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A NEW PROBLEM OF SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce a new class of nonlinear singular fractional differential equations. We are interested in the singularity by using the contraction mapping principle and Schauder fixed point theorem. We present new results on the existence and uniqueness of solutions. Moreover, we investigate the Ulam-Hyers stability and the generalized Ulam-Hyers stability for this fractional equations. Some examples are provided to illustrate the application of our results.

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1. Introduction

Recently, the fractional calculus has attracted interest of researchers in several areas including mathematics, physics, chemistry, engineering and even finance and social sciences. Where there has been an explosion of research activities on the application of fractional calculus. For details, see [11, 18, 21, 22] and the references therein. Indeed, the study of fractional order differential equations is very significant because this kind can often occur in applications. Thence, there have

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been much papers dealing with the existence and uniqueness of solutions for the fractional differential equations, see [3, 4, 6, 7, 8, 9, 12, 13, 16, 23]. Also there are an important articles, [1, 2, 19] has been done for the boundary value problems of singular fractional differential equation. Moreover, from the Ulam-Hyers stability of fractional differential equations, we can inspect several topics in nonlinear analysis problems, see [5, 10, 14, 15, 17, 20].

Let us now introduce some results that have motivaeted our work. We begin by [2], where R.P. Agarwal et al. investigated the existence of solutions for the following singular fractional boundary value problem:

$$\begin{cases} D^{\alpha}u\left(t\right)+f\left(t,u\left(t\right),D^{\mu}u\left(t\right)\right)=0,\ 0<t<1, \\ u\left(0\right)=u\left(1\right)=0, \end{cases}$$

where $1<\alpha<2$, $0<\mu<\alpha-1$, D^{α} is the standard Riemann Liouville fractional derivative, f is a positive Carathéodory function on $[0,1]\times(0,\infty)\times\mathbb{R}$, and $f(t,x,y)$ is singular at $x=0$.

In [13], M. Houas and Z. Dahmani established a new existence and uniqueness result of solutions of the following problem:

$$\begin{cases} D^{\alpha}x\left(t\right)=f\left(t,x\left(t\right),D^{\beta}x\left(t\right)\right),\ t\in J, \\ x\left(0\right)=x\left(1\right)=0, \\ \lambda_1 x^{''}\left(\eta\right)-\lambda_2 x^{'''}\left(\eta\right)=0,\ \lambda_3 x^{''}\left(\xi\right)+\lambda_4 x^{'''}\left(\xi\right)=0, \end{cases}$$

under the conditions $\alpha\in\left]3,4\right]$, $\beta\leq\alpha-1$, $\eta,\xi\in\left]0,1\right[$, D^{α}, D^{β} are the Caputo fractional derivatives, $I=[0,1]$, λ_i, $i=1,2,3,4$, are real constants with $\lambda_2\lambda_3+\lambda_1\lambda_4+\lambda_1\lambda_3\left(\xi-\eta\right)\neq 0$ and f is a continuous function on $[0,1]\times\mathbb{R}^2$.

In [15], R.W. Ibrahim discussed the existence and uniqueness of solutions. Furthermore she presented different types of Ulam-Hyers stability for the following cauchy differential equation of fractional order:

$$\begin{cases} D_z^{\alpha}u\left(z\right)=f\left(z,u\left(z\right)\right), \\ au\left(0\right)+bu\left(1\right)=cu\left(\xi\right),\ z,\xi\in U,\ a+b\neq c, \end{cases}$$

where $u:U\to B$ is an analytic function for all z in the unit disk $U:=\{z:|z|<1\}$ and $f:U\times B\to B$ is an analytic function in $z\in U$. B is the space of all analytic and bounded functions in the unit disk.

In [19], R. Li investigated the existence and uniqueness of solutions for the problem:

$$\begin{cases} {}^{c}D_{0+}^{q}u\left(t\right)+f\left(t,u\left(t\right),{}^{c}D_{0+}^{\sigma}u\left(t\right)\right)=0,\ 0<t<1, \\ u\left(0\right)=u^{'}\left(0\right)=0,\ u^{'}\left(1\right)=\ {}^{c}D_{0+}^{\sigma}u\left(1\right), \end{cases}$$

where $2<q<3$, $0<\sigma<1$, $f:(0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous, $f\left(t,x,y\right)$ may be singular at $t=0$, ${}^{c}D_{0+}^{q}$ is the standard Caputo derivative.

This paper is devoted to build the existence and uniqueness of solutions and the generalized Ulam-Hyers stability for the nonlinear problem:

$$(1)\quad \begin{cases} D^{\alpha_n}x\left(t\right)+f\left(t,x\left(t\right),D^{\alpha_1}x\left(t\right),D^{\alpha_2}x\left(t\right),...,D^{\alpha_{n-1}}x\left(t\right)\right)=0, \\ 0<t\leq 1, \\ k-1<\alpha_k<k,\ k=1,2,...,n, \\ x^{(j)}\left(0\right)=\mu_j,\ j=0,1,...,n-2,\ D^{\delta}x\left(1\right)=\mu_{n-1},\ n-2<\delta<n-1, \end{cases}$$

where $n\in\mathbb{N}^{*}-\{1\}$. The function $f:]0,1]\times\mathbb{R}^{n}\to\mathbb{R}$ continuous, singular at $t=0$, and $\lim\limits_{t\to 0^{+}}f\left(t\right)=\infty$.

To the best of our knowledge, there are no papers that have developed the nonlinear term in multi-arbitrary fractional orders on the derivatives.
First, we present some concepts which we need in this paper.

Definition 1. [18] *The Caputo derivative of order α for a function $x:[0,\infty)\to\mathbb{R}$, which is at least n-times differentiable can be defined as:*

$$(2)\qquad D^{\alpha}x(t)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_{0}^{t}\left(t-s\right)^{n-\alpha-1}x^{(n)}\left(s\right)ds=J^{n-\alpha}x^{(n)}(t),$$

for $n-1<\alpha<n$, $n\in\mathbb{N}^{*}$.

Definition 2. [18] *The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$ for a continuous function f on $[0,\infty[$ is defined as:*

$$(3)\qquad J^{\alpha}f(t)=\begin{cases} \frac{1}{\Gamma(\alpha)}\int_{0}^{t}\left(t-s\right)^{\alpha-1}f\left(s\right)ds,\ \alpha>0, \\ f(t),\ \alpha=0, \end{cases}$$

where $t\geq 0$, and $\Gamma\left(\alpha\right):=\int_{0}^{\infty}e^{-u}u^{\alpha-1}du$.

Lemma 3. [18] *Let $\alpha,\beta>0$, and $n-1<\alpha<n$. Then, the following relations hold:*

$$D^{\alpha}t^{\beta-1}=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta-\alpha\right)}t^{\beta-\alpha-1},\ \beta>n$$

and

$$D^{\alpha}t^{k}=0,\ k=0,1,...,n-1.$$

Lemma 4. [18] *Let $q>p>0$ and $f\in L^{1}([a,b])$. Then for all $t\in[a,b]$, we have*

$$D^{p}J^{q}f(t)=J^{q-p}f(t),\ t\in[a,b].$$

Lemma 5. [18] *For $n-1<\alpha<n$, $n\in\mathbb{N}^{*}$, the general solution of the fractional differential equation $D^{\alpha}x(t)=0$, is given by*

$$x(t)=\sum_{j=0}^{n-1}c_{j}\,t^{j},\ \ (c_{j})_{j=0,1,...,n-1}\in\mathbb{R}. \tag{4}$$

Lemma 6. [18] *Let $n\in\mathbb{N}^{*}$, and $n-1<\alpha<n$, then*

$$J^{\alpha}D^{\alpha}x(t)=x(t)+\sum_{j=0}^{n-1}c_{j}\,t^{j},\ \ (c_{j})_{j=0,1,...,n-1}\in\mathbb{R}. \tag{5}$$

Lemma 7. *(Shauder fixed point theorem) Let (E,d) be a complete metric space, let U be a closed convex subset of E, and let $T:E\to E$ be a mapping such that the set $V:=\{Tx:x\in U\ \}$ is relatively compact in E. Then T has at last one fixed point.*

The following auxiliary result is important to give the integral solution of (1).

