

ON THE STABILITY OF THE TRIANGULAR POINTS IN THE RELATIVISTIC R3BP WITH A BIGGER TRIAXIAL PRIMARY

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ABSTRACT. This paper studies the motion of a third body (test particle) in the vicinity of the triangular points $L_{4,5}$ by considering the more massive as a triaxial body in the frame work of the relativistic restricted three-body problem (R3BP). It is seen that the positions and stability of the triangular points are affected by both relativistic and triaxiality factors. It turns out both the coordinates of the infinitesimal mass are affected. It is seen that for these points, the range of stability region increases or decreases according as $p > 0$ or $p < 0$ where p depends upon the triaxiality and relativistic factors. Furthermore we have studied the periodic orbits around the triangular points in the range $0 < \mu < \mu_c$. It is found that these orbits are elliptical; the frequencies of long and short orbits of the periodic motion, the eccentricities, semi-major and semi-minor axes, orientation and coefficients of long and short periodic terms are all affected by triaxiality and relativistic factors.

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1. INTRODUCTION

The three-body problem refers to three bodies, which move under their mutual gravitational attraction. There does not exist a general analytical solution to this problem but chaotic solutions and numerical ones based on iterative methods. Regarding the general three-body problem, we may cite Chenciner (2007) [5] and Valtonen and Karttunen (2006) [23]. There are also various forms of three-body problem in general relativity, regarding this we may cite Renzetti (2012) [17].

If two of the three bodies move in circular and coplanar orbit around their common barycentre, and additionally, the third mass is small compared to the other two masses so that it does not affect the movements of the two, then one speaks of planar circular restricted three-body problem (CR3BP). It possesses five points of equilibrium: three collinear denoted by L_1, L_2, L_3 and two triangular denoted by L_4, L_5 , where the gravitational and centrifugal forces just balance each other. The collinear points are unstable, while the triangular points are stable for the mass ratio $\mu < 0.03852\dots$ (Szebehely, 1967 [22]). Their stability occurs in spite of the fact that the potential energy has a maximum rather than a minimum at the latter points. The bodies in the R3BP are strictly spherical in shape, but in nature, celestial bodies are not perfect spheres. They are either oblate or triaxial. The lack of sphericity, triaxiality or oblateness of the celestial bodies causes large perturbation in a two body orbit. It must be noted here that the asphericity of the primary has a number of applications in several fields of astronomical sciences (e.g testing alternative gravities) and also in fundamental physics (Renzetti, 2012 [17]; Iorio, 2009 [11]). The asphericity issue inspired several researchers (Subbarao and Sharma, 1975 [21]; Elise and Ferrer, 1985 [9]; El-Shaboury and El-Tantawy, 1993 [8]) to include non-sphericity of the bodies in their studies of the R3BP. When one or two of the primaries are triaxial bodies this problem was discussed by El-Shaboury et al. (1991) [7], Khanna and Bhatnagar (1999) [14], Hallan et al. (2001) [10], Sharma et al. (2001) [18, 19], Singh (2013) [20].

Abouelmagd and EL-Shaboury (2012) [1] examined the existence of equilibrium points and their linear stability when the three participating bodies are axisymmetric and the primaries are radiating, they found that the collinear points remain unstable, and the triangular points are stable for the region $0 < \mu < \mu_c$; the range of stability for these points decreases. They studied periodic orbits around the

triangular points and found that these orbits are elliptical; the frequencies of long and short orbits of the periodic motion are affected by the terms which involve the parameters that characterize the oblateness and radiation repulsive forces.

The general theory of relativity was developed by Einstein a century ago. Since then, it has become the standard theory of gravity, especially important to the field of fundamental astrometry, astrophysics, cosmology and experimental gravitational physics. Today, the application of general relativity is also essential for many practical purposes involving astrometry, navigation, geodesy and time synchronization. It is worthwhile to examine the three-body (or, more generally, N- body) problem in general relativity.

However, it is difficult to work out in general relativity compared with Newton gravity, because the Einstein equation is much more complicated (Misner et al. 1973) [16].

Regarding the three-body relativistic effects we may also cite: the geodesic possession of the orbit of two-body system about a third mass in general relativity (Renzetti, 2012 [17]) and for the post-Newtonian tidal effects (Iorio, 2014) [12]. Brumberg (1972, 1991) [3, 4] studied the relativistic N-body problem for three bodies in more detail and collected most of the important results on relativistic mechanics. He did not only obtain the equations of motion for the general problem of three-bodies but also deduced the equations of motion for the restricted problem of three-bodies.

Bhatnagar and Hallan (1998) [2] investigated the existence and linear stability of the triangular points $L_{4,5}$ in the relativistic R3BP, and found that $L_{4,5}$ are always unstable in the whole range $0 \leq \mu \leq \frac{1}{2}$ in contrast to the classical R3BP where they are stable for $\mu < \mu_0$, where μ is the mass ratio and $\mu_0 = 0.03852\dots$ is the Routh's value. Douskos and Perdios (2002) [6] examined the stability of the triangular points in the relativistic R3BP and contrary to the results of Bhatnagar and Hallan (1998) [2], they obtained a region of linear stability in the parameter space $0 \leq \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2}$ where $\mu_0 = 0.03852\dots$ is Routh's value. They also determined the positions of the collinear points and showed that they are always unstable.

Katour et al. (2014) [13] have studied the photogravitational restricted three-body within the framework of the Post-Newtonian approximation. The mass of the primaries are assumed to change under the effect of continuous radiation process and

oblateness effects of the two primaries. They computed the new perturbed locations of the triangular points.

The objective of this paper is to investigate the triangular points and their linear stability under the relativistic treatment of R3BP when the bigger primary is triaxial.

This paper is organized as follows: In Sect. 2, the equations governing the motion are presented; Sect. 3 describes the positions of triangular points, while their linear stability and periodic orbits are analyzed in Sects 4 and 5 respectively, the discussion of results is given in Sect. 6, finally Sect. 7 conveys the main findings of this paper.

