

STABILITY OF TRIANGULAR POINTS IN THE RELATIVISTIC R3BP WHEN THE BIGGER PRIMARY IS AN OBLATE SPHEROID

NAKONE BELLO¹ AND JAGADISH SINGH²

¹DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, USMANU DANFODIYO
UNIVERSITY, P.M.B. 2346, SOKOTO, NIGERIA

²DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, AHMADU BELLO
UNIVERSITY, ZARIA, NIGERIA

E-MAILS: BNAKONE@YAHOO.COM, JGDS2004@YAHOO.COM

(Received: 21 August 2015, Accepted: 25 November 2015)

ABSTRACT. In this paper, we study the effect of oblateness of the more massive primary in the relativistic R3BP. We observe that the locations of the triangular points and their stability are affected by the relativistic and oblateness factors. It is also noticed that the oblateness factor possesses destabilizing behavior. Therefore, the size of the region of stability decreases with increase in the value of the oblateness factor.

Further, a numerical study on the locations of the triangular points and the critical mass for the Earth-Moon, Jupiter and its Moons, Saturn and its Moons systems is given.

AMS Classification: 70F15

Keywords: Celestial mechanics; Oblateness; Relativity; R3BP

1. INTRODUCTION

The three-body problem is one of the most important problem of celestial mechanics and has been analytically and numerically studied for centuries. The problem of three bodies in its most general form means that three participating bodies are free

to move in space and initially move in any given manner under the influence of a given gravitational force field. Regarding the general three-body problem, we may cite Chenciner 2007, Valtonen and Karttunen 2006.

There does not exist a general analytical solution to this problem but among others chaotic solutions and numerical ones based on iterative methods. If two of the three bodies move in circular and coplanar orbits around their common barycentre and additionally, the third mass is small compared to the other two masses so that it does not affect the movements of the other bodies; one speaks of the circular restricted three-body, e.g. in our solar system like a spacecraft in the gravitational field of Earth and Sun or a space probe flying to Jupiter or Saturn being exposed to the gravitational attraction of the Sun and planet. This problem possesses three collinear and two triangular equilibrium points. The collinear points, denoted by L_1, L_2, L_3 , are in general unstable; while the triangular points, denoted by L_4, L_5 are stable for the mass ratio $\mu < 0.038520...$ (Szebehely 1967). Various authors (SubbaRao and Sharma 1975, Perdiou et al. 2012, Douskos and Markellos 2006, AbdulRaheem and Singh 2006, Abouelmagd 2012) have made studies on Lagrangian points in the restricted three-body problem by considering the oblateness of one or both the primaries with their equatorial planes coincident with the plane of motion.

The theory of general relativity is currently the most successful gravitational theory describing the nature of space and time, and well confirmed by observations. Especially, it has passed “Classical test” such as the deflection of light, the perihelion shift of Mercury, and the Shapiro time delay and also a systematic test using the remarkable binary pulsar “PSR 1913+16” (Klioner, 2003). Exact solutions of the field equations and the form of the rigorous equations of motion of the N-body problem are not known in general relativity even for $N=2$. To solve this problem one has to apply the approximate methods and construct a solution in the form of series in powers of small parameters. Contopoulos (1976) treated the relativistic R3BP in rotating coordinates. He derived the Lagrangian of the system and the deviations of the triangular points as well.

Brumberg (1972, 1991) studied the problem in more details and collected most of the important results on the relativistic celestial mechanics. He did not obtain only the equation of motion for the general problem of three bodies but also deduced the

equation of motion for the restricted problem of three bodies. The existence and linear stability of the collinear libration points in the relativistic R3BP were studied by Ragos et al. (2001). Douskos and Perdios (2002) investigated the stability of the triangular points in the relativistic R3BP and contrary to the result of Bhatnagar and Hallan (1998), they obtained a region of linear stability in the parameter space as $0 \leq \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2}$ where $\mu_0 = 0.03852\dots$ is Routh's value. Singh and Bello (2014a) investigated the motion of a test particle in the vicinity of the triangular points $L_{4,5}$ by considering the more massive primary as a source of radiation in the framework of the relativistic restricted three-body problem (R3BP). They found that the positions and stability of the triangular points affected by both relativistic and radiation factors. Singh and Bello (2014b) studied the existence of triangular points and their linear stability in the perturbed relativistic restricted three-body problem (R3BP). The problem is perturbed in the sense that a small perturbation is given to the centrifugal force. They found that positions and stability of the triangular points are affected by both relativistic factor and a small perturbation in the centrifugal force. Abd El-Salam and Abd El-Bar (2014) studied the photogravitational restricted three-body problem within the framework of the post-Newtonian approximation. The mass of the primaries are assumed to change under the effect of continuous radiation process. The locations of the triangular points are computed. Series forms of these locations are obtained as new analytical results. Katour et al. (2014) extended this work by including the effects of oblateness of the two primaries. They have only obtained the locations of triangular points; but their results seem to be erroneous due to the use of incorrect expression for the mean motion. However, to the present authors' knowledge no study has been done on the stability of any equilibrium points under oblateness effects of primaries of this problem. This motivated us to study linear stability of the triangular points of the above problem under the oblateness of the bigger primary.

