iii) that arising from the centrifugal force. The linear stability of this configuration is examined. It is observed that there exist only six equilibrium solutions of the system, provided they lie within the Roche ellipsoid. The equilibrium solutions of m_3 and m_4 lying on x_1-axis are unstable for $\epsilon > 0, \epsilon' > 0$ and $\epsilon < 0, \epsilon' > 0$ and stable for $\epsilon > 0, \epsilon' < 0$ and $\epsilon < 0, \epsilon' < 0$,using the data of submarines in the Earth -Moon system. The equilibrium solutions of m_3 and m_4 respectively when the displacement is given in the direction of x_2 or x_3- axis are conditionally stable.We observe that the conditions of stability are influenced by the small perturbations in the Coriolis and centrifugal forces.

AMS Classification: 70F15, 37N05

Keywords: Robe's Restricted Problem; Roche Ellipsoid; Triaxial; Stability

1. Introduction

Robe [10] has investigated a new kind of restricted three-body problem in which one of the primaries of mass m_1 is a rigid spherical shell filled with a homogeneous incompressible fluid of density ρ_1. The smaller primary is a mass point m_2 outside the shell. The third body of mass m_3, supposed moving inside the shell, is a small solid sphere of density ρ_3, with the assumption that the mass and the radius of the third body are infinitesimal. He assumed that the mass m_2 describes a Keplerian orbit around the mass m_1. He has proved that the center of the first primary, is the only equilibrium solution for all values of the density parameter K, mass parameter

μ, eccentricity parameter e. Further, he has discussed the linear stability of this equilibrium solution. He has explicitly discussed two cases. In the first case, the orbit of m_2 around m_1 is circular and in the second case, the orbit is elliptic, but the shell is empty (i.e. no fluid inside it) or the densities of m_1 and m_3 are equal. In each case, the domain of stability has been investigated for the whole range of parameters occurring in the problem.

In the above problem, the existence of only one equilibrium solution namely, centre of the first primary is discussed. Hallan & Rana [3] studied the existence and the linear stability of all the equilibrium solutions in the Robe's restricted three-body problem. They have proved that besides the center , there are other equilibrium solutions too which exist only when the density parameter $K \neq 0$ and the second primary moves around the first in a circular orbit.

Szebehely [15] considered the effect of small perturbations in the Coriolis force on the stability of equilibrium points in the classical restricted problem, keeping the centrifugal force constant and proved that the collinear points remain unstable. The range of stability of the triangular equilibrium points increase or decrease depending upon whether the change ϵ in the Coriolis force is positive or negative thereby establishing that the Coriolis force is a stabilizing force. In the same problem, Bhatnagar & Hallan [1] studied the effect of the changes ϵ and ϵ' in the Coriolis and centrifugal forces respectively on the stability of equilibrium points and concluded that the collinear points remain unstable For the triangular equilibrium points, the range of stability increases or decreases depending upon whether the point (ϵ, ϵ') lie in one or the other of the two parts in which the line $36\epsilon - 19\epsilon' = 0$ divides the (ϵ, ϵ') plane.

The Robe's problem has been varied by the introduction of perturbing forces by many researchers. Shrivastava and Garain [11] examined the effect of small perturbations in the Coriolis and centrifugal forces on the position of the equilibrium point. In obtaining position of the equilibrium point, they considered the circular case with equal densities (i.e., $\rho_1 = \rho_3$). While evaluating the buoyancy force, both Robe [10] and Shrivastava and Garain [11] assumed that the pressure field of the fluid has spherical symmetry around the center of the shell and they considered only one of the three components of the pressure field, which is , that due to the own gravitational field of the fluid ρ_1. The other two components which are those

arising from the centrifugal force and attraction of m_2 were ignored. Plastino & Plastino [9] took into account all these components of the pressure field. They assumed the hydrostatic equilibrium figure of the first primary as a Roche ellipsoid (Chandrashekhar [2]).

Hallan & Mangang [5] examined Robe's circular restricted three-body problem by considering the buoyancy force as done by Plastino & Plastino [9]. They assumed the hydrostatic equilibrium figure of the first primary as an oblate spheroid and considered the full potential due to the second primary. They found conditions for the existence of an infinite number of equilibrium points and analyzed their stability.

Singh and Mohammed [12] studied the Robe's circular restricted three-body problem under oblate and triaxial primaries. Singh and Sandah [13] studied the existence and linear stability of equilibrium points in the Robe's restricted three-body problem with oblateness. Singh and Mohammed [14] studied the out-of-plane equilibrium points and their stability in the Robe's problem with oblateness and triaxiality. Hallan & Rana [4] studied the effect of perturbations in the Coriolis and centrifugal forces on the location and stability of the equilibrium points in the Robe's circular restricted problem with density parameter having arbitrary value.

Many authors have worked on problem of 2+2 bodies. Significant work is that of Whipple [16]. He studied equilibrium solutions of the restricted problem of 2+2 bodies. He further studied the linear stability of all the equilibrium solutions. Bhavneet & Rajiv [7] extended the Robe's restricted three-body problem to $2 + 2$ bodies. Bhavneet & Rajiv [6] studied the Robe's restricted problem of 2+2 bodies when the bigger primary is a Roche ellipsoid. They studied the equilibrium solutions of the infinitesimal masses and analysed their linear stability. Bhavneet & Rajiv [8] studied Robe's restricted problem of 2+2 bodies when the bigger primary is a Roche ellipsoid and the smaller primary is an oblate body. They studied the equilibrium solutions of the infinitesimal masses and analyzed their linear stability.