2. EQUATIONS OF MOTION

The pertinent equations of motion of an infinitesimal mass in the relativistic R3BP in a barycentric synodic coordinate system (ξ, η) and dimensionless variables can be written as Brumberg (1972) [3] and Bhatnagar and Hallan (1998) [2]:

$$\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \frac{\partial W}{\partial \xi} - \frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\xi}} \right) \\ \ddot{\eta} + 2n\dot{\xi} &= \frac{\partial W}{\partial \eta} - \frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\eta}} \right) \end{aligned} \quad (1)$$

with

$$\begin{aligned} W &= \frac{1}{2}(\xi^2 + \eta^2) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} + \frac{1}{c^2} \left[-\frac{3}{2} \left(1 - \frac{1}{3}\mu(1-\mu) \right) (\xi^2 + \eta^2) \right. \\ &+ \frac{1}{8} \left\{ \dot{\xi}^2 + \dot{\eta}^2 + 2(\xi\dot{\eta} - \eta\dot{\xi}) + (\xi^2 + \eta^2) \right\}^2 + \frac{3}{2} \left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} \right) \times \\ &\left(\dot{\xi}^2 + \dot{\eta}^2 + 2(\xi\dot{\eta} - \eta\dot{\xi}) + (\xi^2 + \eta^2) \right) \\ &- \frac{1}{2} \left(\frac{(1-\mu)^2}{\rho_1^2} + \frac{\mu^2}{\rho_2^2} \right) + \mu(1-\mu) \left\{ \left(4\dot{\eta} + \frac{7}{2}\xi \right) \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \right. \\ &\left. \left. - \frac{\eta^2}{2} \left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3} \right) + \left(\frac{-1}{\rho_1\rho_2} + \frac{3\mu-2}{2\rho_1} + \frac{1-3\mu}{2\rho_2} \right) \right\} \right] \end{aligned} \quad (2)$$

$$n = 1 - \frac{3}{2c^2} \left(1 - \frac{1}{3}\mu(1-\mu) \right) \quad (3)$$

$$\rho_1^2 = (\xi + \mu)^2 + \eta^2 \quad (4)$$

$$\rho_2^2 = (\xi + \mu - 1)^2 + \eta^2$$

where $0 < \mu \leq \frac{1}{2}$ is the ratio of the mass of the smaller primary to the total mass of the primaries, ρ_1 and ρ_2 are distances of the infinitesimal mass from the bigger and

smaller primary, respectively; n is the mean motion of the primaries; c is the velocity of light.

We now introduce the triaxiality factors of the bigger primary with the help of the parameter $\sigma_i < 1, (i = 1, 2)$ where $\sigma_1 = \frac{a^2 - c^2}{5R^2}, \sigma_2 = \frac{b^2 - c^2}{5R^2}$. (McCuskey 1963 [15]). Here σ_1, σ_2 characterize the triaxiality of the bigger primary with a, b, c as lengths of its semi-axes and R is the distance between the primaries.

Neglecting second and higher power of $\sigma_i (i = 1, 2)$ and also their product, we take equations of motion as:

$$(5) \quad \begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \frac{\partial W}{\partial \xi} - \frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\xi}} \right) \\ \ddot{\eta} + 2n\dot{\xi} &= \frac{\partial W}{\partial \eta} - \frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\eta}} \right) \end{aligned}$$

where W is the potential-like function of the relativistic R3BP. As Katour et al. (2014) [13], we have not included the parameters $\sigma_i (i = 1, 2)$, in the relativistic part of W since the magnitude of these terms is so small due to c^{-2} where c is the speed of light.

Hence

$$(6) \quad \begin{aligned} W &= \frac{1}{2} \left(1 + \frac{3}{2} (2\sigma_1 - \sigma_2) \right) (\xi^2 + \eta^2) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} + \frac{1-\mu}{2\rho_1^3} (2\sigma_1 - \sigma_2) \\ &+ \frac{3(1-\mu)\eta^2}{2\rho_1^5} (\sigma_2 - \sigma_1) + \frac{1}{c^2} \left[-\frac{3}{2} \left\{ 1 - \frac{1}{3}\mu(1-\mu) \right\} (\xi^2 + \eta^2) \right. \\ &+ \frac{1}{8} \left\{ \dot{\xi}^2 + \dot{\eta}^2 + 2 \left(\xi\dot{\eta} - \eta\dot{\xi} \right) + (\xi^2 + \eta^2) \right\}^2 + \frac{3}{2} \left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} \right) \times \\ &\left\{ \dot{\xi}^2 + \dot{\eta}^2 + 2 \left(\xi\dot{\eta} - \eta\dot{\xi} \right) + (\xi^2 + \eta^2) \right\} - \frac{1}{2} \left(\frac{(1-\mu)^2}{\rho_1^2} + \frac{\mu^2}{\rho_2^2} \right) \\ &+ \mu(1-\mu) \left\{ \left(4\dot{\eta} + \frac{7}{2}\xi \right) \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) - \frac{\eta^2}{2} \left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3} \right) \right. \\ &\left. \left. + \left(\frac{-1}{\rho_1\rho_2} + \frac{3\mu-2}{2\rho_1} + \frac{1-3\mu}{2\rho_2} \right) \right\} \right], \end{aligned}$$

and n the perturbed mean motion of the primaries is given by

$$(7) \quad n = 1 + \frac{3}{4} (2\sigma_1 - \sigma_2) - \frac{3}{2c^2} \left(1 - \frac{1}{3}\mu(1-\mu) \right)$$

3. LOCATION OF TRIANGULAR POINTS

The libration points are obtained from equations (5) after putting

$$\dot{\xi} = \dot{\eta} = \ddot{\xi} = \ddot{\eta} = 0.$$

These points are the solutions of the equations

$$\frac{\partial W}{\partial \xi} = 0 = \frac{\partial W}{\partial \eta} \quad \text{with} \quad \dot{\xi} = \dot{\eta} = 0.$$

That is

$$(8) \quad \begin{aligned} & \xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{\mu(\xi-1+\mu)}{\rho_2^3} + (3\sigma_1 - \frac{3}{2}\sigma_2) \xi - \frac{3(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)}{2\rho_1^5} \\ & - \frac{15(1-\mu)(\xi+\mu)(\sigma_2-\sigma_1)\eta^2}{2\rho_1^4} + \frac{1}{c^2} \left[-3\xi \left\{ 1 - \frac{\mu(1-\mu)}{3} \right\} + \frac{1}{2}\xi(\xi^2 + \eta^2) \right. \\ & - \frac{3}{2}(\xi^2 + \eta^2) \left\{ \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} + \frac{\mu(\xi-1+\mu)}{\rho_2^3} \right\} + 3 \left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} \right) \xi \\ & + \frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4} + \frac{\mu^2(\xi-1+\mu)}{\rho_2^4} + \mu(1-\mu) \left\{ \frac{7}{2} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) + \frac{7}{2}\xi \left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3} \right) \right. \\ & + \frac{3}{2}\eta^2 \left(\frac{\mu(\xi+\mu)}{\rho_1^5} + \frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5} \right) + \frac{(\xi+\mu)}{\rho_1^3\rho_2} + \frac{(\xi-1+\mu)}{\rho_1\rho_2^3} \\ & \left. \left. - \frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3} - \frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3} \right\} \right] = 0 \end{aligned}$$