This paper is organized as follows: In Sect. 2, the equations governing the motion are presented; Sect. 3 describes the positions of triangular points, while their linear stability is analyzed in Sect. 4. A discussion of these results and their numerical applications are given in Section 5 and 6 respectively. Finally Section 7 summarizes the results of this paper.

2. EQUATIONS OF MOTION

The pertinent equations of motion of an infinitesimal mass in the relativistic R3BP when the bigger primary is an oblate spheroid, in a barycentric synodic coordinate system (ξ, η) and dimensionless variables, can be written as Brumberg (1972) and Bhatnagar and Hallan (1998):

$$(1) \quad \begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \frac{\partial W}{\partial \xi} - \frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\xi}} \right), \\ \ddot{\eta} + 2n\dot{\xi} &= \frac{\partial W}{\partial \eta} - \frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\eta}} \right), \end{aligned}$$

where,

$$(2) \quad \begin{aligned} W &= \frac{1}{2}n^2(\xi^2 + \eta^2) + \frac{1-\mu}{\rho_1} \left(1 + \frac{A_1}{2\rho_1^2} \right) + \frac{\mu}{\rho_2} \\ &+ \frac{1}{c^2} \left[\frac{1}{8} \left\{ \dot{\xi}^2 + \dot{\eta}^2 + 2n(\xi\dot{\eta} - \eta\dot{\xi}) + n^2(\xi^2 + \eta^2) \right\}^2 \right. \\ &+ \frac{3}{2} \left(\frac{1-\mu}{\rho_1} \left(1 + \frac{A_1}{2\rho_1^2} \right) + \frac{\mu}{\rho_2} \right) \left\{ \dot{\xi}^2 + \dot{\eta}^2 + 2n(\xi\dot{\eta} - \eta\dot{\xi}) + n^2(\xi^2 + \eta^2) \right\} - \\ &\frac{1}{2} \left(\frac{(1-\mu)^2}{\rho_1^2} \left(1 + \frac{A_1}{2\rho_1^2} \right)^2 + \frac{\mu^2}{\rho_2^2} \right) + \mu(1-\mu) \left\{ n \left(4\dot{\eta} + \frac{7}{2}n\xi \right) \left(\frac{1}{\rho_1} \left(1 + \frac{A_1}{2\rho_1^2} \right) - \frac{1}{\rho_2} \right) \right. \\ &\left. \left. - \frac{n^2}{2} \eta^2 \left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3} \right) + n^2 \left(\frac{-1}{\rho_1\rho_2} + \frac{\mu}{2\rho_1} - \frac{(1-\mu)}{\rho_1} \left(1 + \frac{A_1}{2\rho_1^2} \right) + \frac{1-3\mu}{2\rho_2} \right) \right\} \right], \end{aligned}$$

$$(3) \quad \begin{aligned} \rho_1^2 &= (\xi + \mu)^2 + \eta^2 \\ \rho_2^2 &= (\xi + \mu - 1)^2 + \eta^2 \end{aligned}$$

where $0 < \mu \leq \frac{1}{2}$ is the ratio of the mass of the smaller primary to the total mass of the primaries, ρ_1 and ρ_2 are distances of the infinitesimal mass from the bigger and smaller primary, respectively; c is the velocity of light. A_1 characterizes the oblateness of the bigger primary and is given by $A_1 = \frac{(AE^2 - AP^2)}{5R^2} \ll 1$. (McCuskey, 1963), where AE and AP are the equatorial and polar radii of the bigger primary, and R is the distance between the primaries.