Our aim in this paper is to study the effect of perturbations in the Coriolis and centrifugal forces on the location and stability of equilibrium points in Robe's restricted problem of 2 +2 bodies considering the effect of the full buoyancy force of the fluid of the first primary and taking it as Roche ellipsoid and the second primary as a triaxial rigid body.

2. Statement of the Problem and Equations of Motion

Let the first primary of mass m_1 be a Roche ellipsoid filled with homogeneous incompressible fluid of density ρ_1 and the second primary a triaxial rigid body of mass $m_2(m_1 > m_2)$ outside the ellipsoid. The two infinitesimal bodies (of mass m_3 and m_4 respectively) are small solid spheres of density ρ_3 and ρ_4 respectively inside the ellipsoid. Let R denote the distance between the centers of mass of m_1 and m_2. Let m_2 describe a circular orbit of radius R around m_1 with constant angular velocity $\boldsymbol{\omega}$. The masses m_3 and m_4 mutually attract each other, do not influence the motions of m_1 and m_2 but are influenced by them.We assume that (i) m_3 and m_4 never reach the surface of the Ellipsoid (ii)their position vectors at any time t are not the same. Consider a uniformly rotating coordinate system $Ox_1x_2x_3$ with origin of the coordinate system at the center of the bigger primary ,Ox_1 pointing towards m_2 and Ox_1x_2 being the orbital plane of m_2 around m_1 coinciding with the equatorial plane of m_1. The coordinate system $Ox_1x_2x_3$ is as shown in the Figure 1. Let the synodic system of coordinates initially coincident with the inertial system rotate with angular velocity $\boldsymbol{\omega}$. This is the same as the angular velocity of m_2 which is describing a circle around m_1. Let initially the principal axes of m_2 be parallel to the synodic axes and their axes of symmetry be perpendicular to the plane of motion. Since m_2 is revolving without rotation about m_1 with the same angular velocity as that of the synodic axes, the principal axes of m_2 will remain parallel to them throughout the motion.

The equations of motion of m_3 and m_4 in the Robe's circular restricted problem of $2+2$ bodies in the Roche ellipsoid -Triaxial body setup when perturbations are given to the Coriolis and centrifugal forces are given by

$$\ddot{x_1^{(i)}} - 2\alpha\omega\dot{x_2^{(i)}} = U^{(i)}_{x_1^{(i)}}, \tag{1}$$

$$\ddot{x_2^{(i)}} + 2\alpha\omega\dot{x_1^{(i)}} = U^{(i)}_{x_2^{(i)}}, \tag{2}$$

$$\ddot{x_3^{(i)}} = U^{(i)}_{x_3^{(i)}}, \tag{3}$$

where

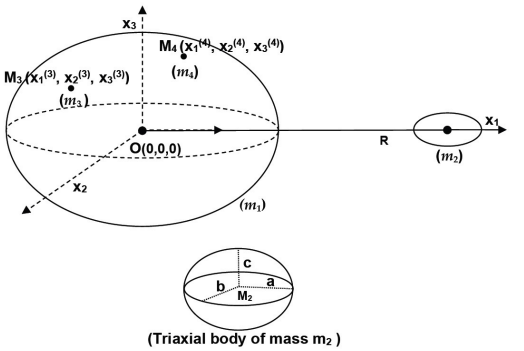


FIGURE 1. **Geometry of the Robe's restricted problem of $2+2$ bodies when the bigger primary is a Roche Ellipsoid and the smaller primary a triaxial body**

$$U^{(i)} = V^{(i)} + \frac{1}{2}\beta\omega^2\left[\left(x_1^{(i)} - \frac{m_2 R}{m_1+m_2}\right)^2 + \left(x_2^{(i)}\right)^2\right],$$

$$V^{(i)} = \frac{Gm_4}{R_{34}} + B_i + B_i' - \frac{\rho_1}{\rho_3}\left[B_i + B_i' + \frac{1}{2}\beta\omega^2\left\{\left(x_1^{(i)} - \frac{m_2 R}{m_1+m_2}\right)^2 + \left(x_2^{(i)}\right)^2\right\}\right],$$

$$B_i = \pi G\rho_1\left(I' - A_1\left(x_1^{(i)}\right)^2 - A_2\left(x_2^{(i)}\right)^2 - A_3\left(x_3^{(i)}\right)^2\right)$$

where

$$I' = a_1^2 A_1 + a_2^2 A_2 + a_3^2 A_3 \tag{4}$$

and

$$A_i = a_1 a_2 a_3 \int_0^\infty \frac{du}{\delta\left(a_i^2+u\right)}\,(i=1,2,3)$$

with

$$\delta^2 = \left(a_1^2+u\right)\left(a_2^2+u\right)\left(a_3^2+u\right). \tag{5}$$

The perturbations in the Coriolis and centrifugal forces are expressed with the help of parameters α, β where the unperturbed value of each is taken as unity.