Lemma 8. *Assume that $n-1<\alpha_{n}<n$, $n\in\mathbb{N}^{*}-\{1\}$ and $Q(t)\in C([0,1])$. Then, the unique solution of the following problem:*

$$\begin{cases} D^{\alpha_{n}}x(t)+Q(t)=0,\ 0<t<1,\\ x^{(j)}(0)=\mu_{j},\ j=0,1,...,n-2,\\ D^{\delta}x(1)=\mu_{n-1},\ n-2<\delta<n-1, \end{cases} \tag{6}$$

is given by

$$\begin{aligned} x(t)=&-\int_{0}^{t}\frac{(t-s)^{\alpha_{n}-1}}{\Gamma(\alpha_{n})}Q(s)\,ds+\sum_{j=0}^{n-2}\frac{\mu_{j}}{j!}\,t^{j}\\ &+\tfrac{\Gamma(n-\delta)}{(n-1)!}\left(\mu_{n-1}+\int_{0}^{1}\tfrac{(1-s)^{\alpha_{n}-\delta-1}}{\Gamma(\alpha_{n}-\delta)}Q(s)\,ds\right)t^{n-1}. \end{aligned} \tag{7}$$

Proof. By Lemma 6, we get

$$x(t)=-\int_{0}^{t}\frac{(t-s)^{\alpha_{n}-1}}{\Gamma(\alpha_{n})}Q(s)\,ds-\sum_{j=0}^{n-1}c_{j}t^{j},\ c_{j}\in\mathbb{R}. \tag{8}$$

Applying the conditions in (6), we observe that

$$\begin{cases} x^{(j)}(0) = -j!c_j = \mu_j,\ j = 0,1,...,n-2, \\ D^{\delta}x(1) = -\int_0^1 \frac{(1-s)^{\alpha_n-\delta-1}}{\Gamma(\alpha_n-\delta)}Q(s)\,ds - \frac{\Gamma(n)}{\Gamma(n-\delta)}c_{n-1} = \mu_{n-1}. \end{cases} \tag{9}$$

Hence, we get

$$\begin{aligned} c_j &= -\frac{\mu_j}{j!},\ j = 0,1,...,n-2, \\ c_{n-1} &= -\frac{\Gamma(n-\delta)}{\Gamma(n)}\left(\mu_{n-1} + \int_0^1 \frac{(1-s)^{\alpha_n-\delta-1}}{\Gamma(\alpha_n-\delta)}Q(s)\,ds\right). \end{aligned} \tag{10}$$

Substituting (10) in (8), we have (7). This ends the proof of Lemma 8. ■

Now, let us introduce the Banach space

$S := \{x : x \in C([0,1],\mathbb{R}),\ D^{\alpha_k}x \in C([0,1],\mathbb{R}),\ k = 1,2,...,n-1\}$ endowed with the norm: $\|x\|_S = \max_{1\le k\le n-1}(\|x\|_\infty, \|D^{\alpha_k}x\|_\infty)$.

2. Existence and Uniqueness

In this section, we shall obtain sufficient conditions for the existence and uniqueness of solutions to (1).

Define the nonlinear operator $T : S \to S$ as follows:

$$\begin{aligned} Tx(t) &:= -\int_0^t \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)}\varphi(s)\,ds \\ &+ \sum_{j=0}^{n-2}\frac{\mu_j}{j!}t^j + \frac{\Gamma(n-\delta)}{(n-1)!}\left(\mu_{n-1} + \int_0^1 \frac{(1-s)^{\alpha_n-\delta-1}}{\Gamma(\alpha_n-\delta)}\varphi(s)\,ds\right)t^{n-1},\ \forall t \in [0,1], \end{aligned} \tag{11}$$

where

$$\varphi(s) = f(s, x(s), D^{\alpha_1}x(s), D^{\alpha_2}x(s), ..., D^{\alpha_{n-1}}x(s)). \tag{12}$$

Lemma 9. *Let $n-1 < \alpha_n < n$, $0 < \rho < 1$, $\varphi :]0,1] \to \mathbb{R}$ is continuous, and $\lim_{t\to 0^+}\varphi(t) = \infty$. Assume that $t^{\rho}\varphi(t)$ is continuous on $[0,1]$. Then*

$$\begin{aligned} x(t) &= -\int_0^t \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)}\varphi(s)\,ds \\ &+ \sum_{j=0}^{n-2}\frac{\mu_j}{j!}t^j + \frac{\Gamma(n-\delta)}{(n-1)!}\left(\mu_{n-1} + \int_0^1 \frac{(1-s)^{\alpha_n-\delta-1}}{\Gamma(\alpha_n-\delta)}\varphi(s)\,ds\right)t^{n-1}, \end{aligned} \tag{13}$$

is continuous on $[0,1]$.

Proof. By the continuity of $t^{\rho}\varphi(t)$ and

$$x(t) = -\int_0^t \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)} s^{-\rho} s^{\rho}\varphi(s)\,ds$$

$$+\sum_{j=0}^{n-2} \frac{\mu_j}{j!}\,t^j + \frac{\Gamma(n-\delta)}{(n-1)!}\left(\mu_{n-1} + \int_0^1 \frac{(1-s)^{\alpha_n-\delta-1}}{\Gamma(\alpha_n-\delta)} s^{-\rho} s^{\rho}\varphi(s)\,ds\right) t^{n-1},$$

it is clear that $x(0) = \mu_0$. Now, we proceed the proof into three cases. Case 1: For $t_0 = 0$ and $\forall t \in (0,1]$, by the continuity of $t^{\rho}\varphi(t), \exists L > 0 : |t^{\rho}\varphi(t)| \le L, \forall t \in [0,1]$. Then

$$|x(t) - x(0)|$$

$$= \left| \begin{array}{c} -\int_0^t \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)} s^{-\rho} s^{\rho}\varphi(s)\,ds + \sum_{j=0}^{n-2} \frac{\mu_j}{j!}\,t^j \\ \\ +\frac{\Gamma(n-\delta)}{(n-1)!}\left(\mu_{n-1} + \int_0^1 \frac{(1-s)^{\alpha_n-\delta-1}}{\Gamma(\alpha_n-\delta)} s^{-\rho} s^{\rho}\varphi(s)\,ds\right) t^{n-1} - \mu_0 \end{array} \right|$$

$$\le \frac{L}{\Gamma(\alpha_n)} \int_0^t (t-s)^{\alpha_n-1} s^{-\rho} ds + \sum_{j=1}^{n-2} \frac{|\mu_j|}{j!} t^j + \frac{\Gamma(n-\delta)\,t^{n-1}}{(n-1)!}$$

$$\times \left(|\mu_{n-1}| + \frac{L}{\Gamma(\alpha_n-\delta)}\right) \int_0^1 (1-s)^{\alpha_n-\delta-1} s^{-\rho} ds. \tag{14}$$

By using Beta function relation, we get

$$|x(t) - x(0)|$$

$$\le \frac{Lt^{\alpha_n-\rho}}{\Gamma(\alpha_n)} \int_0^1 (1-u)^{\alpha_n-1} u^{-\rho} du + \sum_{j=1}^{n-2} \frac{|\mu_j|}{j!} t^j$$
$$+\frac{\Gamma(n-\delta)}{(n-1)!}\left(|\mu_{n-1}| + \frac{L\beta(\alpha_n-\delta, 1-\rho)}{\Gamma(\alpha_n-\delta)}\right) t^{n-1}$$

$$\le \frac{L\beta(\alpha_n, 1-\rho)\,t^{\alpha_n-\rho}}{\Gamma(\alpha_n)} + \sum_{j=1}^{n-2} \frac{|\mu_j|}{j!} t^j$$
$$+\frac{\Gamma(n-\delta)}{(n-1)!}\left(|\mu_{n-1}| + \frac{L\beta(\alpha_n-\delta, 1-\rho)}{\Gamma(\alpha_n-\delta)}\right) t^{n-1}$$

$$\to 0, \quad \text{as } t \to 0. \tag{15}$$

Case 2: For $t_0 \in (0,1)$ and $\forall t \in (t_0, 1]$, we have

$$|x(t) - x(t_0)|$$

(16)
$$\leq \left| -\int_0^t \frac{(t-s)^{\alpha_n - 1}}{\Gamma(\alpha_n)} s^{-\rho} s^{\rho} \varphi(s)\, ds + \int_0^{t_0} \frac{(t_0-s)^{\alpha_n - 1}}{\Gamma(\alpha_n)} s^{-\rho} s^{\rho} \varphi(s)\, ds \right|$$
$$+ \sum_{j=0}^{n-2} \frac{|\mu_j|}{j!} \left(t^j - t_0^j\right) + \frac{\Gamma(n-\delta)}{(n-1)!} \left| \mu_{n-1} + \int_0^1 \frac{(1-s)^{\alpha_n - \delta - 1}}{\Gamma(\alpha_n - \delta)} s^{-\rho} s^{\rho} \varphi(s)\, ds \right| \left(t^{n-1} - t_0^{n-1}\right)$$

$$\leq \frac{L}{\Gamma(\alpha_n)} \left| \int_0^t (t-s)^{\alpha_n - 1} s^{-\rho} ds - \int_0^{t_0} (t_0 - s)^{\alpha_n - 1} s^{-\rho} ds \right|$$
$$+ \sum_{j=0}^{n-2} \frac{|\mu_j|}{j!} \left(t^j - t_0^j\right) + \frac{\Gamma(n-\delta)}{(n-1)!} \left(|\mu_{n-1}| + \frac{L\beta(\alpha_n - \delta, 1-\rho)}{\Gamma(\alpha_n - \delta)} \right) \left(t^{n-1} - t_0^{n-1}\right).$$

Hence,

$$|x(t) - x(t_0)|$$

$$\leq \frac{L\left(t^{\alpha_n - \rho} - t_0^{\alpha_n - \rho}\right)\beta(\alpha_n, 1-\rho)}{\Gamma(\alpha_n)} + \sum_{j=0}^{n-2} \frac{|\mu_j|}{j!} \left(t^j - t_0^j\right)$$
$$+ \frac{\Gamma(n-\delta)}{(n-1)!} \left(|\mu_{n-1}| + \frac{L\beta(\alpha_n - \delta, 1-\rho)}{\Gamma(\alpha_n - \delta)} \right) \left(t^{n-1} - t_0^{n-1}\right)$$

(17)
$$\to 0, \quad \text{as } t \to t_0.$$

Case 3: For t_0 $(0,1]$ and $\forall t \in [0, t_0]$: The proof is similar to case 2, here we just leave it out. This ends the proof. ■