and (8)

$$\eta F = 0,$$

$$\begin{aligned} F = & \left(1 - \frac{1-\mu}{\rho_1^3} - \frac{\mu}{\rho_2^3} \right) + (3\sigma_1 - \frac{3}{2}\sigma_2) + \frac{3(1-\mu)}{\rho_1^5} \left(\frac{3}{2}\sigma_2 - 2\sigma_1 \right) - \frac{15(1-\mu)(\sigma_2-\sigma_1)\eta^2}{2\rho_1^4} + \\ & \frac{1}{c^2} \left[-3 \left(1 - \frac{\mu(1-\mu)}{3} \right) + \frac{1}{2}(\xi^2 + \eta^2) + 3 \left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3} \right) - \frac{3}{2}(\xi^2 + \eta^2) \left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3} \right) \right. \\ & + \left(\frac{(1-\mu)^2}{\rho_1^4} + \frac{\mu^2}{\rho_2^4} \right) + \mu(1-\mu) \left\{ \frac{7}{2}\xi \left(-\frac{1}{\rho_1^3} + \frac{1}{\rho_2^3} \right) - \left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3} \right) + \frac{3}{2}\eta^2 \left(\frac{\mu}{\rho_1^5} + \frac{1-\mu}{\rho_2^5} \right) \right. \\ & \left. \left. + \frac{1}{\rho_1^3\rho_2} + \frac{1}{\rho_1\rho_2^3} - \frac{(3\mu-2)}{2\rho_1^3} - \frac{(1-3\mu)}{2\rho_2^3} \right\} \right] \end{aligned}$$

The triangular points are the solutions of Equations (8) with $\eta \neq 0$. Since $\frac{1}{c^2} \ll 1$ and in the case $\frac{1}{c^2} \rightarrow 0$ and the primaries are spherical (i.e. $\sigma_1 = \sigma_2 = 0$), one can obtain $\rho_1 = \rho_2 = 1$; we assume in the relativistic R3BP that $\rho_1 = 1 + x$ and $\rho_2 = 1 + y$ where, $x, y \ll 1$ may be depending upon the relativistic and triaxiality factors. Substituting these values in the equations (4), solving them for ξ, η and ignoring terms of second and higher powers of x and y , we get

$$\xi = x - y + \frac{1-2\mu}{2},$$

$$\eta = \pm \left(\frac{\sqrt{3}}{2} + \frac{x+y}{\sqrt{3}} \right).$$

Substituting the values of $\rho_1, \rho_2, \xi, \eta$ from the above equations (8) with $\eta \neq 0$, and neglecting second and higher terms in $x, y, \frac{x}{c^2}, \frac{y}{c^2}, \sigma_1, \sigma_2$ and their product, we have

$$(9) \quad \frac{3}{2}(1-\mu)x - \frac{3\mu}{2}y + \frac{(57\sigma_2 - 69\sigma_1)}{16}\mu + \frac{45}{16}(\sigma_1 - \sigma_2) + \frac{1}{c^2} \left(-\frac{9\mu}{16} + \frac{27\mu^2}{16} - \frac{9\mu^3}{8} \right) = 0$$

$$3(1-\mu)x + 3\mu y + \frac{3}{8}(\sigma_1 + 3\sigma_2)\mu + \frac{21}{8}(\sigma_1 + \sigma_2) + \frac{21(\mu - \mu^2)}{8c^2} = 0$$

Solving these equations for x and y , we get

$$(10) \quad x = -\frac{\mu(2+3\mu)}{8c^2} + \frac{11}{8}(\sigma_2 - \sigma_1)$$

$$y = -\frac{(1-\mu)(5-3\mu)}{8c^2} + \left(\frac{1}{2\mu} - \frac{3}{2}\right)\sigma_1 + \left(-\frac{1}{2\mu} + 1\right)\sigma_2$$

Thus, the coordinates of the triangular points $(\xi, \pm\eta)$ denoted by L_4 and L_5 respectively are,

$$(11) \quad \xi = \frac{1-2\mu}{2} \left(1 + \frac{5}{4c^2}\right) + \left(\frac{1}{8} - \frac{1}{2\mu}\right)\sigma_1 + \left(\frac{1}{2\mu} + \frac{3}{8}\right)\sigma_2$$

$$\eta = \pm \frac{\sqrt{3}}{2} \left[1 + \frac{1}{12c^2}(-5 + 6\mu - 6\mu^2) + \frac{2}{3} \left\{ \left(-\frac{23}{8} + \frac{1}{2\mu}\right)\sigma_1 + \left(\frac{19}{8} - \frac{1}{2\mu}\right)\sigma_2 \right\} \right].$$

4. STABILITY OF L_4

Let (a, b) be the coordinates of the triangular points L_4

We set $\xi = a + \alpha$, $\eta = b + \beta$, $(\alpha, \beta < 1)$ in the equations (5) of motion.

First, we compute the terms of their R.H.S, neglecting second and higher order terms, we get

$$\left(\frac{\partial W}{\partial \xi}\right)_{\xi=a+\alpha, \eta=b+\beta} = A\alpha + B\beta + C\dot{\alpha} + D\dot{\beta}$$

where,

$$A = \frac{3}{4} \left\{1 + \frac{1}{2c^2}(2 - 19\mu + 19\mu^2)\right\} + \frac{3(15\mu^2 + 19\mu - 8)}{16\mu}\sigma_1 - \frac{3(31\mu^2 + \mu - 8)}{16\mu}\sigma_2,$$

$$B = \frac{3\sqrt{3}}{4}(1 - 2\mu)\left(1 - \frac{2}{3c^2}\right) - \frac{\sqrt{3}(89\mu^2 - 47\mu + 8)}{16\mu}\sigma_1 + \frac{\sqrt{3}(37\mu^2 - 9\mu + 8)}{16\mu}\sigma_2,$$