Here n is the perturbed mean motion of the primaries and is given as [Singh and Ishwar (1999) and Brumberg (1972)]:

$$(4) \quad n^2 = \frac{G(m_1 + m_2)}{R} \left\{ \frac{1}{R^2} + \frac{3A_1}{2R^4} - \frac{G[3(m_1 + m_2)^2 - m_1 m_2]}{R^3 c^2 (m_1 + m_2)} \right\}$$

where m_1 and m_2 are masses of the bigger and smaller primaries, respectively and G is the gravitational constant. The dimensionless form of (4) is

$$(5) \quad n^2 = 1 + \frac{3}{2}A_1 - \frac{3-\mu(1-\mu)}{c^2}$$

$$(6) \quad n = \sqrt{1 + \frac{3}{2}A_1 - \frac{3-\mu(1-\mu)}{c^2}} = 1 + \frac{3}{4}A_1 - \frac{3}{2c^2}(1 - \frac{\mu(1-\mu)}{3})$$

In the absence of the relativistic terms (6) reduces to

$$(7) \quad n = \sqrt{1 + \frac{3}{2}A_1}$$

(see e.g. SubbaRao and Sharma 1975; Douskos and Markellos 2006) Note that in the absence of relativistic term, the expression for the mean motion n in Katour et al. (2014) is different from that of SubbaRao and Sharma (1975) and that of Douskos and Markellos (2006). It seems that there are errors in the denominators of the coefficients of A_1 and A_2 .

3. LOCATION OF TRIANGULAR POINTS

The libration points are obtained from equation (1) after putting $\dot{\xi} = \dot{\eta} = \ddot{\xi} = \ddot{\eta} = 0$.

These points are the solutions of the equations $\frac{\partial W}{\partial \xi} = 0 = \frac{\partial W}{\partial \eta}$ with $\dot{\xi} = \dot{\eta} = 0$.

That is,

$$\begin{aligned} & \xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{\mu(\xi-1+\mu)}{\rho_2^3} + \frac{3}{2}A_1 \left\{ \xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^5} \right\} + \frac{1}{c^2} \left[-3\xi \left(1 - \frac{1}{3}\mu(1-\mu) \right) \right. \\ & + \frac{1}{2}\xi(\xi^2 + \eta^2) - \frac{3}{2}(\xi^2 + \eta^2) \left\{ \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} + \frac{\mu(\xi-1+\mu)}{\rho_2^3} \right\} + 3 \left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} \right) \xi \\ & + \frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4} + \frac{\mu^2(\xi-1+\mu)}{\rho_2^4} + \mu(1-\mu) \left\{ \frac{7}{2} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) + \frac{7}{2}\xi \left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3} \right) \right. \\ & + \frac{3}{2}\eta^2 \left(\frac{\mu(\xi+\mu)}{\rho_1^5} + \frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5} \right) + \frac{(\xi+\mu)}{\rho_1^3\rho_2} + \frac{(\xi-1+\mu)}{\rho_1\rho_2^3} - \frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3} - \frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3} \left. \right\} \\ & + A_1 \left\{ -\frac{9}{4} \left(1 - \frac{\mu(1-\mu)}{3} \right) \xi + \frac{3}{2}(\xi^2 + \eta^2) \xi \right. \\ & + \left(-\frac{9(1-\mu)(\xi+\mu)}{4\rho_1^5} - \frac{9(1-\mu)(\xi+\mu)}{4\rho_1^3} - \frac{9\mu(\xi-1+\mu)}{4\rho_2^3} \right) \times \\ & (\xi^2 + \eta^2) + \left(\frac{3(1-\mu)}{2\rho_1^3} + \frac{9(1-\mu)}{2\rho_1} + \frac{9\mu}{2\rho_2} \right) \xi + \frac{2(1-\mu)^2(\xi+\mu)}{\rho_1^6} \left. \right\} \\ & + \mu(1-\mu)A_1 \left\{ \frac{7}{4\rho_1^3} + \frac{21}{4} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) + \frac{7}{2}\xi \left(-\frac{3(\xi+\mu)}{2\rho_1^5} \right) + \frac{21}{4}\xi \left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3} \right) \right. \\ & + \frac{9}{4}\eta^2 \left(\frac{\mu(\xi+\mu)}{\rho_1^5} + \frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5} \right) + \frac{3(1-\mu)(\xi+\mu)}{2\rho_1^5} + \frac{3(\xi+\mu)}{2\rho_1^3\rho_2} + \frac{3(\xi-1+\mu)}{2\rho_1\rho_2^3} - \frac{3\mu(\xi+\mu)}{4\rho_1^3} \\ & \left. \left. - \frac{3(1-3\mu)(\xi-1+\mu)}{4\rho_2^3} + \frac{3(1-\mu)(\xi+\mu)}{2\rho_1^3} \right\} \right] = 0 \end{aligned}$$