Lemma 10. *Let $n-1 < \alpha_n < n$, $n \in \mathbb{N}^* - \{1\}$, $0 < \rho < 1$, $f :]0,1] \times \mathbb{R}^n \to \mathbb{R}$ is continuous, and $\lim_{t \to 0^+} f(t, ...) = \infty$. Suppose that $t^{\rho} f(t, ...)$ is continuous on $[0,1] \times \mathbb{R}^n$. Then,*

(18)
$$D^{\alpha_k} Tx(t) = -\int_0^t \frac{(t-s)^{\alpha_n - \alpha_k - 1}}{\Gamma(\alpha_n - \alpha_k)} f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right) ds$$
$$+ \sum_{j=k}^{n-2} \frac{\mu_j}{\Gamma(j+1-\alpha_k)} t^{j-\alpha_k} + \frac{\Gamma(n-\delta)}{\Gamma(n-\alpha_k)}$$
$$\times \left(\mu_{n-1} + \int_0^1 \frac{(1-s)^{\alpha_n - \delta - 1}}{\Gamma(\alpha_n - \delta)} f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right) ds \right) t^{n-\alpha_k - 1},$$

where $k = 1, 2, ..., n-2$, and

(19)
$$D^{\alpha_{n-1}} Tx(t) = -\int_0^t \frac{(t-s)^{\alpha_n - \alpha_{n-1} - 1}}{\Gamma(\alpha_n - \alpha_{n-1})} f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right) ds$$
$$+ \frac{\Gamma(n-\delta)}{\Gamma(n-\alpha_{n-1})}$$
$$\times \left(\mu_{n-1} + \int_0^1 \frac{(1-s)^{\alpha_n - \delta - 1}}{\Gamma(\alpha_n - \delta)} f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right) ds \right) t^{n-\alpha_{n-1} - 1},$$

are continuous on $[0,1]$.

Proof. Let $x \in S$, then $x(t) \in C([0,1])$, $D^{\alpha_k}x(t) \in C([0,1])$, $k = 1,2,...,n-1$. Therefore, there exist n positives constants G_k, such that $\forall t \in [0,1]$, $|x(t)| \leq G_0$ and $|D^{\alpha_k}x(t)| \leq G_k$, $k = 1,2,...,n-1$. Also since $t^{\rho}f(t,...)$ is continuous on $[0,1] \times \mathbb{R}^n$, there exists $M > 0 : M = \|t^{\rho}f(t,x,D^{\alpha_1}x,...,D^{\alpha_{n-1}}x)\|_{\infty}$, for $-G_0 \leq x \leq G_0$ and $-G_k \leq D^{\alpha_k}x \leq G_k$. Then,

$$|D^{\alpha_k}Tx(t)|$$

$$= \left| \begin{array}{c} -\int_0^t \frac{(t-s)^{\alpha_n-\alpha_k-1}}{\Gamma(\alpha_n-\alpha_k)} s^{-\rho}s^{\rho}f(s,x(s),D^{\alpha_1}x(s),...,D^{\alpha_{n-1}}x(s))\,ds \\ +\sum_{j=k}^{n-2} \frac{\mu_j}{\Gamma(j+1-\alpha_k)} t^{j-\alpha_k} + \frac{\Gamma(n-\delta)}{\Gamma(n-\alpha_k)} t^{n-\alpha_k-1} \\ \times \left(\mu_{n-1} + \int_0^1 \frac{(1-s)^{\alpha_n-\delta-1}}{\Gamma(\alpha_n-\delta)} s^{-\rho}s^{\rho}f(s,x(s),D^{\alpha_1}x(s),...,D^{\alpha_{n-1}}x(s))\,ds\right) \end{array} \right|$$

$$\leq \frac{M}{\Gamma(\alpha_n-\alpha_k)} \int_0^t (t-s)^{\alpha_n-\alpha_k-1} s^{-\rho} ds + \sum_{j=k}^{n-2} \frac{|\mu_j|}{\Gamma(j+1-\alpha_k)} t^{j-\alpha_k}$$

$$+\frac{\Gamma(n-\delta)}{\Gamma(n-\alpha_k)} \left(|\mu_{n-1}| + \frac{M}{\Gamma(\alpha_n-\delta)} \int_0^1 (1-s)^{\alpha_n-\delta-1} s^{-\rho} ds\right) t^{n-\alpha_k-1}. \tag{20}$$

Hence,

$$|D^{\alpha_k}Tx(t)| \leq$$

$$\frac{M\beta(\alpha_n-\alpha_k,1-\rho)}{\Gamma(\alpha_n-\alpha_k)} t^{\alpha_n-\alpha_k-\rho} + \sum_{j=k}^{n-2} \frac{|\mu_j|}{\Gamma(j+1-\alpha_k)} t^{j-\alpha_k}$$

$$+\frac{\Gamma(n-\delta)}{\Gamma(n-\alpha_k)} \left(|\mu_{n-1}| + \frac{M\beta(\alpha_n-\delta,1-\rho)}{\Gamma(\alpha_n-\delta)}\right) t^{n-\alpha_k-1}, \tag{21}$$

where $k = 1,2,...,n-2$.

With the same arguments for $k = n-1$, we have

$$|D^{\alpha_{n-1}}Tx(t)| \leq \frac{M\beta(\alpha_n-\alpha_{n-1},1-\rho)}{\Gamma(\alpha_n-\alpha_{n-1})} t^{\alpha_n-\alpha_{n-1}-\rho}$$

$$+\frac{\Gamma(n-\delta)}{\Gamma(n-\alpha_{n-1})} \left(|\mu_{n-1}| + \frac{M\beta(\alpha_n-\delta,1-\rho)}{\Gamma(\alpha_n-\delta)}\right) t^{n-\alpha_{n-1}-1}. \tag{22}$$

From (21) and (22), we can see that $t^{\alpha_n-\alpha_k-\rho}$, $t^{j-\alpha_k}$, $t^{n-\alpha_k-1}$, $t^{\alpha_n-\alpha_{n-1}-\rho}$ and $t^{n-\alpha_{n-1}-1}$ are continuous on $[0,1]$. Using the same method as in Lemma 9, we can show that $D^{\alpha_k}Tx$, $k = 1,2,...,n-1$, are continuous on $[0,1]$. ∎

Lemma 11. *Let $n-1<\alpha_n<n$, $n\in\mathbb{N}^*-\{1\}$, $0<\rho<1$, $f:]0,1]\times\mathbb{R}^n\to\mathbb{R}$ is continuous, and $\lim\limits_{t\to0^+}f(t,...)=\infty$. Suppose that $t^\rho f(t,...)$ is continuous on $[0,1]\times\mathbb{R}^n$. Then the operator $T:S\to S$ is completely continuous.*

Proof. We will prove the Lemma through three steps:
For $x\in S$, we have

$$Tx(t):=-\int_0^t\frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)}f\left(s,x(s),D^{\alpha_1}x(s),D^{\alpha_2}x(s),...,D^{\alpha_{n-1}}x(s)\right)ds$$
$$+\sum_{j=0}^{n-2}\frac{\mu_j}{j!}t^j+\frac{\Gamma(n-\delta)}{(n-1)!}t^{n-1}$$
$$\times\left(\mu_{n-1}+\int_0^1\frac{(1-s)^{\alpha_n-\delta-1}}{\Gamma(\alpha_n-\delta)}f\left(s,x(s),D^{\alpha_1}x(s),D^{\alpha_2}x(s),...,D^{\alpha_{n-1}}x(s)\right)ds\right).$$

Then by Lemma 9 and Lemma 10, we have $T:S\to S$.

Step 1: We will show that the operator $T:S\to S$ is continuous.

Let $x_0\in S$ with $\|x_0\|_S=a_0$, and let $x\in S$; $\|x-x_0\|_S<1$, then, $\|x\|_S<1+a_0=a$. We have $t^\rho f(t,x,D^{\alpha_1}x,...,D^{\alpha_{n-1}}x)$ is uniformly continuous on $[0,1]\times[-a,a]^n$, in view of the continuity of $t^\rho f(t,x,D^{\alpha_1}x,...,D^{\alpha_{n-1}}x)$. So, $\forall t\in[0,1]$, $\forall\epsilon>0,\exists\upsilon>0$ $(\upsilon<1)$:

$$\left|\begin{array}{c}t^\rho f\left(t,x(t),D^{\alpha_1}x(t),...,D^{\alpha_{n-1}}x(t)\right)\\-t^\rho f\left(t,x_0(t),D^{\alpha_1}x_0(t),...,D^{\alpha_{n-1}}x_0(t)\right)\end{array}\right|<\epsilon, \tag{23}$$

such that $x\in S$, and $\|x-x_0\|_S<\upsilon$. Then,

$$|Tx(t)-Tx_0(t)|\leq$$

$$\int_0^t\frac{(t-s)^{\alpha_n-1}s^{-\rho}}{\Gamma(\alpha_n)}\left|\begin{array}{c}s^\rho f\left(s,x(s),D^{\alpha_1}x(s),...,D^{\alpha_{n-1}}x(s)\right)\\-s^\rho f\left(s,x_0(s),D^{\alpha_1}x_0(s),...,D^{\alpha_{n-1}}x_0(s)\right)\end{array}\right|ds$$

$$+\frac{\Gamma(n-\delta)t^{n-1}}{(n-1)!} \tag{24}$$

$$\times\left(\int_0^1\frac{(1-s)^{\alpha_n-\delta-1}s^{-\rho}}{\Gamma(\alpha_n-\delta)}\left|\begin{array}{c}s^\rho f\left(s,x(s),D^{\alpha_1}x(s),...,D^{\alpha_{n-1}}x(s)\right)\\-s^\rho f\left(s,x_0(s),D^{\alpha_1}x_0(s),...,D^{\alpha_{n-1}}x_0(s)\right)\end{array}\right|ds\right).$$