$$C = \frac{\sqrt{3}}{2c^2}(1 - 2\mu),$$

$$D = \frac{6 - 5\mu + 5\mu^2}{2c^2}.$$

Similarly, we obtain

$$\left(\frac{\partial W}{\partial \eta}\right)_{\xi=a+\alpha, \eta=b+\beta} = A_1\alpha + B_1\beta + C_1\dot{\alpha} + D_1\dot{\beta}$$

where,

$$A_1 = \frac{3\sqrt{3}}{4}(1 - 2\mu)\left(1 - \frac{2}{3c^2}\right) - \frac{\sqrt{3}(89\mu^2 - 47\mu + 8)}{16\mu}\sigma_1 + \frac{\sqrt{3}(37\mu^2 - 9\mu + 8)}{16\mu}\sigma_2,$$

$$B_1 = \frac{9}{4} \left\{1 + \frac{7}{6c^2}(-2 + 3\mu - 3\mu^2)\right\} - \frac{3(15\mu^2 - 29\mu - 8)}{16\mu}\sigma_1 + \frac{3(15\mu^2 - 7\mu - 8)}{16\mu}\sigma_2$$

$$C_1 = \frac{1}{2c^2}(-4 + \mu - \mu^2),$$

$$D_1 = -\frac{\sqrt{3}(1-2\mu)}{2c^2}.$$

$$\frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\xi}}\right)_{\xi=a+\alpha, \eta=b+\beta} = A_2\dot{\alpha} + B_2\dot{\beta} + C_2\ddot{\alpha} + D_2\ddot{\beta}$$

where,

$$A_2 = \frac{\sqrt{3}}{2c^2} (1 - 2\mu),$$

$$B_2 = \frac{1}{2c^2} (-4 + \mu - \mu^2),$$

$$C_2 = \frac{1}{4c^2} (17 - 2\mu + 2\mu^2),$$

$$D_2 = -\frac{\sqrt{3}}{4c^2} (1 - 2\mu).$$

$$\frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\eta}} \right)_{\xi=a+\alpha, \eta=b+\beta} = A_3 \dot{\alpha} + B_3 \dot{\beta} + C_3 \ddot{\alpha} + D_3 \ddot{\beta}$$

where

$$A_3 = \frac{1}{2c^2} (6 - 5\mu + 5\mu^2),$$

$$B_3 = -\frac{\sqrt{3}}{2c^2} (1 - 2\mu),$$

$$C_3 = -\frac{\sqrt{3}}{4c^2} (1 - 2\mu),$$

$$D_3 = \frac{3(5-2\mu+2\mu^2)}{4c^2}.$$

Thus, the variational equations of motion corresponding to equations (5), on making use of equation (7), can be obtained as

$$(12) \quad P_1 \ddot{\alpha} + P_2 \ddot{\beta} + P_3 \dot{\alpha} + P_4 \dot{\beta} + P_5 \alpha + P_6 \beta = 0,$$

$$q_1 \ddot{\alpha} + q_2 \ddot{\beta} + q_3 \dot{\alpha} + q_4 \dot{\beta} + q_5 \alpha + q_6 \beta = 0.$$

where

$$P_1 = 1 + C_2, P_2 = D_2, P_3 = A_2 - C, P_4 = \left\{ B_2 - 2 \left(1 + \frac{3}{4} (2\sigma_1 - \sigma_2) - \frac{3}{2c^2} \left(1 - \frac{1}{3} \mu (1 - \mu) \right) \right) - D \right\}, P_5 = -A_1, P_6 = -B$$

$$q_1 = C_3, q_2 = 1 + D_3, q_3 = 2 \left(1 + \frac{3}{4} (2\sigma_1 - \sigma_2) - \frac{3}{2c^2} \left(1 - \frac{1}{3} \mu (1 - \mu) \right) \right) - C_1 + A_3, q_4 = B_3 - D_1, q_5 = -A_1, q_6 = -B_1$$

Then, the characteristic equation is

$$(13) \quad (P_1 q_2 - P_2 q_1) \lambda^4 + (P_1 q_6 + P_5 q_2 + P_3 q_4 - P_6 q_1 - P_2 q_5 - P_4 q_3) \lambda^2 + P_5 q_6 - P_6 q_5 = 0$$

Substituting the values of $P_i, q_i, i = 1, 2, \dots, 6$ in (3), the characteristic equation (3) after normalizing becomes

$$(14) \quad \lambda^4 + b\lambda^2 + d = 0$$

where,

$$b = \left(1 - \frac{9}{c^2}\right) + 3\sigma_1 + \frac{3}{2}(2\mu - 3)\sigma_2,$$

$$d = \frac{27\mu(1-\mu)}{4} + \frac{9\mu(-65+77\mu-24\mu^2+12\mu^3)}{8c^2} + \frac{9(-89\mu^2+99\mu-10)}{16}\sigma_1$$

$$+ \frac{9(37\mu^2-47\mu+10)}{16}\sigma_2$$

When $\frac{1}{c^2} \rightarrow 0$ and in the absence of the triaxiality (*i.e.* $\sigma_1 = \sigma_2 = 0$), (14) reduces to its well-known classical restricted problem form (see e.g. Szebehely, 1967 [22]):

$$\lambda^4 + \lambda^2 + \frac{27\mu(1-\mu)}{4} = 0$$

The discriminant of (4) is

$$(15) \quad \Delta = \frac{-54}{c^2}\mu^4 + \frac{108}{c^2}\mu^3 + \left(27 + \frac{801}{4}\sigma_1 - \frac{333}{4}\sigma_2 - \frac{693}{2c^2}\right)\mu^2$$

$$+ \left(-27 - \frac{891}{4}\sigma_1 + \frac{447}{4}\sigma_2 + \frac{585}{2c^2}\right)\mu + 1 - \frac{18}{c^2} + \frac{57}{2}\sigma_1 - \frac{63}{2}\sigma_2$$

Its roots are

$$(16) \quad \lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2}$$

where,

$$b = \left(1 - \frac{9}{c^2}\right) + 3\sigma_1 + \frac{3}{2}(2\mu - 3)\sigma_2$$

From (15), we have

$$(17) \quad \frac{d\Delta}{d\mu} = \frac{-216}{c^2}\mu^3 + \frac{324}{c^2}\mu^2 + 2\left(27 + \frac{801}{4}\sigma_1 - \frac{333}{4}\sigma_2 - \frac{693}{2c^2}\right)\mu$$

$$+ \left(-27 - \frac{891}{4}\sigma_1 + \frac{447}{4}\sigma_2 + \frac{585}{2c^2}\right) < 0 \quad \forall \mu \in \left(0, \frac{1}{2}\right].$$

From (17), it can be easily seen that Δ is monotone decreasing in $\left(0, \frac{1}{2}\right]$.