We take α and β as

$$\alpha = 1 + \epsilon, |\epsilon| \ll 1,$$
$$\beta = 1 + \epsilon', |\epsilon'| \ll 1.$$

Here $V^{(i)}$ is the potential that explains the combined forces upon the infinitesimal masses, which are , attraction m_4, attraction of the fluid, gravitational field due to the triaxial body of mass m_2, buoyancy force arising in the fluid. $U^{(i)}_{x_1^{(i)}}, U^{(i)}_{x_2^{(i)}}, U^{(i)}_{x_3^{(i)}}$ are the partial derivatives of $U^{(i)}$ with respect to $x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, (i = 3, 4)$ which are the coordinates of m_3 and m_4 respectively,B_i is the potential at any internal point $(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}), (i = 3, 4)$of the homogenous ellipsoid and I' is its polar moment of inertia with index symbol $A_i, (i = 1, 2, 3)$. B_i' stands for the potential due to the second primary, R is the distance between the primaries, G is the gravitational constant, a_1, a_2, a_3 are the semiaxes of the ellipsoidal figure and parallel, respectively, to the coordinate axes, ω is the angular velocity. The last term in $V^{(i)}$ represents the buoyancy force per unit mass can be expressed as

$$E = -\frac{\rho_1}{\rho_3}\nabla\left[B_i + B_i' + \frac{1}{2}\beta\omega^2\left\{\left(x_1^{(i)} - \frac{m_2 R}{m_1 + m_2}\right)^2 + \left(x_2^{(i)}\right)^2\right\}\right] \tag{6}$$

The angular velocity $\boldsymbol{\omega}$ of the triaxial body of mass m_2 is given by

$$\omega^2 = \frac{G\,(m_1 + m_2)}{R^3}\left[1 + \frac{3}{2}\,(2\sigma_1 - \sigma_2)\right]$$

where

$$\sigma_1 = \frac{a^2 - c^2}{5R^2}, \sigma_2 = \frac{b^2 - c^2}{5R^2}, \quad \sigma_i << 1 (i = 1, 2)$$

We fix the units such that $m_1 + m_2 = 1, R = 1$. We choose unit of time t in such a way that $G = 1$.

The quantity μ becomes numerically equal to the ratio

$$\frac{m_2}{m_1 + m_2}.$$

In the new units $\omega^2 = 1 + \frac{3}{2}\,(2\sigma_1 - \sigma_2)$.

Hence,the equations of motion of m_3 and m_4 in the Robe's circular restricted problem of $2 + 2$ bodies when the bigger primary is a Roche ellipsoid and the smaller

primary a triaxial body in the perturbed system in dimensionless cartesian coordinates are given by

$$\ddot{x_1^{(i)}} - 2\alpha\omega\dot{x_2^{(i)}} = U^{(i)}_{x_1^{(i)}}, \tag{7}$$

$$\ddot{x_2^{(i)}} + 2\alpha\omega\dot{x_1^{(i)}} = U^{(i)}_{x_2^{(i)}}, \tag{8}$$

$$\ddot{x_3^{(i)}} = U^{(i)}_{x_3^{(i)}} \tag{9}$$

where

$$\begin{aligned} U^{(i)} =& \frac{\mu_j}{R_{ij}} + D_i\Big[B_i + \mu\left\{\left(x_1^{(i)}\right)^2 - \frac{1}{2}\left(x_2^{(i)}\right)^2 - \frac{1}{2}\left(x_3^{(i)}\right)^2\right\} \\ &+ \frac{\beta\omega^2}{2}\left\{\left(x_1^{(i)}\right)^2 + \left(x_2^{(i)}\right)^2\right\} + \frac{\mu}{2}(2\sigma_1 - \sigma_2)\left\{6\left(x_1^{(i)}\right)^2\right. \\ &\left. - \frac{3}{2}\left(x_2^{(i)}\right)^2 - \frac{3}{2}\left(x_3^{(i)}\right)^2\right\} - \frac{3}{2}\mu(\sigma_1 - \sigma_2)\left(x_2^{(i)}\right)^2 \\ &- \frac{3}{2}\mu\sigma_1\left(x_3^{(i)}\right)^2 + \frac{\mu}{2}(\mu\omega^2 + (2\sigma_1 - \sigma_2) + 2)\Big], \end{aligned} \tag{10}$$

where

$$0 < \mu < 1, \sigma_1, \sigma_2 << 1, \mu_j = \frac{m_j}{m_1 + m_2}, D_i = \left(1 - \frac{\rho_1}{\rho_i}\right) \qquad i, j = 3, 4; i \neq j. \tag{11}$$

Here μ, μ_3, μ_4 are mass ratios and $D_i, (i = 3, 4)$ are density parameters of the two infinitesimal masses.