Thanks to (23), we get

$$|Tx(t)-Tx_0(t)|$$

$$\begin{aligned}
&\leq \frac{\epsilon}{\Gamma(\alpha_n)}\int_0^t (t-s)^{\alpha_n-1}s^{-\rho}ds + \frac{\epsilon\Gamma(n-\delta)t^{n-1}}{(n-1)!\Gamma(\alpha_n-\delta)}\int_0^1 (1-s)^{\alpha_n-\delta-1}s^{-\rho}ds \\
&\leq \epsilon\left(\frac{\beta(\alpha_n,1-\rho)t^{\alpha_n-\rho}}{\Gamma(\alpha_n)} + \frac{\Gamma(n-\delta)\beta(\alpha_n-\delta,1-\rho)t^{n-1}}{(n-1)!\Gamma(\alpha_n-\delta)}\right) \\
(25)&\leq \epsilon\left(\frac{\beta(\alpha_n,1-\rho)}{\Gamma(\alpha_n)} + \frac{\Gamma(n-\delta)\beta(\alpha_n-\delta,1-\rho)}{(n-1)!\Gamma(\alpha_n-\delta)}\right).
\end{aligned}$$

We pose:

$$F_0 := \frac{\beta(\alpha_n,1-\rho)}{\Gamma(\alpha_n)} + \frac{\Gamma(n-\delta)\beta(\alpha_n-\delta,1-\rho)}{(n-1)!\Gamma(\alpha_n-\delta)}.$$

Therefore,

$$\|Tx - Tx_0\|_\infty \leq \epsilon F_0. \tag{26}$$

On the other hand for all $k = 1, 2, ..., n-1$, we have

$$|D^{\alpha_k}Tx(t) - D^{\alpha_k}Tx_0(t)| \leq$$

$$\int_0^t \frac{(t-s)^{\alpha_n-\alpha_k-1}s^{-\rho}}{\Gamma(\alpha_n-\alpha_k)} \left| \begin{array}{c} s^\rho f\left(s, x(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}x(s)\right) \\ -s^\rho f\left(s, x_0(s), D^{\alpha_1}x_0(s), ..., D^{\alpha_{n-1}}x_0(s)\right) \end{array} \right| ds$$

$$+\frac{\Gamma(n-\delta)}{\Gamma(n-\alpha_k)}t^{n-\alpha_k-1} \tag{27}$$

$$\times\left(\int_0^1 \frac{(1-s)^{\alpha_n-\delta-1}s^{-\rho}}{\Gamma(\alpha_n-\delta)} \left| \begin{array}{c} s^\rho f\left(s, x(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}x(s)\right) \\ -s^\rho f\left(s, x_0(s), D^{\alpha_1}x_0(s), ..., D^{\alpha_{n-1}}x_0(s)\right) \end{array} \right| ds\right).$$

By (23), we obtain

$$\begin{aligned}
&|D^{\alpha_k}Tx(t) - D^{\alpha_k}Tx_0(t)| \\
&\leq \epsilon\int_0^t \frac{(t-s)^{\alpha_n-\alpha_k-1}s^{-\rho}}{\Gamma(\alpha_n-\alpha_k)}ds + \frac{\epsilon\Gamma(n-\delta)t^{n-\alpha_k-1}}{\Gamma(n-\alpha_k)\Gamma(\alpha_n-\delta)} \\
&\quad \times\int_0^1 (1-s)^{\alpha_n-\delta-1}s^{-\rho}ds \\
&\leq \epsilon\left(\begin{array}{c} \frac{\beta(\alpha_n-\alpha_k,1-\rho)t^{\alpha_n-\alpha_k-\rho}}{\Gamma(\alpha_n-\alpha_k)} \\ \\ +\frac{\Gamma(n-\delta)\beta(\alpha_n-\delta,1-\rho)t^{n-\alpha_k-1}}{\Gamma(n-\alpha_k)\Gamma(\alpha_n-\delta)} \end{array}\right) \\
&\leq \epsilon\left(\frac{\beta(\alpha_n-\alpha_k,1-\rho)}{\Gamma(\alpha_n-\alpha_k)} + \frac{\Gamma(n-\delta)\beta(\alpha_n-\delta,1-\rho)}{\Gamma(n-\alpha_k)\Gamma(\alpha_n-\delta)}\right). \qquad (28)
\end{aligned}$$

Also we pose:

$$\digamma_k := \frac{\beta(\alpha_n - \alpha_k, 1-\rho)}{\Gamma(\alpha_n - \alpha_k)} + \frac{\Gamma(n-\delta)\beta(\alpha_n - \delta, 1-\rho)}{\Gamma(n-\alpha_k)\Gamma(\alpha_n - \delta)},$$

where $k = 1, 2, ..., n-1$. Then, we obtain

$$\left\|D^{\alpha_k}Tx - D^{\alpha_k}Tx_0\right\|_\infty \leq \epsilon \digamma_k,\ k = 1, 2, ..., n-1. \tag{29}$$

From (26) and (29), we get

$$\left\|Tx - Tx_0\right\|_S \leq \epsilon \max_{1\leq k\leq n-1}(\digamma_0, \digamma_k). \tag{30}$$

Thus, $\left\|Tx - Tx_0\right\|_S \to 0$ as $\left\|x - x_0\right\|_S \to 0$. That is $T : S \to S$ a continuous operator. Step 2: We will prove that T maps a bounded set into a bounded set. Let $B \subset S$ be bounded, then $\exists \sigma > 0 : \left\|x\right\|_S \leq \sigma, \forall x \in B$. Since $t^\rho f(t, x(t), D^{\alpha_1}x(t), ..., D^{\alpha_{n-1}}x(t))$ is continuous on $[0,1] \times [-\sigma, \sigma]^n$, there exists a positive constant C :

$$|t^\rho f(t, x(t), D^{\alpha_1}x(t), ..., D^{\alpha_{n-1}}x(t))| \leq C,\ \forall t \in [0,1],\ \forall x \in B. \tag{31}$$

Therefore, we get

$$|Tx(t)| \leq$$

$$\begin{array}{c} \int_0^t \frac{(t-s)^{\alpha_n - 1}}{\Gamma(\alpha_n)} s^{-\rho} \left|s^\rho f(s, x(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}x(s))\right| ds \\ + \sum_{j=0}^{n-2} \frac{|\mu_j|}{j!} t^j + \frac{\Gamma(n-\delta)}{(n-1)!} t^{n-1} \\ \times \left(|\mu_{n-1}| + \int_0^1 \frac{(1-s)^{\alpha_n - \delta - 1}}{\Gamma(\alpha_n - \delta)} s^{-\rho} \left|s^\rho f(s, x(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}x(s))\right| ds\right). \end{array}$$

Using (31), we get

$$\left\|Tx\right\|_\infty$$

$$\begin{aligned} &\leq \frac{C}{\Gamma(\alpha_n)} \sup_{t\in J} \int_0^t (t-s)^{\alpha_n - 1} s^{-\rho} ds + \sum_{j=0}^{n-2} \frac{|\mu_j|}{j!} \\ &\quad + \frac{\Gamma(n-\delta)}{(n-1)!} \left(|\mu_{n-1}| + \frac{C}{\Gamma(\alpha_n - \delta)} \int_0^1 (1-s)^{\alpha_n - \delta - 1} s^{-\rho} ds\right) \\ &\leq C\left(\frac{\beta(\alpha_n, 1-\rho) t^{\alpha_n - \rho}}{\Gamma(\alpha_n)} + \frac{\Gamma(n-\delta)\beta(\alpha_n - \delta, 1-\rho)}{(n-1)!\Gamma(\alpha_n - \delta)}\right) \\ &\quad + \sum_{j=0}^{n-2} \frac{|\mu_j|}{j!} + \frac{\Gamma(n-\delta)|\mu_{n-1}|}{(n-1)!} \\ &\leq C\digamma_0 + \sum_{j=0}^{n-2} \frac{|\mu_j|}{j!} + \frac{\Gamma(n-\delta)|\mu_{n-1}|}{(n-1)!}. \end{aligned} \tag{32}$$

On the other hand, we get

$$|D^{\alpha_k}Tx(t)|$$

$$\begin{aligned}
&\leq \int_0^t \frac{(t-s)^{\alpha_n-\alpha_k-1}}{\Gamma(\alpha_n-\alpha_k)} s^{-\rho} \left|s^{\rho} f\left(s, x\left(s\right), D^{\alpha_1}x\left(s\right), ..., D^{\alpha_{n-1}}x\left(s\right)\right)\right| ds \\
&\qquad + \sum_{j=k}^{n-2} \frac{|\mu_j|}{\Gamma(j+1-\alpha_k)} t^{j-\alpha_k} + \frac{\Gamma(n-\delta)}{\Gamma(n-\alpha_k)} t^{n-\alpha_k-1} \\
&\times \left(|\mu_{n-1}| + \int_0^1 \frac{(1-s)^{\alpha_n-\delta-1}}{\Gamma(\alpha_n-\delta)} s^{-\rho} \left|s^{\rho} f\left(s, x\left(s\right), D^{\alpha_1}x\left(s\right), ..., D^{\alpha_{n-1}}x\left(s\right)\right)\right| ds\right).
\end{aligned} \tag{33}$$