and

$$(8) \quad \eta F = 0$$

with

$$\begin{aligned} F = & 1 - \frac{(1-\mu)}{\rho_1^3} - \frac{\mu}{\rho_2^3} + \frac{3}{2}A_1 \left(1 - \frac{(1-\mu)}{\rho_1^5}\right) + \frac{1}{c^2} \left[-3 \left(1 - \frac{1}{3}\mu(1-\mu)\right) + \frac{1}{2}(\xi^2 + \eta^2)\right. \\ & - \frac{3}{2}(\xi^2 + \eta^2) \left\{\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right\} + 3 \left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right) + \left(\frac{(1-\mu)^2}{\rho_1^4} + \frac{\mu^2}{\rho_2^4}\right) \\ & + \mu(1-\mu) \left\{\frac{7}{2}\xi \left(-\frac{1}{\rho_1^3} + \frac{1}{\rho_2^3}\right) - \left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3}\right) + \frac{3}{2}\eta^2 \left(\frac{\mu}{\rho_1^5} + \frac{(1-\mu)}{\rho_2^5}\right) + \frac{1}{\rho_1^3\rho_2} + \frac{1}{\rho_1\rho_2^3}\right. \\ & \left. - \frac{(3\mu-2)}{2\rho_1} - \frac{(1-3\mu)}{2\rho_2^3}\right\} + A_1 \left\{-\frac{9}{4} \left(1 - \frac{\mu(1-\mu)}{3}\right) + \frac{3}{2}(\xi^2 + \eta^2)\right. \\ & + \left(-\frac{9(1-\mu)}{4\rho_1^5} - \frac{9(1-\mu)}{4\rho_1^3} - \frac{9\mu}{4\rho_2^3}\right) (\xi^2 + \eta^2) + \frac{3(1-\mu)}{2\rho_1^3} + \frac{9(1-\mu)}{2\rho_1} \\ & + \frac{9\mu}{2\rho_2} + \frac{2(1-\mu)^2}{\rho_1^6}\left\} + \mu(1-\mu)A_1 \left\{\frac{21}{4}\xi \left(-\frac{1}{\rho_1^5} - \frac{1}{\rho_1^3} + \frac{1}{\rho_2^3}\right) - \frac{3}{2} \left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3}\right)\right. \right. \\ & \left. \left. + \frac{9}{4}\eta^2 \left(\frac{\mu}{\rho_1^5} + \frac{1-\mu}{\rho_2^5}\right) + \frac{3}{2} \left(\frac{1-\mu}{\rho_1^5} + \frac{1-\mu}{\rho_1^3} + \frac{1}{\rho_1^3\rho_2} + \frac{1}{\rho_1\rho_2^3} - \frac{\mu}{2\rho_1^3} - \frac{(1-3\mu)}{2\rho_2^3}\right)\right\}\right] \end{aligned}$$

The triangular points are the solutions of equation (8) with $\eta \neq 0$. Since $\frac{1}{c^2} \ll 1$ and in the case $\frac{1}{c^2} \rightarrow 0$ and in the absence of oblateness (i.e. $A_1 = 0$), one can obtain $\rho_1 = \rho_2 = 1$, we assume in the relativistic R3BP that $\rho_1 = 1 + x$ and $\rho_2 = 1 + y$ where, $x, y \ll 1$ may be depending upon the relativistic and oblateness factors. Substituting these values in the equations (3), solving them for ξ, η and ignoring terms of second and higher powers of x and y , we get

$$\xi = x - y + \frac{1-2\mu}{2},$$

$$\eta = \pm \left(\frac{\sqrt{3}}{2} + \frac{x+y}{\sqrt{3}} \right),$$

Now substituting the values of $\rho_1, \rho_2, \xi, \eta$ from the above in equations (8) with $\eta \neq 0$, and neglecting second and higher terms in $x, y, \frac{1}{c^2}, A_1$, we have

$$\begin{aligned} (9) \quad & \left\{ \frac{3}{2}(1-\mu) + \frac{3}{2}A_1 \left(\frac{5}{2} - \frac{3\mu}{2}\right) \right\} x - \left\{ \frac{3\mu}{2} + \frac{3\mu}{2}A_1 \right\} y - \frac{3\mu}{4}A_1 + \frac{1}{c^2} \left\{ \left(-\frac{9\mu}{16} + \frac{27\mu^2}{16} - \frac{9\mu^3}{8}\right) \right. \\ & + \left. \left(\frac{11}{8} - \frac{125\mu}{32} + \frac{183\mu^2}{32} - \frac{63\mu^3}{16}\right) A_1 \right\} = 0 \\ & \left\{ 3(1-\mu) + \frac{15}{2}A_1(1-\mu) \right\} x + 3\mu y + \frac{3\mu}{2}A_1 + \frac{1}{c^2} \left\{ \frac{21}{8}(\mu - \mu^2) \right. \\ & + \left. \left(\frac{11}{4} + \frac{53\mu}{16} - \frac{49\mu^2}{16} - \frac{3\mu^3}{2}\right) A_1 \right\} = 0 \end{aligned}$$

