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SOME CURVATURE PROPERTIES OF LORENTZIAN α-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study the pseudo-projective ϕ-recurrent and generalized projective ϕ-recurrent Lorentzian α-Sasakian manifolds. Here we show that pseudo-projective ϕ-recurrent Lorentzian α-Sasakian Manifold is an Einstein manifold and in the case of generalized projective ϕ-recurrent Lorentzian α-Sasakian manifold, we find a relation between the associated 1-forms A and B. We have also proved that the characteristic vector field ξ and vector field ρ associated to the 1-forms A and B are co-directional. We also study quasi-projectively flat Lorentzian α-Sasakian manifolds.

AMS Classification: 53C25, 53C15, 53D15
Keywords: Pseudo projective ϕ-recurrent manifold; generalized projective ϕ-recurrent manifold; Einstein manifold; quasi-projectively flat Lorentzian α-Sasakian manifold

1. Introduction

In 1977, T. Takahashi ([8]) introduced the notion of a locally ϕ-symmetric Sasakian manifold and obtained some interesting properties. The authors like([3]) and ([10])

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have extended this notion to 3-dimensional Kenmotsu and trans-Sasakian manifolds respectively. Also ϕ-recurrent Sasakian Kenmotsu manifolds were studied by the author([2]).

In ([9]), S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of a plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes:

1) Homogeneous normal contact Riemannian manifolds with $c > 0$;

2) Global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and;

3) A warped product space $R \times_f C$ if $c > 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure.

In 2005, Yildiz and Murathan ([11]) studied Lorentzian α-Sasakian manifolds and proved that conformally flat and quasi conformally flat Lorentzian α-Sasakian manifolds are locally isometric with a sphere. In 2012, Yadav and Suthar ([12]) studied Lorentzian α-Sasakian manifolds.

Definition 1.1. *A Lorentzian α-Sasakian manifold is said to be locally ϕ-symmetric if*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0 \tag{1}$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced by Takahashi for Sasakian manifolds ([8]).

Definition 1.2. *A Lorentzian α-Sasakian manifold is said to be locally projective ϕ-symmetric if*

$$\phi^2((\nabla_W P)(X,Y)Z) = 0 \tag{2}$$

for all vector fields X, Y, Z, W orthogonal to ξ, where P is the projective curvature tensor ([5]) defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y] \tag{3}$$

for $X, Y, Z \in \chi(M)$, where S is the Ricci tensor of the manifold.

Definition 1.3. *A Lorentzian α-Sasakian manifold is said to be locally pseudo projective ϕ-symmetric if*

$$\phi^2((\nabla_W \tilde{P})(X,Y)Z) = 0 \tag{4}$$

for all vector fields X, Y, Z, W orthogonal to ξ, where $\tilde{P}$ is the pseudo projective curvature tensor ([6]) defined as

$$\begin{aligned} \tilde{P}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{2n+1}[\frac{a}{2n} + b][g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{5}$$

where a and b are non-zero constants. If $a = 1$ and $b = -\frac{1}{2n}$, then the equation (1.5) becomes (1.3). Hence the projective curvature P is a particular case of the pseudo projective curvature tensor $\tilde{P}$, where R is the Riemann curvature tensor S is Ricci tensor and r is the scalar curvature.

Definition 1.4. *A Lorentzian α-Sasakian manifold M is said to be projective ϕ-recurrent if there exists a non-zero 1-form A such that*

$$\phi^2((\nabla_W P)(X,Y)Z) = A(W)P(X,Y)Z \tag{6}$$

for arbitrary vector fields X, Y, Z, W, where P is the projective curvature defined as (1.3).

If the 1-form A vanishes, then the manifold reduces to locally projective ϕ-symmetric manifold.

Definition 1.5. *A Lorentzian α-Sasakian manifold is said to be pseudo projective ϕ-recurrent if there exists a non-zero 1-form A such that*

$$\phi^2((\nabla_W \tilde{P})(X,Y)Z) = A(W)\tilde{P}(X,Y)Z \tag{7}$$

for arbitrary vector fields X, Y, Z, W, where $\tilde{P}$ is the projective curvature defined as (1.5).