Using (31) , we have

$$\|D^{\alpha_k}Tx\|_{\infty}$$

$$\begin{aligned}
&\leq \frac{C}{\Gamma\left(\alpha_n-\alpha_k\right)} \int_0^t \left(t-s\right)^{\alpha_n-\alpha_k-1} s^{-\rho} ds + \sum_{j=k}^{n-2} \frac{|\mu_j|}{\Gamma\left(j+1-\alpha_k\right)} \\
&\quad + \frac{\Gamma\left(n-\delta\right)}{\Gamma\left(n-\alpha_k\right)} \left(|\mu_{n-1}| + \frac{C}{\Gamma\left(\alpha_n-\delta\right)} \int_0^1 \left(1-s\right)^{\alpha_n-\delta-1} s^{-\rho} ds\right) \\
&\leq C\left(\frac{\beta\left(\alpha_n-\alpha_k, 1-\rho\right) t^{\alpha_n-\alpha_k-\rho}}{\Gamma\left(\alpha_n-\alpha_k\right)} + \frac{\Gamma\left(n-\delta\right)\beta\left(\alpha_n-\delta, 1-\rho\right)}{\Gamma\left(n-\alpha_k\right)\Gamma\left(\alpha_n-\delta\right)}\right) \\
&\quad + \sum_{j=k}^{n-2} \frac{|\mu_j|}{\Gamma\left(j+1-\alpha_k\right)} + \frac{\Gamma\left(n-\delta\right)|\mu_{n-1}|}{\Gamma\left(n-\alpha_k\right)} \\
&\leq CF_k + \sum_{j=k}^{n-2} \frac{|\mu_j|}{\Gamma\left(j+1-\alpha_k\right)} + \frac{\Gamma\left(n-\delta\right)|\mu_{n-1}|}{\Gamma\left(n-\alpha_k\right)},
\end{aligned} \tag{34}$$

where $k = 1, 2, ..., n-2$. With the same arguments, we get

$$\|D^{\alpha_{n-1}}Tx\|_{\infty} \leq CF_{n-1} + \frac{\Gamma\left(n-\delta\right)|\mu_{n-1}|}{\Gamma\left(n-\alpha_{n-1}\right)}. \tag{35}$$

Combining (32) , (34) and (35) , yields

$$\|Tx\|_S \leq \max_{1\leq k\leq n-2} \begin{pmatrix} CF_0 + \sum_{j=0}^{n-2} \frac{|\mu_j|}{j!} + \frac{\Gamma(n-\delta)|\mu_{n-1}|}{(n-1)!}, \\ CF_k + \sum_{j=k}^{n-2} \frac{|\mu_j|}{\Gamma(j+1-\alpha_k)} + \frac{\Gamma(n-\delta)|\mu_{n-1}|}{\Gamma(n-\alpha_k)}, \\ CF_{n-1} + \frac{\Gamma(n-\delta)|\mu_{n-1}|}{\Gamma(n-\alpha_{n-1})} \end{pmatrix}. \tag{36}$$

Thus, $T\left(B\right)$ is bounded.

For each $x \in B$, and $\forall t_1, t_2 \in [0,1]$ satisfy $t_1 < t_2$, we have:

$$|Tx(t_2) - Tx(t_1)| \leq$$

$$\left| \begin{array}{l} -\int_0^{t_2} \frac{(t_2-s)^{\alpha_n-1} s^{-\rho}}{\Gamma(\alpha_n)} s^{\rho} f\left(s, x(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}x(s)\right) ds \\ \\ +\int_0^{t_1} \frac{(t_1-s)^{\alpha_n-1} s^{-\rho}}{\Gamma(\alpha_n)} s^{\rho} f\left(s, x(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}x(s)\right) ds \end{array} \right|$$

$$+\sum_{j=0}^{n-2} \frac{|\mu_j|\left(t_2^j - t_1^j\right)}{j!} + \frac{\Gamma(n-\delta)\left(t_2^{n-1} - t_1^{n-1}\right)}{(n-1)!}$$

$$\times \left(|\mu_{n-1}| + \int_0^1 \frac{(1-s)^{\alpha_n-\delta-1} s^{-\rho}}{\Gamma(\alpha_n - \delta)} \left| s^{\rho} f\left(s, x(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}x(s)\right)\right| ds \right).$$

Hence,

$$\|Tx(t_2) - Tx(t_1)\|_{\infty}$$

$$\begin{aligned} &\leq \frac{C\beta(\alpha_n, 1-\rho)\left(t_2^{\alpha_n-\rho} - t_1^{\alpha_n-\rho}\right)}{\Gamma(\alpha_n)} + \sum_{j=0}^{n-2} \frac{|\mu_j|\left(t_2^j - t_1^j\right)}{j!} \\ &\quad + \frac{\Gamma(n-\delta)\left(t_2^{n-1} - t_1^{n-1}\right)}{(n-1)!} \left(|\mu_{n-1}| + \frac{C\beta(\alpha_n - \delta, 1-\rho)}{\Gamma(\alpha_n - \delta)} \right). \end{aligned} \tag{37}$$

And for each $k = 1, 2, ..., n-2$, we get

$$|D^{\alpha_k} Tx(t_2) - D^{\alpha_k} Tx(t_1)|$$

$$\leq \left| \begin{array}{l} -\int_0^{t_2} \frac{(t_2-s)^{\alpha_n-\alpha_k-1} s^{-\rho}}{\Gamma(\alpha_n-\alpha_k)} s^{\rho} f\left(s, x(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}x(s)\right) ds \\ \\ +\int_0^{t_1} \frac{(t_1-s)^{\alpha_n-\alpha_k-1} s^{-\rho}}{\Gamma(\alpha_n-\alpha_k)} s^{\rho} f\left(s, x(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}x(s)\right) ds \end{array} \right|$$

$$+\sum_{j=k}^{n-2} \frac{|\mu_j|\left(t_2^{j-\alpha_k} - t_1^{j-\alpha_k}\right)}{\Gamma(j+1-\alpha_k)} + \frac{\Gamma(n-\delta)\left(t_2^{n-\alpha_k-1} - t_1^{n-\alpha_k-1}\right)}{\Gamma(n-\alpha_k)}$$

$$\times \left(|\mu_{n-1}| + \int_0^1 \frac{(1-s)^{\alpha_n-\delta-1} s^{-\rho}}{\Gamma(\alpha_n - \delta)} \left| s^{\rho} f\left(s, x(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}x(s)\right)\right| ds \right).$$

Therefore,

$$\|D^{\alpha_k} Tx(t_2) - D^{\alpha_k} Tx(t_1)\|_{\infty}$$

$$\leq \frac{C\beta\left(\alpha_n-\alpha_k,1-\rho\right)\left(t_2^{\alpha_n-\alpha_k-\rho}-t_1^{\alpha_n-\alpha_k-\rho}\right)}{\Gamma\left(\alpha_n-\alpha_k\right)}+\sum_{j=k}^{n-2}\frac{|\mu_j|\left(t_2^{j-\alpha_k}-t_1^{j-\alpha_k}\right)}{\Gamma\left(j+1-\alpha_k\right)}$$

$$+\frac{\Gamma\left(n-\delta\right)\left(t_2^{n-\alpha_k-1}-t_1^{n-\alpha_k-1}\right)}{\Gamma\left(n-\alpha_k\right)}\left(|\mu_{n-1}|+\frac{C\beta\left(\alpha_n-\delta,1-\rho\right)}{\Gamma\left(\alpha_n-\delta\right)}\right), \tag{38}$$

and

$$\left\|D^{\alpha_{n-1}}Tx\left(t_2\right)-D^{\alpha_{n-1}}Tx\left(t_1\right)\right\|_\infty$$

$$\begin{aligned}\leq\ & \frac{C\beta\left(\alpha_n-\alpha_{n-1},1-\rho\right)\left(t_2^{\alpha_n-\alpha_{n-1}-\rho}-t_1^{\alpha_n-\alpha_{n-1}-\rho}\right)}{\Gamma\left(\alpha_n-\alpha_{n-1}\right)}\\ &+\frac{\Gamma\left(n-\delta\right)\left(t_2^{n-\alpha_{n-1}-1}-t_1^{n-\alpha_{n-1}-1}\right)}{\Gamma\left(n-\alpha_{n-1}\right)}\\ &\times\left(|\mu_{n-1}|+\frac{C\beta\left(\alpha_n-\delta,1-\rho\right)}{\Gamma\left(\alpha_n-\delta\right)}\right).\end{aligned} \tag{39}$$