The Equations of motion of m_3 and m_4 can be rewritten as

$$\begin{aligned} \ddot{x_1^{(i)}} - 2\omega\dot{x_2^{(i)}} =& -\mu_j\frac{\left(x_1^{(i)} - x_1^{(j)}\right)}{R_{ij}^3} \\ &+ D_i\left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right]x_1^{(i)}, \end{aligned} \tag{12}$$

$$\begin{aligned} \ddot{x_2^{(i)}} + 2\omega\dot{x_1^{(i)}} =& -\mu_j\frac{\left(x_2^{(i)} - x_2^{(j)}\right)}{R_{ij}^3} \\ &+ D_i\left[(1 - \mu - C_2) + \frac{3}{2}(1 - 2\mu)(2\sigma_1 - \sigma_2) + \frac{3}{2}\mu\sigma_2 + \epsilon'\right]x_2^{(i)}, \end{aligned} \tag{13}$$

$$\ddot{x_3^{(i)}} = -\mu_j\frac{\left(x_3^{(i)} - x_3^{(j)}\right)}{R_{ij}^3} + D_i\left[-\mu - C_3 - \frac{3}{2}\mu(4\sigma_1 - \sigma_2)\right]x_3^{(i)} \tag{14}$$

where

$$i, j = 3, 4; i \neq j.$$

and

$$\begin{aligned} C_l &= 2\left(\pi G \rho_1\right) A_l \\ &= 2\left(\frac{\mu}{\mu^*}\right) A_l \left(l = 1, 2, 3\right). \end{aligned}$$

and

$$\mu^* = \frac{\mu}{\pi G \rho_1}.$$

The equations (12),(13),(14) are the equations of motion of the infinitesimal masses m_3 and m_4 under the full influence of buoyancy force of the fluid , the gravitational attractions of the primaries and each other. We observe that the triaxiality of the second primary and small perturbations in the Coriolis and centrifugal forces have significant effect on the equations of motion.

3. Equilibrium Solutions

The equilibrium solutions of m_3 and m_4 are given by

$$U^{(i)}_{x_1^{(i)}} = U^{(i)}_{x_2^{(i)}} = U^{(i)}_{x_3^{(i)}} = 0 \quad (i = 3, 4).$$

i.e. ,

(15)

$$-\mu_j \frac{\left(x_1^{(i)} - x_1^{(j)}\right)}{R_{ij}^3} + D_i \left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right] x_1^{(i)} = 0,$$

(16)

$$-\mu_j \frac{\left(x_2^{(i)} - x_2^{(j)}\right)}{R_{ij}^3} + D_i \left[(1 - \mu - C_2) + \frac{3}{2}(1 - 2\mu)(2\sigma_1 - \sigma_2) + \frac{3}{2}\mu\sigma_2 + \epsilon'\right] x_2^{(i)} = 0,$$

(17)

$$-\mu_j \frac{\left(x_3^{(i)} - x_3^{(j)}\right)}{R_{ij}^3} + D_i \left[-\mu - C_3 - \frac{3}{2}\mu\left(4\sigma_1 - \sigma_2\right)\right] x_3^{(i)} = 0.$$

with

$$i, j = 3, 4; i \neq j.$$

Collinear Equilibrium Solutions

Equilibrium Solutions lying on x_1- axis

The Equations (16),(17) are satisfied with $x_2^{(i)}, x_3^{(i)}; i = 3, 4$ equal to zero. Hence we determine $x_1^{(i)}; i = 3, 4$ from the Equation (15) by putting $x_2^{(i)}, x_3^{(i)}; i = 3, 4$ equal to zero, we get

(18)
$$-\mu_4 \frac{\left(x_1^{(3)} - x_1^{(4)}\right)}{|x_1^{(3)} - x_1^{(4)}|^3} + D_i \left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right] x_1^{(3)} = 0.$$

(19)
$$-\mu_3 \frac{\left(x_1^{(4)} - x_1^{(3)}\right)}{|x_1^{(4)} - x_1^{(3)}|^3} + D_i \left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right] x_1^{(4)} = 0.$$

Multiplying the Equation (18) by μ_3 and (19) by μ_4 and adding, we get

$$x_1^{(4)} = -\lambda x_1^{(3)} \tag{20}$$

where

$$\lambda = \frac{D_3 \mu_3}{D_4 \mu_4} \tag{21}$$

Substituting in the Equation (18) ,we get

$$x_1^{(3)} = \pm \left[\frac{\mu_4}{D_3 \left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right](1 + \lambda)^2}\right]^{\frac{1}{3}}, \tag{22}$$

and

$$x_1^{(4)} = \mp\lambda \left[\frac{\mu_4}{D_3 \left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right](1 + \lambda)^2}\right]^{\frac{1}{3}}. \tag{23}$$

Hence $\left(x_1^{(3)}, 0, 0\right)$ and $\left(x_1^{(4)}, 0, 0\right)$ are the equilibrium solutions for m_3 and m_4 provided $|x_1^{(3)}| < a_1$ and $|x_1^{(4)}| < a_1$ respectively where a_1 is the equatorial axis of the Roche Ellipsoid.