Solving these equations for x and y , we get

$$(10) \quad \begin{aligned} x = & -\frac{\mu(2+3\mu)}{8c^2} - \frac{(-44+51\mu-22\mu^2+30\mu^3)}{48(\mu-1)c^2} A_1 \\ y = & -\frac{(1-\mu)(5-3\mu)}{8c^2} - \left\{ \frac{1}{2} + \frac{74-86\mu+51\mu^2}{48c^2} \right\} A_1 \end{aligned}$$

Thus, the coordinates of the triangular points $(\xi, \pm\eta)$ denoted by L_4 and L_5 respectively, are,

$$(11) \quad \begin{aligned} \xi &= \frac{1-2\mu}{2} \left(1 + \frac{5}{4c^2}\right) + \left\{ \frac{1}{2} - \frac{(30-109\mu+115\mu^2-21\mu^3)}{48(\mu-1)c^2} \right\} A_1 \\ \eta &= \pm \left\{ \frac{\sqrt{3}}{2} \left(1 + \frac{1}{12c^2} (-5 + 6\mu - 6\mu^2)\right) + \left(-\frac{\sqrt{3}}{6} - \frac{\sqrt{3}(-118+211\mu-159\mu^2+81\mu^3)}{144(\mu-1)c^2} \right) A_1 \right\} \end{aligned}$$

4. STABILITY OF L_4

Let (a,b) be the coordinates of the triangular points L_4

We set $\xi = a + \alpha$, $\eta = b + \beta$, $(\alpha, \beta < 1)$ in the equations (5) of motion.

First, we compute the terms of their R.H.S, neglecting second and higher order terms, we get

$$\left(\frac{\partial W}{\partial \xi} \right)_{\xi=a+\alpha, \eta=b+\beta} = A\alpha + B\beta + C\dot{\alpha} + D\dot{\beta}$$

where,

$$\begin{aligned} A &= \frac{3}{4} \left\{ 1 + \frac{1}{2c^2} (2 - 19\mu + 19\mu^2) \right\} + \left\{ -\frac{3}{8} (8\mu - 9) - \frac{(226-1036\mu+1758\mu^2-1065\mu^3+87\mu^4)}{32(\mu-1)c^2} \right\} A_1, \\ B &= \frac{3\sqrt{3}}{4} (1 - 2\mu) \left(1 - \frac{2}{3c^2} \right) + \left\{ -\frac{\sqrt{3}}{8} (26\mu - 19) - \frac{\sqrt{3}(-46+393\mu-599\mu^2-135\mu^3+417\mu^4)}{96(\mu-1)c^2} \right\} A_1, \\ C &= \frac{\sqrt{3}}{2c^2} (1 - 2\mu) + \left\{ -\frac{\sqrt{3}}{24c^2} (46\mu - 35) \right\} A_1, \\ D &= \frac{6-5\mu+5\mu^2}{2c^2} + \left\{ \frac{22-33\mu+45\mu^2}{8c^2} \right\} A_1. \end{aligned}$$

Similarly, we obtain

$$\left(\frac{\partial W}{\partial \eta} \right)_{\xi=a+\alpha, \eta=b+\beta} = E\alpha + B_1\beta + C_1\dot{\alpha} + D_1\dot{\beta},$$

where,

$$\begin{aligned} E &= \frac{3\sqrt{3}}{4} (1 - 2\mu) \left(1 - \frac{2}{3c^2} \right) + \left\{ -\frac{\sqrt{3}}{8} (26\mu - 19) + \frac{\sqrt{3}(46+393\mu+599\mu^2-135\mu^3+417\mu^4)}{96(\mu-1)c^2} \right\} A_1, \\ B_1 &= \frac{9}{4} \left\{ 1 + \frac{7(-2+3\mu-3\mu^2)}{6c^2} \right\} + \left\{ \frac{33}{8} + \frac{(290-1292\mu+2082\mu^2-1221\mu^3+1118\mu^4)}{32(\mu-1)c^2} \right\} A_1, \\ C_1 &= \frac{1}{2c^2} (-4 + \mu - \mu^2) + \left\{ -\frac{(20-13\mu+9\mu^2)}{8c^2} \right\} A_1, \\ D_1 &= -\frac{\sqrt{3}}{2c^2} (1 - 2\mu) + \left\{ \frac{-\sqrt{3}(35+46\mu)}{24c^2} \right\} A_1. \end{aligned}$$