From (37), (38) and (39), we conclude that

$$\left\|Tx\left(t_2\right)-Tx\left(t_1\right)\right\|_S \leq$$

(40)

$$\max_{1\leq k\leq n-2}\left(\begin{array}{c}\frac{C\beta(\alpha_n,1-\rho)\left(t_2^{\alpha_n-\rho}-t_1^{\alpha_n-\rho}\right)}{\Gamma(\alpha_n)}+\sum\limits_{j=0}^{n-2}\frac{|\mu_j|\left(t_2^j-t_1^j\right)}{j!}\\ +\frac{\Gamma(n-\delta)}{(n-1)!}\left(|\mu_{n-1}|+\frac{C\beta(\alpha_n-\delta,1-\rho)}{\Gamma(\alpha_n-\delta)}\right)\left(t_2^{n-1}-t_1^{n-1}\right),\\ \\ \frac{C\beta(\alpha_n-\alpha_k,1-\rho)\left(t_2^{\alpha_n-\alpha_k-\rho}-t_1^{\alpha_n-\alpha_k-\rho}\right)}{\Gamma(\alpha_n-\alpha_k)}+\sum\limits_{j=k}^{n-2}\frac{|\mu_j|\left(t_2^{j-\alpha_k}-t_1^{j-\alpha_k}\right)}{\Gamma(j+1-\alpha_k)}\\ +\frac{\Gamma(n-\delta)}{\Gamma(n-\alpha_k)}\left(|\mu_{n-1}|+\frac{C\beta(\alpha_n-\delta,1-\rho)}{\Gamma(\alpha_n-\delta)}\right)\left(t_2^{n-\alpha_k-1}-t_1^{n-\alpha_k-1}\right),\\ \\ \frac{C\beta(\alpha_n-\alpha_{n-1},1-\rho)\left(t_2^{\alpha_n-\alpha_{n-1}-\rho}-t_1^{\alpha_n-\alpha_{n-1}-\rho}\right)}{\Gamma(\alpha_n-\alpha_{n-1})}\\ +\frac{\Gamma(n-\delta)}{\Gamma(n-\alpha_k)}\left(|\mu_{n-1}|+\frac{C\beta(\alpha_n-\delta,1-\rho)}{\Gamma(\alpha_n-\delta)}\right)\left(t_2^{n-\alpha_{n-1}-1}-t_1^{n-\alpha_{n-1}-1}\right)\end{array}\right).$$

The right-hand side of (40) is independent of x and tend to zero as $t_1 \to t_2$, so $T\left(B\right)$ is equicontinuous. Thus, by Arzela-Ascoli theorem, T is a completely continuous operator. ■

Theorem 12. *Assume that*

(H_1) : *There exist nonnegative constants* $(\omega_i)_{i=1,2,...,n}$, *such that*

$$t^{\rho}\left|f\left(t,u_1,u_2,...,u_n\right)-f\left(t,v_1,v_2,...,v_n\right)\right|\leq\sum_{i=1}^{n}\omega_i\left|u_i-v_i\right|,$$

for all $t\in[0,1]$ *and all* $(u_1,u_2,...,u_n)$, $(v_1,v_2,...,v_n)\in\mathbb{R}^n$.

$(H_2):\Lambda:=\max\limits_{1\leq k\leq n-1}\sum\limits_{i=1}^{n}\omega_i\left(\digamma_0,\digamma_k\right)<1.$

Then system (1) *has a unique solution.*

Proof. We will show that the operator T is contractive on S :

Let x, $y\in S$ and $t\in[0,1]$, we have

$$\left\|Tx-Ty\right\|_{\infty}$$

$$\leq\sup_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_n-1}s^{-\rho}}{\Gamma(\alpha_n)}s^{\rho}\left|\begin{array}{c}f\left(s,x\left(s\right),D^{\alpha_1}x\left(s\right),...,D^{\alpha_{n-1}}x\left(s\right)\right)\\-f\left(s,y\left(s\right),D^{\alpha_1}y\left(s\right),...,D^{\alpha_{n-1}}y\left(s\right)\right)\end{array}\right|ds$$

$$+\sup_{t\in[0,1]}\frac{\Gamma(n-\delta)}{(n-1)!}t^{n-1} \tag{41}$$

$$\times\left(\int_0^1\frac{(1-s)^{\alpha_n-\delta-1}s^{-\rho}}{\Gamma(\alpha_n-\delta)}s^{\rho}\left|\begin{array}{c}f\left(s,x\left(s\right),D^{\alpha_1}x\left(s\right),...,D^{\alpha_{n-1}}x\left(s\right)\right)\\-f\left(s,y\left(s\right),D^{\alpha_1}y\left(s\right),...,D^{\alpha_{n-1}}y\left(s\right)\right)\end{array}\right|ds\right).$$

Using (H_1) , we get

$$\left\|Tx-Ty\right\|_{\infty}$$

$$\begin{array}{c}\leq\left(\omega_1\left\|x-y\right\|_{\infty}+\omega_2\left\|D^{\alpha_1}\left(x-y\right)\right\|_{\infty}+...+\omega_n\left\|D^{\alpha_{n-1}}\left(x-y\right)\right\|_{\infty}\right)\\\times\left(\int_0^t\frac{(t-s)^{\alpha_n-1}s^{-\rho}}{\Gamma(\alpha_n)}ds+\frac{\Gamma(n-\delta)}{(n-1)!}\int_0^1\frac{(1-s)^{\alpha_n-\delta-1}s^{-\rho}}{\Gamma(\alpha_n-\delta)}ds\right)\end{array}$$

$$\begin{array}{c}\leq\sum\limits_{i=1}^{n}\omega_i\max\left(\left\|x-y\right\|_{\infty},\left\|D^{\alpha_1}\left(x-y\right)\right\|_{\infty},...,\left\|D^{\alpha_{n-1}}\left(x-y\right)\right\|_{\infty}\right)\\\times\left(\frac{\beta(\alpha_n,1-\rho)t^{\alpha_n-\rho}}{\Gamma(\alpha_n)}+\frac{\Gamma(n-\delta)\beta(\alpha_n-\delta,1-\rho)}{(n-1)!\Gamma(\alpha_n-\delta)}\right).\end{array} \tag{42}$$

Thus,

$$\left\|Tx-Ty\right\|_{\infty}\leq\digamma_0\sum_{i=1}^{n}\omega_i\left\|x-y\right\|_S. \tag{43}$$

On the other hand for all $k=1,2,...,n-1$, we have

$$\left\|D^{\alpha_k}Tx-D^{\alpha_k}Ty\right\|_{\infty}$$

$$\leq \sup_{t\in[0,1]} \int_0^t \frac{(t-s)^{\alpha_n-\alpha_k-1}s^{-\rho}}{\Gamma(\alpha_n-\alpha_k)} s^{\rho} \left| \begin{array}{c} f\left(s,x\left(s\right),D^{\alpha_1}x\left(s\right),...,D^{\alpha_{n-1}}x\left(s\right)\right) \\ -f\left(s,y\left(s\right),D^{\alpha_1}y\left(s\right),...,D^{\alpha_{n-1}}y\left(s\right)\right) \end{array} \right| ds$$

$$+ \sup_{t\in[0,1]} \frac{\Gamma(n-\delta)}{\Gamma(n-\alpha_k)} t^{n-\alpha_k-1} \tag{44}$$

$$\times \left(\int_0^1 \frac{(1-s)^{\alpha_n-\delta-1}s^{-\rho}}{\Gamma(\alpha_n-\delta)} s^{\rho} \left| \begin{array}{c} f\left(s,x\left(s\right),D^{\alpha_1}x\left(s\right),...,D^{\alpha_{n-1}}x\left(s\right)\right) \\ -f\left(s,y\left(s\right),D^{\alpha_1}y\left(s\right),...,D^{\alpha_{n-1}}y\left(s\right)\right) \end{array} \right| ds \right).$$

Thanks to (H_1), we can write

$$\left\|D^{\alpha_k}Tx - D^{\alpha_k}Ty\right\|_{\infty}$$

$$\begin{array}{l} \leq \left(\omega_1 \left\|x-y\right\|_{\infty} + \omega_2 \left\|D^{\alpha_1}\left(x-y\right)\right\|_{\infty} + ... + \omega_n \left\|D^{\alpha_{n-1}}\left(x-y\right)\right\|_{\infty}\right) \\ \quad \times \left(\int_0^t \frac{(t-s)^{\alpha_n-\alpha_k-1}s^{-\rho}}{\Gamma(\alpha_n-\alpha_k)} ds + \frac{\Gamma(n-\delta)}{\Gamma(n-\alpha_k)} \left(\int_0^1 \frac{(1-s)^{\alpha_n-\delta-1}s^{-\rho}}{\Gamma(\alpha_n-\delta)} ds \right) \right) \\ \leq \sum_{i=1}^{n} \omega_i \max\left(\left\|x-y\right\|_{\infty}, \left\|D^{\alpha_1}\left(x-y\right)\right\|_{\infty}, ..., \left\|D^{\alpha_{n-1}}\left(x-y\right)\right\|_{\infty}\right) \\ \quad \times \left(\frac{\beta(\alpha_n-\alpha_k,1-\rho)t^{\alpha_n-\alpha_k-\rho}}{\Gamma(\alpha_n-\alpha_k)} + \frac{\Gamma(n-\delta)\beta(\alpha_n-\delta,1-\rho)}{\Gamma(n-\alpha_k)\Gamma(\alpha_n-\delta)} \right). \end{array} \tag{45}$$

Hence,

$$\left\|D^{\alpha_k}Tx(t) - D^{\alpha_k}Ty(t)\right\|_{\infty} \leq \sum_{i=1}^{n} \omega_i F_k \left\|x-y\right\|_S, \tag{46}$$

where $k = 1, 2, ..., n-1$.