Equilibrium Solutions lying on x_2- axis

If $x_1^{(3)} = x_3^{(3)} = 0$, then

$$x_2^{(3)} = \pm \left[\frac{\mu_4}{D_3 \left[(1 - \mu - C_2) + \frac{3}{2}(1 - 2\mu)(2\sigma_1 - \sigma_2) + \frac{3}{2}\mu\sigma_2 + \epsilon'\right](1 + \lambda)^2}\right]^{\frac{1}{3}},$$

and

$$x_2^{(4)} = \mp \frac{D_3\mu_3}{D_4\mu_4}\left[\frac{\mu_4}{D_3\left[(1-\mu-C_2)+\frac{3}{2}(1-2\mu)(2\sigma_1-\sigma_2)+\frac{3}{2}\mu\sigma_2+\epsilon'\right](1+\lambda)^2}\right]^{\frac{1}{3}}.$$

Hence $\left(0, x_2^{(3)}, 0\right)$ and $\left(0, x_2^{(4)}, 0\right)$ are the equilibrium solutions for m_3 and m_4 provided $|x_2^{(3)}| < a_2$ and $|x_2^{(4)}| < a_2$ respectively where a_2 is the equatorial axis of the Roche Ellipsoid.

Equilibrium Solutions lying on x_3- axis

If $x_1^{(3)} = x_2^{(3)} = 0$, then

$$x_3^{(3)} = \pm\left[\frac{\mu_4}{D_3\left[-\mu-C_3-\frac{3}{2}\mu(4\sigma_1-\sigma_2)\right](1+\lambda)^2}\right]^{\frac{1}{3}}, \tag{24}$$

and

$$x_3^{(4)} = \mp \frac{D_3\mu_3}{D_4\mu_4}\left[\frac{\mu_4}{D_3\left[-\mu-C_3-\frac{3}{2}\mu(4\sigma_1-\sigma_2)\right](1+\lambda)^2}\right]^{\frac{1}{3}}. \tag{25}$$

Hence $\left(0, 0, x_3^{(3)}\right)$ and $\left(0, 0, x_3^{(4)}\right)$ are the equilibrium solutions for m_3 and m_4 provided $|x_3^{(3)}| < a_3$ and $|x_3^{(4)}| < a_3$ respectively where a_3 is the polar axis of the Roche Ellipsoid.

There exist two equilibrium solutions of the system lying each on x_1- axis or x_2- axis or x_3- axis. Hence, there exist six equilibrium solutions of the system, provided

$$\frac{x_1^{(3)}}{a_1^2} + \frac{x_2^{(3)}}{a_2^2} + \frac{x_3^{(3)}}{a_3^2} < 1 \tag{26}$$

holds. The equilibrium solutions of m_3 are shown in Figure 2.

Non Collinear Equilibrium Solutions

From the Equations (15), (16), (17) we get a relation of the type $x_i^{(4)} = -\lambda x_i^{(3)}, i = 1, 2, 3$ which on back substitution in the above equations give us three different

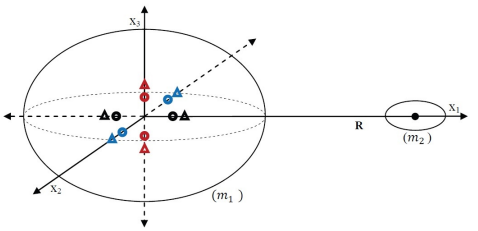


FIGURE 2. **Location of Equilibrium Solutions in the Robe's Restricted Problem of 2+2 Bodies in a Roche Ellipsoid - Triaxial system when perturbations are given to the Coriolis and centrifugal forces . Circles denote the positions of m_3 and triangles denote the positions of m_4.**

values of R_{13} as shown below

$$R_{13} = \left[\frac{\mu_4}{\left[D_3\left(1+2\mu-C_1\right)+\frac{3}{2}(1+4\mu)\left(2\sigma_1-\sigma_2\right)+\epsilon'\right]\left(1+\lambda\right)^2}\right]^{\frac{1}{3}}$$

$$R_{13} = \left[\frac{\mu_4}{\left[D_3\left(1-\mu-C_2\right)+\frac{3}{2}(1-2\mu)\left(2\sigma_1-\sigma_2\right)+\frac{3}{2}\mu\sigma_2+\epsilon'\right]\left(1+\lambda\right)^2}\right]^{\frac{1}{3}}$$

$$R_{13} = \left[\frac{\mu_4}{D_3\left[-\mu-C_3-\frac{3}{2}\mu\left(4\sigma_1-\sigma_2\right)\right]\left(1+\lambda\right)^2}\right]^{\frac{1}{3}}$$

which is not possible.Hence, there are no equilibrium solutions possible in this case. In all there exist only six equilibrium solutions of the system, provided they lie within Roche ellipsoid.

4. Stability of Equilibrium Solutions

Let m_3 and m_4 be displaced to $x_i^{(j)} = X_i^{(j)} + \alpha_i^{(j)}, i = 1, 2, 3; j = 3, 4$ where $\alpha_i^{(j)}$ are very small and $(X_1^{(j)}, X_2^{(j)}, X_3^{(j)})$; $j = 3, 4$ are the equilibrium solution of m_3 and m_4 respectively.