But

$$(18) \quad (\Delta)_{\mu=0} = 1 + \frac{57}{2}\sigma_1 - \frac{63}{2}\sigma_2 - \frac{18}{c^2} > 0$$

$$(\Delta)_{\mu=\frac{1}{2}} = -\frac{23}{4} - \frac{525}{16}\sigma_1 + \frac{57}{16}\sigma_2 + \frac{207}{4c^2} < 0$$

Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\frac{1}{2}}$ are of opposite signs, and Δ is monotone decreasing and continuous, there is one value of μ , e.g. μ_c in the interval $\left(0, \frac{1}{2}\right]$ for which Δ vanishes.

Solving the equation $\Delta = 0$, using (15), we obtain critical value of the mass parameter as

$$(19) \quad \mu_c = \frac{1}{2} - \frac{1}{18}\sqrt{69} - \frac{17\sqrt{69}}{486c^2} + \frac{1}{2} \left(\frac{5}{6} + \frac{59}{9\sqrt{69}} \right) \sigma_1 - \frac{1}{2} \left(\frac{19}{18} + \frac{85}{9\sqrt{69}} \right) \sigma_2$$

$$\mu_c = \mu_0 - \frac{17\sqrt{69}}{486c^2} + \frac{1}{2} \left(\frac{5}{6} + \frac{59}{9\sqrt{69}} \right) \sigma_1 - \frac{1}{2} \left(\frac{19}{18} + \frac{85}{9\sqrt{69}} \right) \sigma_2$$

μ_c can be written as

$$(20) \quad \mu_c = \mu_0 + p$$

with

$$p = -\frac{17\sqrt{69}}{486c^2} + \frac{1}{2} \left(\frac{5}{6} + \frac{59}{9\sqrt{69}} \right) \sigma_1 - \frac{1}{2} \left(\frac{19}{18} + \frac{85}{9\sqrt{69}} \right) \sigma_2$$

where $\mu_0 = 0.03852\dots$ is the Routh's value.

We consider the following three regions of the values of μ separately.

- (1) When $0 \leq \mu < \mu_c, \Delta > 0$, the values of λ^2 given by (16) are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are stable.
- (2) When $\mu_c < \mu \leq \frac{1}{2}, \Delta < 0$, the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.
- (3) When $\mu = \mu_c, \Delta = 0$, the values of λ^2 given by (16) are the same.

So the solutions of the equations of motion contain secular terms. This induces instability of the triangular points.

Hence, the stability region is

$$(21) \quad 0 < \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2} + \frac{1}{2} \left(\frac{5}{6} + \frac{59}{9\sqrt{69}} \right) \sigma_1 - \frac{1}{2} \left(\frac{19}{18} + \frac{85}{9\sqrt{69}} \right) \sigma_2$$

5. PERIODIC ORBITS

Studying of the periodic orbits is considered as one of the greatest important matters in the restricted three-body problem and starting point for attacking the problem of classifying solutions. Furthermore, if one has a particular solution for the restricted problem, then he can always get a periodic solution. The importance of studying these orbits is situated in their significant appearing in nature, they provide us with exciting information about orbital resonances and spin orbits and they can be used as reference orbits, also to approximate quasi-periodic trajectories.

5.1. The motion in the proximity of the triangular points. The triangular Lagrangian points $L_{4,5}$ ($\xi_0, \pm\eta_0$) are given from (11) as:

$$(22) \quad \xi = \frac{1-2\mu}{2} \left(1 + \frac{5}{4c^2}\right) + \left(\frac{1}{8} - \frac{1}{2\mu}\right) \sigma_1 + \left(\frac{1}{2\mu} + \frac{3}{8}\right) \sigma_2$$

$$\eta = \pm \frac{\sqrt{3}}{2} \left[1 + \frac{1}{12c^2} (-5 + 6\mu - 6\mu^2) + \frac{2}{3} \left\{ \left(-\frac{23}{8} + \frac{1}{2\mu}\right) \sigma_1 + \left(\frac{19}{8} - \frac{1}{2\mu}\right) \sigma_2 \right\} \right].$$

In section 4, it is stated that the triangular point are linearly stable in the range $0 \leq \mu < \mu_c$ and the characteristic equation has four purely imaginary roots. Thus, the motion in this region is bounded and made up of two harmonic motion with frequencies s_1 and s_2 given by:

$$(23) \quad x = C_1 \cos s_1 t + D_1 \sin s_1 t + C_2 \cos s_2 t + D_2 \sin s_2 t$$

$$y = \bar{C}_1 \cos s_1 t + \bar{D}_1 \sin s_1 t + \bar{C}_2 \cos s_2 t + \bar{D}_2 \sin s_2 t$$

$$(24) \quad \bar{C}_i = \Gamma_i \left(2ns_i D_i - W_{\xi\eta}^0 C_i\right) \quad (i = 1, 2)$$

$$\bar{D}_i = -\Gamma_i \left(2ns_i C_i - W_{\xi\eta}^0 D_i\right) \quad (i = 1, 2)$$

with and $\Gamma_i = \frac{s_i^2 + W_{\xi\eta}^0}{4n^2 s_i^2 + (W_{\xi\eta}^0)^2} = \frac{1}{s_i^2 + W_{\eta\eta}^0} \quad (i = 1, 2)$ (Szebehely, 1967 [22]) where s_1 and s_2 are the frequencies with respect to long and short periodic orbits respectively, the terms with coefficients $C_1, D_1, \bar{C}_1, \bar{D}_1$ are the long periodic terms while the coefficients $C_2, D_2, \bar{C}_2, \bar{D}_2$ are the short periodic terms. The four roots of the characteristic equation (14) are written as :

$$(25) \quad \lambda_{1,2} = \pm i s_1$$

and

$$\lambda_{3,4} = \pm i s_2$$

From equation (16), we get

$$(26) \quad s_1^2 = \frac{27\mu(1-\mu)}{4} + \left(\frac{-45}{8} + \frac{891}{16}\mu - \frac{801}{16}\mu^2\right) \sigma_1 + \left(\frac{45}{8} - \frac{423}{16}\mu + \frac{333}{16}\mu^2\right) \sigma_2$$