Using the inequalities (43) and (46), we get

$$\left\|Tx - Ty\right\|_S \leq \max_{1\leq k\leq n-1} \sum_{i=1}^{n} \omega_i \left(F_0, F_k\right) \left\|x-y\right\|_S.$$

By (H_2), we deduce that the operator T is contractive. Consequently, the problem (1) has a unique solution. This completes the proof. ■

Example 13. *Consider the following fractional problem:*

$$\begin{cases} D^{\frac{11}{3}}x\left(t\right) + \frac{\sin t\left|x(t)+D^{\frac{3}{4}}x(t)+D^{\frac{4}{3}}x(t)+D^{\frac{5}{2}}x(t)\right|}{8\pi\sqrt{t}\left(1+\left|x(t)+D^{\frac{3}{4}}x(t)+D^{\frac{4}{3}}x(t)+D^{\frac{5}{2}}x(t)\right|\right)} = 0, \ 0 < t \leq 1, \\ \\ x\left(0\right) = -2\sqrt{3}, \ x^{'}\left(0\right) = \sqrt{2}, \ x^{''}\left(0\right) = \sqrt{5}, \ D^{\frac{8}{3}}x\left(1\right) = 3\sqrt{2}. \end{cases} \tag{47}$$

We have:

$n=4$, $\alpha_4=\frac{11}{3}$, $\alpha_1=\frac{3}{4}$, $\alpha_2=\frac{4}{3}$, $\alpha_3=\frac{5}{2}$, $\delta=\frac{8}{3}$, $\mu_0=-2\sqrt{3}$, $\mu_1=\sqrt{2}$, $\mu_2=\sqrt{5}$, $\mu_3=3\sqrt{2}$.

For all $t\in[0,1]$ *and* (u_1,u_2,u_3,u_4), $(v_1,v_2,v_3,v_4)\in\mathbb{R}^4$, *we have:*

$$t^{\frac{3}{5}}\left|f\left(t,u_1,u_2,u_3,u_4\right)-f\left(t,v_1,v_2,v_3,v_4\right)\right|$$

$$\leq\frac{t^{\frac{1}{10}}\sin t}{8\pi}\left|u_1-v_1\right|+\frac{t^{\frac{1}{10}}\sin t}{8\pi}\left|u_2-v_2\right|+\frac{t^{\frac{1}{10}}\sin t}{8\pi}\left|u_3-v_3\right|+\frac{t^{\frac{1}{10}}\sin t}{8\pi}\left|u_4-v_4\right|,$$

where $\rho=\frac{3}{5}$.

Moreover, we get:

$$\omega_1=\omega_2=\omega_3=\omega_4=\frac{1}{8\pi},\ \sum_{i=1}^{4}\omega_i=\frac{1}{2\pi},$$

$$\digamma_0=0.7119,\ \digamma_1=2.9455,\ \digamma_2=2.5661,\ \digamma_3=3.6696.$$

In addition,

$$\sum_{i=1}^{4}\omega_i\digamma_0=0.1133,\ \sum_{i=1}^{4}\omega_i\digamma_1=0.4688,\ \sum_{i=1}^{4}\omega_i\digamma_2=0.4084,\ \sum_{i=1}^{4}\omega_i\digamma_3=0.5840.$$

Thus, $\Lambda<1$ *and* (47) *has a unique solution on* $[0,1]$.

Theorem 14. *Assume that* $n-1<\alpha_n<n$, $n\in\mathbb{N}^*-\{1\}$, $0<\rho<1$, $f:\left]0,1\right]\times\mathbb{R}^n\to\mathbb{R}$ *is continuous,* $\lim\limits_{t\to0^+}f\left(t,...\right)=\infty$, *and* $t^\rho f\left(t,...\right)$ *is continuous on* $[0,1]\times\mathbb{R}^n$. *Then the nonlinear fractional problem* (1) *has at least one solution on* $[0,1]$.

Proof. Let $C_0=\sup\limits_{t\in[0,1]}t^\rho\left|f\left(t,x(t),D^{\alpha_1}x(t),...,D^{\alpha_{n-1}}x(t)\right)\right|$,

$$r=\max_{1\leq k\leq n-2}\begin{pmatrix}C_0\digamma_0+\sum\limits_{j=0}^{n-2}\frac{|\mu_j|}{j!}+\frac{\Gamma(n-\delta)|\mu_{n-1}|}{(n-1)!},\\ C_0\digamma_k+\sum\limits_{j=k}^{n-2}\frac{|\mu_j|}{\Gamma(j+1-\alpha_k)}+\frac{\Gamma(n-\delta)|\mu_{n-1}|}{\Gamma(n-\alpha_k)},\\ C_0\digamma_{n-1}+\frac{\Gamma(n-\delta)|\mu_{n-1}|}{\Gamma(n-\alpha_{n-1})}\end{pmatrix},\tag{48}$$

and $\Sigma:=\{x\in S:\|x\|_S\leq r\}$. We show that $T:\Sigma\to\Sigma$. For $x\in\Sigma$ and $t\in[0,1]$, we have

$$\|Tx\|_\infty$$

$$\begin{aligned} &\leq \frac{C_0}{\Gamma(\alpha_n)} \sup_{t\in J} \int_0^t (t-s)^{\alpha_n-1} s^{-\rho} ds + \sum_{j=0}^{n-2} \frac{|\mu_j|}{j!} \sup_{t\in J} t^j \\ &\quad + \frac{\Gamma(n-\delta)}{(n-1)!} \sup_{t\in J} t^{n-1} \left(|\mu_{n-1}| + \frac{C_0}{\Gamma(\alpha_n-\delta)} \int_0^1 (1-s)^{\alpha_n-\delta-1} s^{-\rho} ds \right) \end{aligned}$$

$$\leq C_0 F_0 + \sum_{j=0}^{n-2} \frac{|\mu_j|}{j!} + \frac{\Gamma(n-\delta)\,|\mu_{n-1}|}{(n-1)!}. \tag{49}$$

And for all $k = 1, 2, ..., n-2$, we get

$$\|D^{\alpha_k} Tx\|_\infty$$

$$\begin{aligned} &\leq \frac{C_0}{\Gamma(\alpha_n-\alpha_k)} \sup_{t\in J} \int_0^t (t-s)^{\alpha_n-\alpha_k-1} s^{-\rho} ds + \sum_{j=k}^{n-2} \frac{|\mu_j|}{\Gamma(j+1-\alpha_k)} \sup_{t\in J} t^{j-\alpha_k} \\ &\quad + \frac{\Gamma(n-\delta)}{\Gamma(n-\alpha_k)} \sup_{t\in J} t^{n-1-\alpha_k} \left(|\mu_{n-1}| + \frac{C_0}{\Gamma(\alpha_n-\delta)} \int_0^1 (1-s)^{\alpha_n-\delta-1} s^{-\rho} ds \right) \end{aligned}$$

$$\leq C_0 F_k + \sum_{j=k}^{n-2} \frac{|\mu_j|}{\Gamma(j+1-\alpha_k)} + \frac{\Gamma(n-\delta)\,|\mu_{n-1}|}{\Gamma(n-\alpha_k)}, \tag{50}$$

and

$$\|D^{\alpha_{n-1}} Tx\|_\infty \leq C_0 F_{n-1} + \frac{\Gamma(n-\delta)\,|\mu_{n-1}|}{\Gamma(n-\alpha_{n-1})}. \tag{51}$$

Using (49), (50) and (51), yield

$$\|Tx\|_S \leq \max_{1\leq k\leq n-2} \begin{pmatrix} C_0 F_0 + \sum_{j=0}^{n-2} \frac{|\mu_j|}{j!} + \frac{\Gamma(n-\delta)|\mu_{n-1}|}{(n-1)!}, \\ C_0 F_k + \sum_{j=k}^{n-2} \frac{|\mu_j|}{\Gamma(j+1-\alpha_k)} + \frac{\Gamma(n-\delta)|\mu_{n-1}|}{\Gamma(n-\alpha_k)}, \\ C_0 F_{n-1} + \frac{\Gamma(n-\delta)|\mu_{n-1}|}{\Gamma(n-\alpha_{n-1})} \end{pmatrix}.$$

By (48), we have $\|Tx\|_S \leq r$. Also Lemmas 9-10 imply that $Tx(t)$, $D^{\alpha_k} Tx(t) \in C([0,1])$, $k = 1, 2, ..., n-1$. That is, we obtain $T : \Sigma \to \Sigma$. Furthermore from Lemma 11, T is completely continuous. So we have all the assumptions of Lemma 7 are satisfied, which implies that problem (1) has at last one solution on $[0,1]$. Theorem 14 is thus proved. ■

Example 15. *Consider the following nonlinear problem:*

$$\begin{cases} D^{\frac{9}{2}}x(t)+\frac{e^{-t}\cos x(t)D^{\frac{2}{3}}x(t)}{t^{\frac{1}{3}}\left(2\pi-\sin D^{\frac{5}{4}}x(t)\sin D^{\frac{8}{3}}x(t)+\cos D^{\frac{7}{2}}x(t)\right)}=0,\ 0<t\leq 1, \\ x(0)=\sqrt{2},\ x'(0)=3\sqrt{3},\ x''(0)=-1,\ x'''(0)=\frac{1}{4},\ D^{\frac{13}{4}}x(1)=1. \end{cases} \tag{52}$$

We have: $n=5$, $\alpha_5=\frac{9}{2}$, $\alpha_1=\frac{2}{3}$, $\alpha_2=\frac{5}{4}$, $\alpha_3=\frac{8}{3}$, $\alpha_4=\frac{7}{2}$, $\delta=\frac{13}{4}$, $\mu_0=\sqrt{2}$, $\mu_1=3\sqrt{3}$, $\mu_2=-1$, $\mu_3=\frac{1}{4}$, $\mu_4=1$.