The variational equations of m_3 and m_4 are

$$\ddot{\alpha}_1^{(i)} - 2\alpha\omega\dot{\alpha}_2^{(i)} = \alpha_1^{(i)}\left(U^{(i)}_{x_1^{(i)}x_1^{(i)}}\right)^o + \alpha_2^{(i)}\left(U^{(i)}_{x_1^{(i)}x_2^{(i)}}\right)^o + \alpha_3^{(i)}\left(U^{(i)}_{x_1^{(i)}x_3^{(i)}}\right)^o, \tag{27}$$

$$\ddot{\alpha}_2^{(i)} + 2\alpha\omega\dot{\alpha}_1^{(i)} = \alpha_1^{(i)}\left(U^{(i)}_{x_2^{(i)}x_1^{(i)}}\right)^o + \alpha_2^{(i)}\left(U^{(i)}_{x_2^{(i)}x_2^{(i)}}\right)^o + \alpha_3^{(i)}\left(U^{(i)}_{x_2^{(i)}x_3^{(i)}}\right)^o, \tag{28}$$

$$\ddot{\alpha}_3^{(i)} = \alpha_1^{(i)}\left(U^{(i)}_{x_3^{(i)}x_1^{(i)}}\right)^o + \alpha_2^{(i)}\left(U^{(i)}_{x_3^{(i)}x_2^{(i)}}\right)^o + \alpha_3^{(i)}\left(U^{(i)}_{x_3^{(i)}x_3^{(i)}}\right)^o, \tag{29}$$

$$i = 3, 4$$

where

$$U^{(i)}_{x_1^{(i)}x_1^{(i)}} = -\frac{\mu_j}{R_{ij}^3} + \frac{3\mu_j\left(x_1^{(i)} - x_1^{(j)}\right)^2}{R_{ij}^5} + D_i\left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right]$$

$$U^{(i)}_{x_1^{(i)}x_2^{(i)}} = \frac{3\mu_j\left(x_1^{(i)} - x_1^{(j)}\right)\left(x_2^{(i)} - x_2^{(j)}\right)}{R_{ij}^5}$$

$$U^{(i)}_{x_1^{(i)}x_3^{(i)}} = \frac{3\mu_j\left(x_1^{(i)} - x_1^{(j)}\right)\left(x_3^{(i)} - x_3^{(j)}\right)}{R_{ij}^5}$$

$$U^{(i)}_{x_2^{(i)}x_2^{(i)}} = -\frac{\mu_j}{R_{ij}^3} + \frac{3\mu_j\left(x_2^{(i)} - x_2^{(j)}\right)^2}{R_{ij}^5} + D_i\left[(1 - \mu - C_2) + \frac{3}{2}(1 - 2\mu)(2\sigma_1 - \sigma_2) + \frac{3}{2}\mu\sigma_2 + \epsilon'\right].$$

$$U^{(i)}_{x_2^{(i)}x_3^{(i)}} = \frac{3\mu_j\left(x_2^{(i)} - x_2^{(j)}\right)\left(x_3^{(i)} - x_3^{(j)}\right)}{R_{ij}^5}$$

$$U^{(i)}_{x_3^{(i)}x_3^{(i)}} = -\frac{\mu_j}{R_{ij}^3} + \frac{3\mu_j\left(x_3^{(i)} - x_3^{(j)}\right)^2}{R_{ij}^5} + D_i\left[-\mu - C_3 - \frac{3}{2}\mu(4\sigma_1 - \sigma_2)\right]$$

$$i, j = 3, 4; \quad i \neq j.$$

Here the upper suffix $'o'$ denotes the evaluation of the derivatives at the equilibrium solutions under consideration.

Stability of Equilibrium Solutions lying on x_1- axis

Let the equilibrium solution $\left(x_1^{(3)}, 0, 0\right)$ and $\left(x_1^{(4)}, 0, 0\right)$ of m_3 and m_4 be displaced to $\left(x_1^{(3)} + \alpha_1^{(3)}, \alpha_2^{(3)}, \alpha_3^{(3)}\right)$ and $\left(x_1^{(4)} + \alpha_1^{(4)}, \alpha_2^{(4)}, \alpha_3^{(4)}\right)$.

The variational equations of m_3 and m_4 are

$$\ddot{\alpha}_1^{(i)} - 2\alpha\omega\dot{\alpha}_2^{(i)} = \alpha_1^{(i)} p_1^{(i)} \left(\frac{3+\lambda}{1+\lambda}\right) \tag{30}$$

$$\ddot{\alpha}_2^{(i)} + 2\alpha\omega\dot{\alpha}_1^{(i)} = \alpha_2^{(i)} \left(\frac{-p_1^{(i)}}{1+\lambda} + p_2^{(i)}\right) \tag{31}$$

$$\ddot{\alpha}_3^{(i)} = \alpha_3^{(i)} \left(\frac{-p_1^{(i)}}{1+\lambda} + p_3^{(i)}\right) \qquad i = 3, 4 \tag{32}$$

where

$$\begin{aligned}
\lambda &= \frac{D_3\mu_3}{D_4\mu_4}, \\
p_1^{(i)} &= D_i \left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right] \\
p_2^{(i)} &= D_i \left[(1 - \mu - C_2) + \frac{3}{2}(1 - 2\mu)(2\sigma_1 - \sigma_2) + \frac{3}{2}\mu\sigma_2 + \epsilon'\right] \\
p_3^{(i)} &= D_i \left[\left[-\mu - C_3 - \frac{3}{2}\mu(4\sigma_1 - \sigma_2)\right]\right].
\end{aligned}$$