$$+ \frac{1}{8c^2} (108\mu^4 - 216\mu^3 + 693\mu^2 - 585\mu)$$

$$s_2^2 = 1 - \frac{27\mu(1-\mu)}{4} + \left(\frac{69}{8} - \frac{891}{16}\mu + \frac{801}{16}\mu^2\right) \sigma_1 + \left(-\frac{81}{8} + \frac{471}{16}\mu - \frac{333}{16}\mu^2\right) \sigma_2$$

$$- \frac{1}{8c^2} (108\mu^4 - 216\mu^3 + 693\mu^2 - 585\mu + 72)$$

5.2. Elliptic orbits. The Taylor expansion of the function of the potential W around the triangular points $L_{4,5}$ can be written in the following form:

$$(27) \quad W = W^0 + \frac{1}{2}W_{\xi\eta}^0 x^2 + W_{\xi\xi}^0 xy + \frac{1}{2}W_{\eta\eta}^0 y^2 + 0(x^3, y^3)$$

where the terms containing the third and higher powers of x and y are neglected and

$$(28) \quad W^0 = \frac{1}{2}(\mu^2 - \mu + 3) + \left(\frac{11}{8} - \frac{11}{8}\mu + \frac{3}{2}\mu^2\right)\sigma_1 - \frac{1}{8}(6\mu^2 - \mu + 1)\sigma_2 \\ - \frac{1}{8c^2}(3\mu^4 - 6\mu^3 - 2\mu^2 + 5\mu + 3),$$

$$(29)$$

$$W_{\xi\xi}^0 = \frac{3}{4} \left\{ 1 + \frac{1}{2c^2}(2 - 19\mu + 19\mu^2) \right\} + \frac{3(15\mu^2 + 19\mu - 8)}{16\mu}\sigma_1 - \frac{3(31\mu^2 + \mu - 8)}{16\mu}\sigma_2,$$

$$(30)$$

$$W_{\eta\eta}^0 = \frac{9}{4} \left\{ 1 + \frac{7}{6c^2}(-2 + 3\mu - 3\mu^2) \right\} - \frac{3(15\mu^2 - 29\mu - 8)}{16\mu}\sigma_1 + \frac{3(15\mu^2 - 7\mu - 8)}{16\mu}\sigma_2,$$

$$(31)$$

$$W_{\xi\eta}^0 = \frac{3\sqrt{3}}{4}(1 - 2\mu) \left(1 - \frac{2}{3c^2} \right) - \frac{\sqrt{3}(89\mu^2 - 47\mu + 8)}{16\mu}\sigma_1 + \frac{\sqrt{3}(37\mu^2 - 9\mu + 8)}{16\mu}\sigma_2,$$

Equation (27) represents a quadratic form and indicates that the periodic orbits around the triangular points are ellipse, such that

$$H = \begin{vmatrix} \frac{1}{2}W_{\xi\xi}^0 & \frac{1}{2}W_{\xi\eta}^0 \\ \frac{1}{2}W_{\eta\xi}^0 & \frac{1}{2}W_{\eta\eta}^0 \end{vmatrix} = \frac{1}{4}(W_{\xi\xi}^0 W_{\eta\eta}^0 - (W_{\xi\eta}^0)^2) > 0$$

5.2.1. Orientation of the principal axes of the ellipses. With the transformation

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \text{ the directions of the principal axes} \\ \text{are given by:}$$

$$(32) \quad \tan 2\theta = \frac{2W_{\xi\eta}^0}{W_{\xi\xi}^0 - W_{\eta\eta}^0},$$

[Abouelmagd and El-Shaboury (2012) [1]]

Substituting (29), (30) and (21) in (32), we obtain

$$(33)$$

$$\tan 2\theta = \pm\sqrt{3} \left\{ (1 - 2\mu) - \frac{10}{3c^2}(1 - 2\mu)(1 - 3\mu + 3\mu^2) + \left(\frac{45\mu^3 + 7\mu^2 - 40\mu + 16}{6\mu} \right) \sigma_1 \right. \\ \left. - \left(\frac{69\mu^3 - 25\mu^2 - 24\mu + 16}{6\mu} \right) \sigma_2 \right\},$$

such that positive and negative signs refer to the periodic motion of L_4 and L_5 respectively.

5.2.2. *Eccentricities of the ellipses.* The Taylor expansion of the function of potential W around the triangular points $L_{4,5}$ is given by equation (27), but the Jacobian constant $C = 2W$ implies that

$$(34) \quad \begin{aligned} C = & (\mu^2 - \mu + 3) + \left(\frac{11}{4} - \frac{11}{4}\mu + 3\mu^2\right) \sigma_1 \\ & - \frac{1}{4} (1 - \mu + 6\mu^2) \sigma_2 - \frac{1}{4c^2} (3\mu^4 - 6\mu^3 - 2\mu^2 + 5\mu + 3) \\ & + \left\{ \frac{3}{4} \left(1 + \frac{1}{2c^2}\right) (2 - 19\mu + 19\mu^2) + 3 \left(\frac{15\mu^2 + 19\mu - 8}{16\mu}\right) \sigma_1 - 3 \left(\frac{31\mu^2 + \mu - 8}{16\mu}\right) \sigma_2 \right\} x^2 \\ & + \left\{ \frac{3\sqrt{3}}{2} (1 - 2\mu) \left(1 - \frac{2}{3c^2}\right) - \sqrt{3} \left(\frac{89\mu^2 - 47\mu + 8}{8\mu}\right) \sigma_1 + \sqrt{3} \left(\frac{37\mu^2 - 9\mu + 8}{8\mu}\right) \sigma_2 \right\} xy \\ & + \left\{ \frac{9}{4} \left(1 + \frac{7}{6c^2} (-2 + 3\mu - 3\mu^2)\right) - 3 \left(\frac{15\mu^2 - 29\mu - 8}{16\mu}\right) \sigma_1 + 3 \left(\frac{15\mu^2 - 9\mu + 8}{8\mu}\right) \sigma_2 \right\} y^2. \end{aligned}$$

The associated form of the corresponding characteristic equation of (34) is

$$(35) \quad \begin{aligned} & \lambda^2 - \left\{ 3 + 9\sigma_1 - \left(3\mu + \frac{3}{2}\right) \sigma_2 + \frac{3}{4c^2} (-6 + \mu - \mu^2) \right\} \lambda + \frac{27}{4} \mu (1 - \mu) \\ & + \left(-\frac{45}{8} + \frac{891}{16} \mu - \frac{801}{16} \mu^2\right) \sigma_1 + \left(\frac{45}{8} - \frac{423}{16} \mu + \frac{333}{16} \mu^2\right) \sigma_2 - \frac{153}{8c^2} \mu (1 - \mu) = 0 \end{aligned}$$