$$\frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\xi}} \right)_{\xi=a+\alpha, \eta=b+\beta} = A_2 \dot{\alpha} + B_2 \dot{\beta} + C_2 \ddot{\alpha} + D_2 \ddot{\beta}$$

where,

$$A_2 = \frac{\sqrt{3}}{2c^2} (1 - 2\mu) + \left\{ -\frac{\sqrt{3}(46\mu-35)}{24c^2} \right\} A_1,$$

$$B_2 = -\frac{4-\mu+\mu^2}{2c^2} + \left\{ -\frac{(20-13\mu+9\mu^2)}{8c^2} \right\} A_1,$$

$$C_2 = \frac{17-2\mu+2\mu^2}{4c^2} + \left\{ \frac{(23-10\mu+6\mu^2)}{8c^2} \right\} A_1,$$

$$D_2 = -\frac{\sqrt{3}(1-2\mu)}{4c^2} + \left\{ \frac{\sqrt{3}(13+14\mu)}{24c^2} \right\} A_1.$$

$$\frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\eta}} \right)_{\xi=a+\alpha, \eta=b+\beta} = A_3 \dot{\alpha} + B_3 \dot{\beta} + C_3 \ddot{\alpha} + D_3 \ddot{\beta}$$

where,

$$A_3 = \frac{6-5\mu+5\mu^2}{2c^2} + \left\{ \frac{22-33\mu+45\mu^2}{8c^2} \right\} A_1,$$

$$B_3 = -\frac{\sqrt{3}(1-2\mu)}{2c^2} + \left\{ \frac{\sqrt{3}(-35+46\mu)}{24c^2} \right\} A_1,$$

$$C_3 = -\frac{\sqrt{3}(1-2\mu)}{4c^2} + \left\{ \frac{\sqrt{3}(-13+14\mu)}{24c^2} \right\} A_1,$$

$$D_3 = \frac{3(5-2\mu+2\mu^2)}{4c^2} + \left\{ \frac{25-30\mu+18\mu^2}{8c^2} \right\} A_1$$

Thus, the variational equations of motion corresponding to equations (5), on making use of equation (7), can be obtained as

$$(12) \quad \begin{aligned} P_1 \ddot{\alpha} + P_2 \ddot{\beta} + P_3 \dot{\alpha} + P_4 \dot{\beta} + P_5 \alpha + P_6 \beta &= 0, \\ q_1 \ddot{\alpha} + q_2 \ddot{\beta} + q_3 \dot{\alpha} + q_4 \dot{\beta} + q_5 \alpha + q_6 \beta &= 0. \end{aligned}$$

where,

$$P_1 = 1 + C_2, P_2 = D_2, P_3 = A_2 - C,$$

$$P_4 = \left\{ B_2 - 2 \left(1 + \frac{3}{4} A_1 - \frac{3}{2c^2} \left(1 - \frac{1}{3} \mu(1 - \mu) \right) \right) - D \right\} P_5 = -A, P_6 = -B$$

$$q_1 = C_3, q_2 = 1 + D_3, q_3 = 2 \left(1 + \frac{3}{4} A_1 - \frac{3}{2c^2} \left(1 - \frac{1}{3} \mu(1 - \mu) \right) \right) - C_1 + A_3,$$

$$q_4 = B_3 - D_1 q_5 = -E, q_6 = -B_1$$

Then, the corresponding characteristic equation is

$$(13) \quad (P_1 q_2 - P_2 q_1) \lambda^4 + (P_1 q_6 + P_5 q_2 + P_3 q_4 - P_6 q_1 - P_2 q_5 - P_4 q_3) \lambda^2 + P_5 q_6 - P_6 q_5 = 0$$

Substituting the values of $P_i, q_i, i = 1, 2, \dots, 6$ in (13), the characteristic equation (13) after normalizing becomes

$$(14) \quad \lambda^4 + b\lambda^2 + d = 0,$$

$$b = \left(1 - \frac{9}{c^2} \right) + \left\{ -\frac{3}{2} + 3\mu + \frac{-80 - 108\mu + 105\mu^2 - 18\mu^3}{8c^2} \right\} A_1,$$

$$d = \left\{ \frac{27\mu(1-\mu)}{4} + \frac{9\mu(-65+77\mu-24\mu^2+12\mu^3)}{8c^2} \right\} + \left\{ \frac{117}{4}\mu(1-\mu) \right.$$

$$\left. + \frac{3(80-7245\mu+9624\mu^2-3366\mu^3+846\mu^4)}{64c^2} \right\} A_1$$

When $\frac{1}{c^2} \rightarrow 0$ and in the absence of the oblateness (*i.e.* $A_1 = 0$), (14) reduces to its well-known classical restricted problem form (see e.g. Szebehely 1967):