If the 1-form A vanishes, then the manifold reduces to locally pseudo projective ϕ-symmetric manifold.

Definition 1.6. *A Lorentzian α-Sasakian manifold is said to be generalized projective ϕ-recurrent if its curvature tensor R satisfies the condition*

$$(8)\ \phi^2((\nabla_W P)(X,Y)Z) = A(W)P(X,Y)Z + B(W)[g(Y,Z)X - g(Y,Z)X],$$

where A and B are 1-forms, β is non-zero and these are defined by

$$A(W) = g(W,\rho_1), B(W) = g(W,\rho_2),$$

and where ρ_1 and ρ_2 are vector fields associated with 1-forms A and B respectively.

In this paper we show that pseudo projective recurrent Lorentzian α-Sasakian manifold is an Einstein manifold and in a pseudo projective recurrent Lorentzian α-Sasakian manifold, the characteristic vector field ξ and the vector field ρ_1 associated to the form are co-directional. In the fourth section we have studied Generalized Projective ϕ-Recurrent Lorentzian α-Sasakian Manifold. In this section we find a relation between the associated 1-forms A and B and we also proved that the characteristic vector field ξ and the vector field ρ_1 and ρ_2 associated to the forms A and B are co-directional. Section 5 is devoted to study quasi-projectively flat Lorentzian α-Sasakian manifolds. Finally, we give an example 3-dimensional Lorentzian α-Sasakian manifold.

2. **Preliminaries**

A (2n+1)-dimensional differentiable manifold M is said to be a Lorentzian α-Sasakian manifold if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy the following conditions

$$(9)\qquad \phi^2 X = X + \eta(X)\xi,$$

$$\eta(\xi) = -1, \phi\xi = 0, \eta(\phi X) = 0, \tag{10}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{11}$$

$$g(X, \xi) = \eta(X), \tag{12}$$

$$(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi + \eta(Y)X\} \tag{13}$$

$\forall X, Y \in \chi(M)$ and for a non-zero smooth functions α on M, ∇ denotes the covariant differentiation with respect to Lorentzian metric g ([7], [14]).

For a Lorentzian α-Sasakian manifold, it can be shown that ([7], [14]):

$$\nabla_X \xi = -\alpha\phi X, \tag{14}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y), \tag{15}$$

for all $X, Y \in \chi(M)$. Further on a Lorentzian α-Sasakian manifold, the following relations hold ([7])

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{16}$$

$$R(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X], \tag{17}$$

$$R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{18}$$

$$R(\xi, X)\xi = \alpha^2[X + \eta(X)\xi], \tag{19}$$

$$S(X, \xi) = S(\xi, X) = 2n\alpha^2\eta(X), \tag{20}$$

$$S(\xi, \xi) = -2n\alpha^2, \tag{21}$$

$$S(\phi Y, \phi Z) = S(Y, Z) + \alpha^2 2n\eta(Y)\eta(Z), \tag{22}$$

$$Q\xi = 2n\alpha^2\xi, \tag{23}$$

where Q is the Ricci operator, i.e.

$$g(QX, Y) = S(X, Y). \tag{24}$$

3. **Pseudo Projective ϕ-Recurrent Lorentzian α-Sasakian Manifold**

In this section we consider a Lorentzian α-Sasakian manifold which is pseudo projective recurrent Sasakian manifold. Now using the equation (2.1) in (1.7), we obtain,

$$(\nabla_W \tilde{P})(X,Y)Z + \eta((\nabla_W \tilde{P})(X,Y)Z)\xi = A(W)\tilde{P}(X,Y)Z, \tag{25}$$

from which it follows that

$$(\nabla_W \tilde{P})(X,Y)Z,U) + \eta((\nabla_W \tilde{P})(X,Y)Z)\eta(U) = A(W)g(\tilde{P}(X,Y)Z,U). \tag{26}$$