From the Equation (32), we ascertain that the motion of m_3 and m_4 parallel to x_3 axis is stable when

$$\frac{\left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right]}{\left[-\mu - C_3 - \frac{3}{2}(4\sigma_1 - \sigma_2) + \epsilon'\right]} > 1 + \frac{D_3\mu_3}{D_4\mu_4} \tag{33}$$

$$i = 3, 4 \text{ in case } \rho_1 > \rho_3, \rho_1 > \rho_4$$

and

$$\frac{\left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right]}{\left[-\mu - C_3 - \frac{3}{2}(4\sigma_1 - \sigma_2) + \epsilon'\right]} < 1 + \frac{D_3\mu_3}{D_4\mu_4} \tag{34}$$

$$i = 3, 4 \text{ in case } \rho_1 < \rho_3, \rho_1 < \rho_4$$

The remaining Equations (30) and (31) admit solutions of the form $\alpha_i^{(j)} = A_i^{(j)} e^{L_j t}, i = 1, 2, ; j = 3, 4$.

The characteristic equations of m_3 and m_4 are given by

$$\Lambda_i^4 + \Lambda_i^2 \left(4\omega^2(1 + 2\epsilon) - \left(Q_1^{(i)} + Q_2^{(i)}\right)\right) + Q_1^{(i)} Q_2^{(i)} = 0 \tag{35}$$

$$i = 3, 4$$

where

$$Q_1^{(i)} = p_1^{(i)}\left(\frac{3+\lambda}{1+\lambda}\right)$$

$$Q_2^{(i)} = p_2^{(i)} - \frac{p_1^{(i)}}{1+\lambda}.$$

Then, $(\Lambda_1^{(i)})^2 + (\Lambda_2^{(i)})^2 = \left(Q_1^{(i)} + Q_2^{(i)}\right) - 4\omega^2(1+2\epsilon)$,
$(\Lambda_1^{(i)})^2(\Lambda_2^{(i)})^2 = Q_1^{(i)}Q_2^{(i)}$.

$$\begin{aligned}(\Lambda_1^{(i)})^2 + (\Lambda_2^{(i)})^2 &= D_i\left[(1+2\mu-C_1) + \frac{3}{2}(1+4\mu)\,(2\sigma_1-\sigma_2) + \epsilon'\right]\left(\frac{2+\lambda}{1+\lambda}\right)\\ &\quad + D_i\left[(1-\mu-C_2) + \frac{3}{2}(1-2\mu)\,(2\sigma_1-\sigma_2) + \frac{3}{2}\mu\sigma_2 + \epsilon'\right]\\ &\quad - 4 - 6(2\sigma_1-\sigma_2) - 8\epsilon \qquad (36)\end{aligned}$$

$$\begin{aligned}(\Lambda_1^{(i)})^2(\Lambda_2^{(i)})^2 &= D_i^2\left(\frac{3+\lambda}{1+\lambda}\right)\Big[(1+2\mu-C_1)\,(1-\mu-C_2)\\ &\quad + \frac{3}{2}(1+4\mu)\,(2\sigma_1-\sigma_2)\,(1-\mu-C_2)\\ &\quad + \left\{\frac{3}{2}(1-2\mu)\,(2\sigma_1-\sigma_2) + \frac{3}{2}\mu\sigma_2\right\}(1+2\mu-C_1)\\ &\quad - \frac{1}{1+\lambda}\{(1+2\mu-C_1)^2 + 3(1+4\mu)\,(2\sigma_1-\sigma_2)\,(1+2\mu-C_1)\}\\ &\quad + \epsilon'\{(2+\mu-C_1-C_2) - \frac{2}{1+\lambda}\,(1+2\mu-C_1)\}\Big] \qquad (37)\end{aligned}$$

The equilibrium solutions $\left(x_1^{(3)}, 0, 0\right)$ and $\left(x_1^{(4)}, 0, 0\right)$ of m_3 and m_4 respectively when the displacement is given in the direction of x_1- axis or x_2- axis are stable if $(\Lambda_1^{(i)})^2$ and $(\Lambda_2^{(i)})^2$ are real and negative or we must have $(\Lambda_1^{(i)})^2 + (\Lambda_2^{(i)})^2 < 0, (\Lambda_1^{(i)})^2(\Lambda_2^{(i)})^2 > 0$. We compute the numerical value for the sum and product of the roots of the characteristic equation for different values of ϵ, ϵ' for the analysis of the equilibrium solutions of m_3 and m_4. We use the numerical values $\mu = 0.01$, $A_1 = 0.7$, $A_2 = 0.6$, $\rho_1 = 0.246$, $\rho_3 = 0.3$, $\rho_4 = 0.2$, $\sigma_1 = 0.07$, $\sigma_2 = 0.06$, $\lambda = 0.0001$ in the Equations (36) and (37) .We observe that in Table 1 and Table 4 the stability conditions hold and in Table 2 and Table 3 they donot. Hence the equilibrium solutions of m_3 (and similarly of m_4)lying on x_1-axis are unstable for $\epsilon > 0, \epsilon' > 0$ and $\epsilon < 0, \epsilon' > 0$ and stable for $\epsilon > 0, \epsilon' < 0$ and $\epsilon < 0, \epsilon' < 0$. We observe that the condition of stability are influenced by the small perturbations in

the Coriolis and centrifugal forces. Similarly we study the stability of equilibrium solutions lying on x_2 and x_3 axis.