Its roots are

$$(36) \quad \begin{aligned} \lambda_1 = & 3 - \frac{9\mu(1-\mu)}{4} + \left(\frac{87}{8} - \frac{297}{16} \mu + \frac{267}{16} \mu^2\right) \sigma_1 + \left(-\frac{27}{8} + \frac{93}{16} \mu - \frac{111}{16} \mu^2\right) \sigma_2 \\ & + \frac{3}{8c^2} (-12 + 19\mu - 19\mu^2) \\ \lambda_2 = & \frac{9\mu(1-\mu)}{4} + \left(-\frac{15}{8} + \frac{297}{16} \mu - \frac{267}{16} \mu^2\right) \sigma_1 + \left(\frac{15}{8} - \frac{141}{16} \mu + \frac{111}{16} \mu^2\right) \sigma_2 \\ & + \frac{51}{8c^2} (\mu^2 - \mu) \end{aligned}$$

The eccentricities of the ellipses are given by (Szebehely, 1967 [22])

$$(37) \quad e_i = (1 - \alpha_i^2)^{\frac{1}{2}}, \quad \alpha_i = \frac{2s_i}{s_i^2 + \lambda} \quad (i = 1, 2)$$

where $\bar{\lambda}$ is one of the roots of equation (34). For $i = 1, 2$ and making use of equation (26) and $\bar{\lambda} = \lambda_1$, we obtain

(38)

$$\begin{aligned}\alpha_1^2 &= 3\mu(1-\mu) - 9\mu^2(1-\mu)^2 + \left(-\frac{5}{2} + \frac{87}{4}\mu - \frac{671}{4}\mu^2 + 282\mu^3 - \frac{267}{2}\mu^4\right)\sigma_1 \\ &+ \left(\frac{5}{2} - \frac{95}{4}\mu + \frac{391}{4}\mu^2 - 132\mu^3 + \frac{111}{2}\mu^4\right)\sigma_2 + \frac{1}{2c^2}(-47\mu + 518\mu^2 - 1032\mu^3 \\ &+ 741\mu^4 - 270\mu^5 + 90\mu^6) \\ \alpha_2^2 &= \frac{1}{4} - \frac{9\mu(1-\mu)}{8} - \frac{243\mu^2(1-\mu)^2}{32} + \left(-\frac{9}{32} + \frac{1377}{64}\mu - \frac{4713}{32}\mu^2 + \frac{3807}{16}\mu^3 - \frac{7209}{64}\mu^4\right)\sigma_1 \\ &+ \left(-\frac{27}{32} - \frac{1269}{64}\mu + \frac{2685}{32}\mu^2 - \frac{1755}{16}\mu^3 + \frac{2997}{64}\mu^4\right)\sigma_2 - \frac{1}{32c^2}(72 + 414\mu - 20679\mu^2 \\ &+ 41364\mu^3 - 29187\mu^4 + 10206\mu^5 - 3402\mu^6).\end{aligned}$$

Hence,

(39)

$$\begin{aligned}e_1 &= 1 - \frac{3}{2}\mu(1-\mu) + \frac{9}{2}\mu^2(1-\mu) + \left(\frac{5}{4} - \frac{87}{8}\mu + \frac{671}{8}\mu^2 - 141\mu^3 + \frac{267}{4}\mu^4\right)\sigma_1 \\ &+ \left(-\frac{5}{4} + \frac{95}{8}\mu - \frac{391}{8}\mu^2 + 66\mu^3 - \frac{111}{4}\mu^4\right)\sigma_2 + \frac{1}{4c^2}(47\mu - 518\mu^2 + 1032\mu^3 \\ &- 741\mu^4 + 270\mu^5 - 90\mu^6)\end{aligned}$$

6. SEMI-MAJOR AND SEMI-MINOR AXES

The lengths of the Semi-major axis, a_i and the Semi-minor axis, b_i of the long and short period are obtained from

$$\begin{aligned}(40) \quad a_i &= \left(\xi_0^2 + \frac{\eta_0^2}{\alpha_i^2}\right)^{\frac{1}{2}} \\ \text{and} \\ b_i &= \left(\alpha_i^2 \xi_0^2 + \eta_0^2\right)^{\frac{1}{2}},\end{aligned}$$

respectively, with ξ_0 and η_0 as initial conditions (Szebeheley, 1967 [22]).

To obtain the semi-major axis of the orbit of the long period and that of the short period, we make use of equations (22) and (38) in (40), when $i = 1$, respectively.

Thus, we get

$$\begin{aligned}
 (41) \quad a_1 &= \frac{\sqrt{5}}{2} \left\{ \left(1 - \frac{1}{10\mu} + \frac{7\mu^2}{10} \right) + \left(\frac{493}{120} - 7\mu + \frac{33}{10}\mu^2 - \frac{25}{24\mu} + \frac{3}{20\mu^2} \right) \sigma_1 \right. \\
 &\quad + \left(-\frac{221}{120} + 2\mu - \frac{9\mu^2}{10} + \frac{25}{24\mu} - \frac{3}{20\mu^2} \right) \sigma_2 \\
 &\quad \left. + \frac{1}{60c^2} \left(-517 + 1020\mu + \frac{42}{\mu} - 156\mu^2 + 324\mu^3 - 108\mu^4 \right) \right\}. \\
 b_1 &= \frac{\sqrt{3}}{2} \left\{ \left(1 + \frac{\mu}{2} - 4\mu^2 + \frac{39}{3}\mu^3 - \frac{43}{2}\mu^4 + 18\mu^5 - 6\mu^6 \right) \right. \\
 &\quad + \left(-\frac{10}{3} + \frac{277}{24}\mu + \frac{1}{3\mu} - \frac{477}{8}\mu^2 + \frac{1151}{6}\mu^3 - \frac{1991}{6}\mu^4 + \frac{557}{2}\mu^5 - 89\mu^6 \right) \sigma_1 \\
 &\quad + \left(3 - \frac{261}{24}\mu - \frac{1}{3\mu} + \frac{1039}{24}\mu^2 - \frac{215}{2}\mu^3 + \frac{943}{6}\mu^4 - \frac{241}{2}\mu^5 + \frac{111}{3}\mu^6 \right) \sigma_2 \\
 &\quad + \frac{1}{12c^2} \left(-52 + 209\mu + 204\mu^2 - 2714\mu^3 + 6296\mu^4 - 6822\mu^5 \right. \\
 &\quad \left. + 3954\mu^6 - 1440\mu^7 + 360\mu^8 \right) \left. \right\}.
 \end{aligned}$$