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0$$

The discriminant of (14) is

$$(15) \quad \Delta = \left(\frac{-54}{c^2} - \frac{1269A_1}{8c^2} \right) \mu^4 + \left(\frac{108}{c^2} + \frac{5013A_1}{8c^2} \right) \mu^3 + \left(27 - \frac{693}{2c^2} + 117A_1 - \frac{7113A_1}{4c^2} \right) \mu^2$$

$$+ \left(-27 + \frac{585}{2c^2} - 111A_1 + \frac{20439A_1}{16c^2} \right) \mu + 1 - 3A_1 - \frac{18}{c^2} - \frac{8A_1}{c^2}$$

Its roots are

$$(16) \quad \lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2}$$

where,

$$b = \left(1 - \frac{9}{c^2} \right) + \left\{ -\frac{3}{2} + 3\mu + \frac{-80-108\mu+105\mu^2-18\mu^3}{8c^2} \right\} A_1.$$

Now we have from (15),

$$(17) \quad \begin{aligned} \frac{d\Delta}{d\mu} = & 4 \left(\frac{-54}{c^2} - \frac{1269A_1}{8c^2} \right) \mu^3 + 3 \left(\frac{108}{c^2} + \frac{5013A_1}{8c^2} \right) \mu^2 \\ & + 2 \left(27 - \frac{693}{2c^2} + 117A_1 - \frac{7113A_1}{4c^2} \right) \mu \\ & - 27 + \frac{585}{2c^2} - 111A_1 + \frac{20439A_1}{16c^2} < 0 \quad \forall \mu \in \left(0, \frac{1}{2}\right] \end{aligned}$$

From (17), it follows that Δ is decreasing in $\left(0, \frac{1}{2}\right]$.

But

$$(18) \quad \begin{aligned} (\Delta)_{\mu=0} &= 1 - 3A_1 - \frac{18}{c^2} - \frac{8A_1}{c^2} > 0 \\ (\Delta)_{\mu=\frac{1}{2}} &= -\frac{23}{4} - \frac{117A_1}{4} + \frac{207}{4c^2} + \frac{32585A_1}{128c^2} < 0 \end{aligned}$$

Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\frac{1}{2}}$ are of opposite signs, and Δ is monotone and continuous, there is one value of μ , e.g. μ_c in the interval $\left(0, \frac{1}{2}\right)$ for which Δ vanishes. Solving the equation $\Delta = 0$, using (15), we obtain critical value of the mass parameter as

$$(19) \quad \begin{aligned} \mu_c = & \frac{1}{2} - \frac{1}{18}\sqrt{69} - \frac{1}{9} \left(1 + \frac{13}{9\sqrt{69}} \right) A_1 - \frac{17\sqrt{69}}{486c^2} + \left(\frac{-197133 + 15493\sqrt{69}}{536544c^2} \right) A_1 \\ \mu_c = & \mu_0 - \frac{17\sqrt{69}}{486c^2} - \frac{1}{9} \left(1 + \frac{13}{9\sqrt{69}} \right) A_1 + \left(\frac{-197133 + 15493\sqrt{69}}{536544c^2} \right) A_1 \end{aligned}$$

where $\mu_0 = 0.03852\dots$ is the Routh's value.

We consider the following three regions of the values of μ separately.

- (1) When $0 < \mu < \mu_c$, $\Delta > 0$, the values of λ^2 given by (16) are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are stable.
- (2) When $\mu_c < \mu \leq \frac{1}{2}$, $\Delta < 0$, the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.
- (3) When $\mu = \mu_c$, $\Delta = 0$, the values of λ^2 given by (16) are the same

So the solutions of the equations of motion contain secular terms. This induces instability of the triangular points.

Hence, the stability region is

$$(20) \quad 0 < \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2} - \frac{1}{9} \left(1 + \frac{13}{9\sqrt{69}} \right) A_1 + \left(\frac{-197133 + 15493\sqrt{69}}{536544c^2} \right) A_1$$

5. DISCUSSION

In this section we discuss the triangular equilibrium points obtained in our relativistic restricted three-body problem under the assumption that the bigger primary is oblate. It is important to note that these triangular equilibrium points (11) cease to form equilateral triangles as in classical ones since the distances $\rho_1 \neq \rho_2 \neq 1$, but turn into classical ones when $\frac{1}{c^2} \rightarrow 0$ and $A_1 = 0$.