Let $\{e_i\}$, i=1,2,.....2n+1 be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.1) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$(\nabla_W S)(Y,Z) = A(W)[S(Y,Z) - (\frac{r}{2n+1})g(Y,Z)], \tag{27}$$

Replacing Z by ξ in (3.3) and using (2.4) and (2.12), we get

$$(\nabla_W S)(Y,\xi) = A(W)[2n\alpha^2 - (\frac{r}{2n+1})]\eta(Y), \tag{28}$$

We know that

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi), \tag{29}$$

Using (2.6), (2.7) and (2.12) in (3.5), it follows that

$$(\nabla_W S)(Y,\xi) = -2n\alpha^3 g(\phi Y,W) + \alpha S(Y,\phi W), \tag{30}$$

In view of (3.4) and (3.6), we obtain

$$\alpha S(Y,\phi W) = 2n\alpha^3 g(\phi Y,W) + A(W)[2n\alpha^2 - (\frac{r}{2n+1})]\eta(Y), \tag{31}$$

Replacing Y by ϕY and using (2.1) and (2.14) in the above equation, we get

$$S(Y,W) = ag(Y,W), \tag{32}$$

where $a = 2n\alpha^2$ for all $Y, W \in \chi(M)$.
Thus we can state the following theorem:

Theorem 3.1. *A Pseudo Projective ϕ-Recurrent Lorentzian α-Sasakian Manifold is an Einstein manifold.*

From (3.1), we have

$$(\nabla_W \tilde{P})(X,Y)Z = A(W)\tilde{P}(X,Y)Z - \eta((\nabla_W \tilde{P})(X,Y)Z)\xi, \tag{33}$$

Now using (1.5) in (3.9), we have

$$\begin{aligned} a((\nabla_W R)(X,Y)Z) &+ b[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y] \\ &= A(W)aR(X,Y)Z + bA(W)[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{2n+1}[\frac{a}{2n} + b]A(W)[g(Y,Z)X - g(X,Z)Y] \\ &- a\eta((\nabla_W R)(X,Y)Z)\xi - b[(\nabla_W S)(Y,Z)\eta(X) \\ &- (\nabla_W S)(X,Z)\eta(Y)]\xi. \end{aligned} \tag{34}$$

From (3.10) and using Bianchi identity, we get

$$\begin{aligned} aA(W)\eta(R(X,Y)Z) &+ aA(X)\eta(R(Y,W)Z) + aA(Y)\eta(R(W,X)Z) \\ &= -bA(W)[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] \\ &+ \frac{r}{2n+1}[\frac{a}{2n} + b]A(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &- bA(X)[S(W,Z)\eta(Y) - S(Y,Z)\eta(W)] \\ &+ \frac{r}{2n+1}[\frac{a}{2n} + b]A(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)] \\ &- bA(Y)[S(X,Z)\eta(W) - S(W,Z)\eta(X)] \\ &+ \frac{r}{2n+1}[\frac{a}{2n} + b]A(X)[g(X,Z)\eta(W) \\ &- g(W,Z)\eta(X)]. \end{aligned} \tag{35}$$

Now using (2.8) in the above equation, we obtain

$$\begin{aligned} aA(W)\alpha^2[g(Y,Z)\eta(X) &- g(X,Z)\eta(Y)] + aA(X)\alpha^2[g(W,Z)\eta(Y) \\ &- g(Y,Z)\eta(W)] + aA(Y)\alpha^2[g(X,Z)\eta(W) \\ &- g(W,Z)\eta(X)] = -bA(W)[S(Y,Z)\eta(X) \\ &- S(X,Z)\eta(Y)] + \frac{r}{2n+1}[\frac{a}{2n}+b]A(W)[g(Y,Z)\eta(X) \\ &- g(X,Z)\eta(Y)] - bA(X)[S(W,Z)\eta(Y) - S(Y,Z)\eta(W)] \\ &+ \frac{r}{2n+1}[\frac{a}{2n}+b]A(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)] \\ &- bA(Y)[S(X,Z)\eta(W) - S(W,Z)\eta(X)] \\ &+ \frac{r}{2n+1}[\frac{a}{2n}+b]A(X)[g(X,Z)\eta(W) \\ &- g(W,Z)\eta(X)]. \end{aligned} \tag{36}$$