Similarly, we obtain the semi-minor axis of the orbits of the long and short periods

as

$$\begin{aligned}
 (42) \quad a_2 &= \frac{\sqrt{13}}{2} \left\{ \left(1 + \frac{25}{13}\mu + \frac{629}{52}\mu^2 - \frac{729}{26}\mu^3 + \frac{729}{52}\mu^4 \right) + \left(\frac{4}{13} - \frac{4135}{104}\mu + \frac{3}{13\mu} + \frac{21573}{104}\mu^2 \right. \right. \\
 &\quad \left. - \frac{16767}{52}\mu^3 + \frac{8019}{52}\mu^4 \right) \sigma_1 + \left(\frac{323}{52} - \frac{1884}{13}\mu - \frac{3}{13\mu} - \frac{58563}{104}\mu^2 + \frac{37017}{52}\mu^3 - \frac{31347}{104}\mu^4 \right) \sigma_2 \\
 &\quad \left. + \frac{1}{104c^2} \left(402 + 2312\mu - 124901\mu^2 + 251640\mu^3 - 180495\mu^4 + 65610\mu^5 - 21870\mu^6 \right) \right\}. \\
 b_2 &= \frac{\sqrt{13}}{4} \left\{ \left(1 - \frac{17}{52}\mu - \frac{31}{208}\mu^2 + \frac{585}{104}\mu^3 - \frac{1015}{208}\mu^4 + \frac{729}{52}\mu^5 - \frac{243}{52}\mu^6 \right) \right. \\
 &\quad + \left(-\frac{161}{208} + \frac{1889}{416}\mu - \frac{3507}{104}\mu^2 + \frac{11337}{104}\mu^3 - \frac{22533}{104}\mu^4 + \frac{45117}{208}\mu^5 - \frac{7209}{104}\mu^6 \right) \sigma_1 \\
 &\quad + \left(\frac{205}{208} - \frac{1701}{416}\mu - \frac{4}{13\mu} + \frac{8055}{416}\mu^2 - \frac{4443}{208}\mu^3 + \frac{987}{52}\mu^4 - \frac{8565}{208}\mu^5 + \frac{2997}{104}\mu^6 \right) \sigma_2 \\
 &\quad + \frac{1}{416c^2} \left(-312 - 422\mu + 21353\mu^2 - 114036\mu^3 + 247209\mu^4 \right. \\
 &\quad \left. - 263250\mu^5 + 151254\mu^6 - 54432\mu^7 + 13608\mu^8 \right) \left. \right\}
 \end{aligned}$$

7. DISCUSSION

The equations (11) and (19) corresponding to positions of the triangular points and critical mass respectively differ from those obtained by Douskos and Perdios (2002) [6] due to the triaxiality of the bigger primary. If the bigger primary is not triaxial and in the absence of the relativistic effect, equations (11) and (19) correspond to the classical restricted problem. Equation (19) gives the critical value μ_c of mass parameter. It shows the combined effect of relativistic and triaxiality factors. It can be seen that the relativistic factor shrinks the size of the region of stability whereas the triaxiality factor shrinks or expands the size of the region depending whether the term containing the triaxiality coefficients factors σ_1 and σ_2 is less or greater

than zero respectively. It can be also observed from (26) that the frequencies of the orbits of the long and short periodic motion are dependent on all the parameters involved. From equation (27), one can infer that the periodic orbits are elliptical. Also from (33) it can be observed that the orientation of these orbits are affected by triaxiality and relativistic factors. The eccentricities of the long and short periodic orbits are obtained in (39). It is seen that they are also affected by the triaxiality and relativistic factors.

Equations (41) and (42) provide the semi-major and semi-minor axes of the long and short periods. In the absence of relativistic and triaxiality factors i.e. when $\frac{1}{c^2} \rightarrow 0$ and $\sigma_1 = \sigma_2 = 0$, the results of the present study correspond to those of the classical case (Szebehely, 1967 [22]) and the results of equations (26) and (33) coincide with those of Abouelmagd and El-Shaboury (2012) [1] when the oblateness and radiation factors are absent. In the absence of relativistic factor our result also validate those of Sharma et al. (2001b) when the bigger primary is considered only triaxial when μ tends to zero.

In the absence of the triaxiality factors, the results obtained in this study are in agreement with those of Douskos and Perdios (2002) [6] but in disagreement with those of Bhatnagar and Hallan (1998) [2]. In the absence of relativistic terms, it is observed that our results coincide with those of Singh (2013) [20] when the primaries are non-luminous and the bigger primary is triaxial only, of those of Sharma et al. (2001a) when the primaries are non-luminous and the bigger triaxial only and of those of Sharma (2001) [19] when only the bigger primary is triaxial and the smaller spheroid. When the bigger primary is oblate spheroid (i.e. $\sigma_1 = \sigma_2$) and the relativistic factor is absent, the results of this study coincide with those of SubbaRao and Sharma (1975) [21] but disagree with those of Katour et al. (2014) [13] when the primaries are non-luminous and the bigger primary only oblate.

8. CONCLUSION

Taking into account only the triaxiality of the bigger primary in the relativistic CR3BP, we have obtained the locations of the triangular points and have analyzed their linear stability. It is also observed that both factors relativistic and triaxiality have a destabilizing tendency. We also investigated the periodic orbit in the vicinity of the triangular points for $0 < \mu < \mu_c$. The frequencies, the coefficients of long and

short periodic terms, the elliptic orbits, the orientation, the lengths of semi-major and semi-minor axes were found. A practical application of this model could be the study of the motion of a dust grain in the gravitational field of the Pluto-Charon system. We have noticed that the expressions for A , D , A_2 , C_2 in Bhatnagar and Hallan (1998) [2] differ from those the current study in the absence of triaxiality. Hence, makes the expressions P_1 , P_3 , P_4 , P_5 and the characteristic equation are also different. This led them (Bhatnagar and Hallan 1998 [2]) to conclude that triangular points are unstable, contrary to Douskos and Perdios (2002) [6] and our results. Moreover, we have also realized that there is error in the expression of the perturbed mean motion in Katour et al. (2014) [13]. This makes the major difference with their result when the primaries are non-luminous and the bigger primary oblate (i.e. $\sigma_1 = \sigma_2$).

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