In the absence of oblateness (*i.e.* $A_1 = 0$), these positions correspond to those of Douskos and Perdios (2002). Equation (19) represents the critical value of the mass parameter, which depends upon the relativistic and oblateness factors. It is seen that from (20) both the relativistic and oblateness factors shrink the region of linear stability as they have destabilizing characteristic behavior. It can also be seen that from table 3 that the size of region of stability reduces with the presence of relativistic and oblateness factors.

In the absence of oblateness (*i.e.* $A_1 = 0$), the stability results obtained in this study are in accordance with those of Douskos and Perdios (2002) and in disagreement with those of Bhatnagar and Hallan (1998).

It is also important to notice that, the positions of triangular equilibrium points (11) are in apparent disagreement with those of Dakour et al. (2014) on the same problem when the primaries are non-luminous and a bigger primary is oblate only. In the absence of relativistic terms it is noticed that from (11), (14), (19) that our results are in conformity with those of SubbaRao and Sharma (1975) and those of AbdulRaheem and Singh (2006) when the primaries are non-luminous in the absence of small perturbations in Coriolis and centrifugal forces and the bigger primary is oblate only.

6. NUMERICAL APPLICATIONS

The necessary data used to obtain the numerical result for the positions of the triangular points and critical value for various binary systems were borrowed from Sharma and SubbaRao (1975), Ragos et al. (2001) and Murray and Dermott (1999). The numerical computations are presented in Table 2 and table 3.

TABLE 1. **Parameters of the systems**

System	μ	Dimensionless velocity(c)	Obla. Coeff. (A_1)
Earth-Moon	0.01215295291	292624.8185	0.0000003641
Jupiter-Io	0.00004704824459	17294.81937	0.0006885582
Saturn-Mimas	$6.772683635 \times 10^{-8}$	20967.086225	0.0038673922
Saturn-Thethys	0.000001094183090	26424.27518	0.0015345696
Jupiter-Europa	0.000025266534711	21817.21721	0.0002717680

TABLE 2. **Locations of the triangular points**

System	ξ		
	Classical	Relativistic	Eq.(11)
Earth-Moon	0.4878470471	0.4878470471	0.4878472292
Jupiter-Io	0.4999529518	0.4999529538	0.5002972329
Saturn-Mimas	0.4999999323	0.4999999338	0.5019336299
Saturn-Thethys	0.4999989058	0.4999989068	0.5007661916
Jupiter-Europa	0.4999747347	0.4999747362	0.5001106202

System	$\pm\eta$		
	Classical	Relativistic	Eq.(11)
Earth-Moon	0.8660254040	0.8660254040	0.8660252989
Jupiter-Io	0.8660254040	0.8660254028	0.8658266332
Saturn-Mimas	0.8660254040	0.8660254033	0.8649089833
Saturn-Thethys	0.8660254040	0.8660254035	0.8655824114
Jupiter-Europa	0.8660254040	0.8660254033	0.8659469506

TABLE 3. Critical mass μ_c

System	Classical	Relativistic (without oblateness)	Eq. (19)
Earth-Moon	0.0385208965	0.0385208965	0.0385207927
Jupiter-Io	0.0385208965	0.0385208955	0.0383246551
Saturn- Mimas	0.0385208965	0.0385208958	0.0374186822
Saturn- Thethys	0.0385208965	0.0385208960	0.0380835410
Jupiter-Europa	0.0385208965	0.0385208959	0.0384434415

7. CONCLUSION

We study the existence of the triangular points and their linear

Stability by considering the bigger primary is an oblate spheroid whose equatorial plane coincides with the plane of motion in relativistic restricted three-body problem (R3BP). The problem has been studied by Bhatnagar and Hallan (1998) and Douskos and Perdios (2002), their study did not include oblateness. Recently Katour et al. (2014) have obtained the positions of the triangular points of the relativistic R3BP when both the primaries are luminous and oblate.

Regarding the linear stability of these triangular points, our result when the oblateness factor is ignored (*i.e.* $A_1 = 0$) are in accordance with those of Douskos and Perdios (2002) and in disagreement with those of Bhatnagar and Hallan (1998), because it seems that the expression for A , D , A_2 , C_2 and consequently P_1 , P_3 , P_4 , P_5 in their study differ from the ones of the present study. This led them (Bhatnagar and Hallan) to conclude that the triangular points are unstable.

However, concerning the positions of the triangular points, there are apparent disagreements with the results of this study and those of Katour et al. (2014) when the primaries are non-luminous and the bigger primary oblate only. This is as a result of the error in the expression of the perturbed mean motion which they have used in their study and in addition to that we have included the coupling term $\frac{A_1}{c^2}$, while they do not. It is further seen that the oblateness and relativistic factors have prominent effects on the position and the size of the region of stability. This is confirmed from table 2 and table 3.

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