We can take $\rho=\frac{2}{3}$, *then all the assumptios of Theorem 14 are satisfied. Thus,* (52) *has at least one solution on* $[0,1]$.

3. Ulam Stability

In this section, we impose the Ulam-Hyers stability and the generalized Ulam-Hyers stability for problrm (1).

Definition 16. *The fractional problem* (1) *is Ulam-Hyers stable if there exists a real number* $A>0$, *with the following property:*

For each $\epsilon>0$, *and for all solution* $x\in S$, *if*

$$\left|D^{\alpha_n}x(t)+f\left(t,x(t),D^{\alpha_1}x(t),...,D^{\alpha_{n-1}}x(t)\right)\right|<\epsilon, \tag{53}$$

then, there exists $y\in S$ *satisfying*

$$D^{\alpha_n}y(t)+f\left(t,y(t),D^{\alpha_1}y(t),...,D^{\alpha_{n-1}}y(t)\right)=0, \tag{54}$$

with $y^{(j)}(0)=\mu_j$, $j=0,1,...,n-2$, $D^{\delta}y(1)=\mu_{n-1}$, $n-2<\delta<n-1$, *where* $\|x-y\|_S<A\epsilon$.

Definition 17. *The fractional system* (1) *has the generalized Ulam-Hyers stability if there exist* $\Upsilon\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$, *such that for all* $\epsilon>0$, *and for each solution* $x\in S$ *of* (53), *there exists* $y\in S$ *of* (54) *such that* $y^{(j)}(0)=\mu_j$, $j=0,1,...,n-2$, $D^{\delta}y(1)=\mu_{n-1}$, $n-1<\delta<\alpha_n$, *with*

$$\|x-y\|_S<\Upsilon(\epsilon). \tag{55}$$

Theorem 18. *Let the assumptions of Theorem 12 hold. Assume that* $n-1<\alpha_n<n$, $n\in\mathbb{N}^*-\{1\}$, $0<\rho<1$, $f:]0,1]\times\mathbb{R}^n\to\mathbb{R}$ *is continuous,* $\lim\limits_{t\to 0^+}f(t,...)=\infty$,

$t^{\rho}f(t,...)$ is continuous on $[0,1]\times\mathbb{R}^{n}$. Then, the problem (1) *has the generalized Ulam-Hyers stability in S provided that:*

$$\left\|t^{\rho}D^{\alpha_{n}}x\right\|_{\infty} \geq \max_{1\leq k\leq n-2}\begin{pmatrix} C_{0}F_{0}+\sum_{j=0}^{n-2}\frac{|\mu_{j}|}{j!}+\frac{\Gamma(n-\delta)|\mu_{n-1}|}{(n-1)!},\\ C_{0}F_{k}+\sum_{j=k}^{n-2}\frac{|\mu_{j}|}{\Gamma(j+1-\alpha_{k})}+\frac{\Gamma(n-\delta)|\mu_{n-1}|}{\Gamma(n-\alpha_{k})},\\ C_{0}F_{n-1}+\frac{\Gamma(n-\delta)|\mu_{n-1}|}{\Gamma(n-\alpha_{n-1})}\end{pmatrix}$$

$$\sum_{i=1}^{n}\omega_{i} < 1. \tag{56}$$

Proof. Assume that (56) is hold. For each $\epsilon>0$, we suppose that there exists $x\in S$ satisfy (53) . Then,

$$\begin{aligned}&\left\|x\right\|_{\infty}\\ &\leq \sup_{s\in[0,1]}\int_{0}^{t}\frac{(t-s)^{\alpha_{n}-1}s^{-\rho}}{\Gamma(\alpha_{n})}s^{\rho}\left|f\left(s,x\left(s\right),D^{\alpha_{1}}x\left(s\right),...,D^{\alpha_{n-1}}x\left(s\right)\right)\right|ds\\ &\quad+\sum_{j=0}^{n-2}\frac{|\mu_{j}|}{j!}\sup_{t\in[0,1]}t^{j}+\frac{\Gamma(n-\delta)}{(n-1)!}\sup_{t\in[0,1]}t^{n-1}\\ &\times\left(|\mu_{n-1}|+\int_{0}^{1}\frac{(1-s)^{\alpha_{n}-\delta-1}s^{-\rho}}{\Gamma(\alpha_{n}-\delta)}s^{\rho}\left|f\left(s,x\left(s\right),D^{\alpha_{1}}x\left(s\right),...,D^{\alpha_{n-1}}x\left(s\right)\right)\right|ds\right)\\ &\leq C_{0}F_{0}+\sum_{j=0}^{n-2}\frac{|\mu_{j}|}{j!}+\frac{\Gamma(n-\delta)|\mu_{n-1}|}{(n-1)!}.\end{aligned} \tag{57}$$

Also we get, for each $k=1,2,...,n-2$,

$$\left\|D^{\alpha_{k}}x\right\|_{\infty}\leq C_{0}F_{k}+\sum_{j=k}^{n-2}\frac{|\mu_{j}|}{\Gamma\left(j+1-\alpha_{k}\right)}+\frac{\Gamma\left(n-\delta\right)|\mu_{n-1}|}{\Gamma\left(n-\alpha_{k}\right)}, \tag{58}$$

and

$$\left\|D^{\alpha_{n-1}}x\right\|_{\infty}\leq C_{0}F_{n-1}+\frac{\Gamma\left(n-\delta\right)|\mu_{n-1}|}{\Gamma\left(n-\alpha_{n-1}\right)}. \tag{59}$$

Combining (57) , (58) and (59) , yields

$$\left\|x\right\|_{S}\leq\max_{1\leq k\leq n-2}\begin{pmatrix} C_{0}F_{0}+\sum_{j=0}^{n-2}\frac{|\mu_{j}|}{j!}+\frac{\Gamma(n-\delta)|\mu_{n-1}|}{(n-1)!},\\ C_{0}F_{k}+\sum_{j=k}^{n-2}\frac{|\mu_{j}|}{\Gamma(j+1-\alpha_{k})}+\frac{\Gamma(n-\delta)|\mu_{n-1}|}{\Gamma(n-\alpha_{k})},\\ C_{0}F_{n-1}+\frac{\Gamma(n-\delta)|\mu_{n-1}|}{\Gamma(n-\alpha_{n-1})}\end{pmatrix}. \tag{60}$$

Therefore, by (56) and (60) , we get

$$\left\|x\right\|_{S}\leq\left\|t^{\rho}D^{\alpha_{n}}x\right\|_{\infty}. \tag{61}$$

From Theorem 12, problem (1) has a solution $y \in S$ satisfying (54). Using the last inequality (61), we can write

$$\|(x-y)\|_S \leq \|t^\rho D^{\alpha_n}(x-y)\|_\infty$$

$$\leq \left\| \begin{array}{c} t^\rho \left(D^{\alpha_n} x + f\left(s, x, D^{\alpha_1} x, ..., D^{\alpha_{n-1}} x\right)\right) \\ -t^\rho \left(D^{\alpha_n} y + f\left(s, y, D^{\alpha_1} y, ..., D^{\alpha_{n-1}} y\right)\right) \\ +t^\rho \left(f\left(s, y, D^{\alpha_1} y, ..., D^{\alpha_{n-1}} y\right) - f\left(s, x, D^{\alpha_1} x, ..., D^{\alpha_{n-1}} x\right)\right) \end{array} \right\|_\infty . \tag{62}$$

Then,

$$\|(x-y)\|_S \leq \left(\begin{array}{c} \|t^\rho\|_\infty \left\| \left(D^{\alpha_n} x + f\left(s, x, D^{\alpha_1} x, ..., D^{\alpha_{n-1}} x\right)\right) \right\|_\infty \\ + \|t^\rho\|_\infty \left\| -\left(D^{\alpha_n} y + f\left(s, y, D^{\alpha_1} y, ..., D^{\alpha_{n-1}} y\right)\right) \right\|_\infty \\ + \left\| t^\rho \left(\begin{array}{c} f\left(s, y, D^{\alpha_1} y, ..., D^{\alpha_{n-1}} y\right) \\ -f\left(s, x, D^{\alpha_1} x, ..., D^{\alpha_{n-1}} x\right) \end{array} \right) \right\|_\infty \end{array} \right). \tag{63}$$

By the first hypothesis (H_1) of Theorem 12, and from (53) and (54), we get

$$\|(x-y)\|_S$$

$$\leq \epsilon + \sum_{i=1}^{n} \omega_i \max\left(\|x-y\|_\infty, \|D^{\alpha_1}(x-y)\|_\infty, ..., \|D^{\alpha_{n-1}}(x-y)\|_\infty\right)$$

$$\leq \epsilon + \sum_{i=1}^{n} \omega_i \|(x-y)\|_S, \tag{64}$$

which implies that,

$$\|(x-y)\|_S \leq \frac{\epsilon}{1 - \sum_{i=1}^{n} \omega_i} := A\epsilon. \tag{65}$$

Thus, problem (1) has the Ulam-Hyers stability. Taking $\Upsilon(\epsilon) = A\epsilon$, yields that problem (1) has the generalized Ulam-Hyers stability. This completes the proof. ■

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