5. Conclusion

In this paper, we have studied the effect of perturbations in the Coriolis and centrifugal forces on the location and stability of the equilibrium points in Robe's restricted problem of 2+2 bodies. Plastino & Plastino [9] revised the Robe's restricted three-body problem under the assumption that the fluid body assumes the shape of a Roche ellipsoid. They took into consideration all the three components of the pressure field in deriving the expression for the buoyancy force viz, due to the own gravitational field of the fluid, that originating in the attraction of m_2 and that arising from the centrifugal force.They proved that that the center of the Roche Ellipsoid viz,$(0, 0, 0)$ is the only equilibrium solution of the system.The equilibrium solutions in our case as compared to Plastino & Plastino [9] are perturbed along x_1 ,x_2 and x_3 axis.We have assumed throughout the paper that the perturbed coordinates of m_3 and m_4 are such that m_3 and m_4 remain within the Ellipsoid.There exist two equilibrium solutions of the system lying each on x_1- axis or x_2- axis or x_3- axis. Hence, there exist six equilibrium solutions of the system, provided they lie within Roche ellipsoid (Equation (26)). We observe that there are no other equilibrium solutions except these six. We observe that the triaxiality of the second primary and small perturbations in the Coriolis and centrifugal forces have significant effect on the equations of motion. The number of equilibrium solutions in this problem are the same as in the problem with no perturbation (Bhavneet & Rajiv [6]), but positions of the equilibrium solutions have changed. The change in the Coriolis force does not affect the position of the equilibrium solutions. The stability behavior of the equilibrium solution depends on the sign of the small perturbations in the Coriolis and centrifugal forces .The equilibrium solutions of m_3 (and similarly of m_4) lying on x_1, x_2, x_3 axis are unstable for $\epsilon > 0, \epsilon' > 0$;$\epsilon < 0, \epsilon' > 0$ and stable for $\epsilon < 0, \epsilon' < 0$;$\epsilon > 0, \epsilon' < 0$,using the data of submarines in the Earth -Moon system. Similarly, the equilibrium solutions of m_3 and m_4 respectively when the displacement is given in the direction of x_2 or x_3- axis are conditionally stable. Earth is the only celestial body we know of which has fluid and the test particles within the fluid are considered as the submarines. In the solar or the extra solar

system, we donot know of any other celestial body having liquid inside it. Our entire work is applicable in solar or extra solar system when and if such a system is discovered .

TABLE 1. $\epsilon > 0$ and $\epsilon' < 0$.

ϵ	ϵ'	$\Lambda_1^2 + \Lambda_2^2$	$\Lambda_1^2 \times \Lambda_2^2$
0.02	-0.01	-4.60675	0.000121742
0.03	-0.02	-4.68675	0.000200461
0.04	-0.03	-4.76675	0.00027918
0.05	-0.04	-4.84675	0.000357899
0.06	-0.05	-4.92675	0.000436617
0.07	-0.06	-5.00675	0.000515336
0.08	-0.07	-5.08675	0.000594055
0.09	-0.08	-5.16675	0.000672774
0.1	-0.09	-5.24675	0.000751492

TABLE 2. $\epsilon > 0$ and $\epsilon' > 0$.

ϵ	ϵ'	$\Lambda_1^2 + \Lambda_2^2$	$\Lambda_1^2 \times \Lambda_2^2$
0.02	0.01	-4.60675	-0.0000356952
0.03	0.02	-4.68675	-0.000114414
0.04	0.03	-4.76675	-0.000193133
0.05	0.04	-4.84675	-0.000271851
0.06	0.05	-4.92675	-0.00035057
0.07	0.06	-5.00675	-0.000429289
0.08	0.07	-5.08675	-0.000508008
0.09	0.08	-5.16675	-0.000586726
0.1	0.09	-5.24675	-0.000665445

TABLE 3. $\epsilon < 0$ and $\epsilon' > 0$.

ϵ	ϵ'	$\Lambda_1^2 + \Lambda_2^2$	$\Lambda_1^2 \times \Lambda_2^2$
-0.02	0.01	-4.28675	-0.0000356952
-0.03	0.02	-4.20675	-0.000114414
-0.04	0.03	-4.12675	-0.000193133
-0.05	0.04	-4.04675	-0.000271851
-0.06	0.05	-3.96675	-0.00035057
-0.07	0.06	-3.88675	-0.000429289
-0.08	0.07	-3.80675	-0.000508008
-0.09	0.08	-3.72675	-0.000586726
-0.1	0.09	-3.64675	-0.000665445

TABLE 4. $\epsilon < 0$ and $\epsilon' < 0$.

ϵ	ϵ'	$\Lambda_1^2 + \Lambda_2^2$	$\Lambda_1^2 \times \Lambda_2^2$
-0.02	-0.01	-4.28675	0.000121742
-0.03	-0.02	-4.20675	0.000200461
-0.04	-0.03	-4.12675	0.00027918
-0.05	-0.04	-4.04675	0.000357899
-0.06	-0.05	-3.96675	0.000436617
-0.07	-0.06	-3.88675	0.000515336
-0.08	-0.07	-3.80675	0.000594055
-0.09	-0.08	-3.72675	0.000672774
-0.1	-0.09	-3.64675	0.000751492

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