Putting $Y = Z = e_i$ in (3.12) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$A(W)\eta(X) = A(X)\eta(W), \tag{37}$$

for all vector fields X, W. Putting $X = \xi$ in (3.13), we obtain

$$\eta(W)\eta(\rho_1) + A(W) = 0 \tag{38}$$

for all vector field W and $A(W) = g(W, \rho_1)$, where ρ_1 are vector fields associated with 1-forms A and B respectively.
Thus we can state the following theorem:

Theorem 3.2. *In a pseudo projective ϕ-recurrent Lorentzian α-Sasakian manifold (M^{2n+1}) $(n \geq 1)$, the characteristic vector field ξ and the vector field ρ_1 associated to the 1-form A are co-directional and the 1-form A is given by* (3.14).

4. Generalized Projective ϕ-Recurrent Lorentzian α-Sasakian Manifold

Let us consider generalized projective ϕ-recurrent Lorentzian α-Sasakian manifold. By virtue of (1.8) and (2.1), we get

$$\begin{aligned}(\nabla_W P)(X,Y)Z &+ \eta((\nabla_W P)(X,Y)Z)\xi = A(W)P(X,Y)Z \\ &+ B(W)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{39}$$

from which it follows that

$$\begin{aligned}g((\nabla_W P)(X,Y)Z,U) &+ \eta((\nabla_W P)(X,Y)Z)\eta(U) = A(W)g(P(X,Y)Z,U) \\ &+ B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{40}$$

Let $\{e_i\}$, $i = 1, 2, ...2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (4.2) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$\begin{aligned}(\nabla_W S)(X,U) &- \frac{\nabla_W r}{2n} g(X,U) + \frac{(\nabla_W S)(X,U)}{2n} - (\nabla_W S)(X,\xi)\eta(U) \\ &+ \frac{\nabla_W r}{2n}\eta(X)\eta(U) - \frac{(\nabla_W S)(X,U)}{2n}\eta(U) \\ &= A(W)[\frac{2n+1}{2n}S(X,U) - \frac{r}{2n}g(X,U)] \\ &+ 2nB(W)g(X,U). \end{aligned} \tag{41}$$

Putting $U = \xi$ in (4.3) and using the equation (2.2) an (2.12), we obtain

$$A(W)[(2n+1)\alpha^2 - \frac{r}{2n}]\eta(X) + 2nB(W)\eta(X) = 0. \tag{42}$$

Putting $X = \xi$ in (4.4), we get

$$B(W) = [\frac{r}{4n^2} - \frac{2n+1}{2n}\alpha^2]A(W). \tag{43}$$

Thus we can the following theorem:

Theorem 4.1. *In a generalized projective ϕ-ecurrent Lorentzian α-Sasakian manifold (M^{2n+1}) $(n \geq 1)$, the 1-forms A and B are related as* (4.5).

Now from (4.1), we get

$$\begin{aligned}(\nabla_W P)(X,Y)Z &= -\eta((\nabla_W P)(X,Y)Z)\xi + A(W)P(X,Y)Z \\ &+ B(W)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{44}$$

this implies

$$\begin{aligned}((\nabla_W R)(X,Y)Z) &- \frac{1}{2n}[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y] \\ &= A(W)[R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\}] \\ &+ B(W)[g(Y,Z)X - g(X,Z)Y] - \eta(\nabla_W R)(X,Y)Z)\xi \\ &+ \frac{1}{2n}[(\nabla_W S)(Y,Z)\eta(X) - (\nabla_W S)(X,Z)\eta(Y)]\xi. \end{aligned} \tag{45}$$

Using the Bianchi identity of the above equation, we get

$$\begin{aligned}aA(W)\eta(R(X,Y)Z) &+ aA(X)\eta(R(Y,W)Z) + aA(Y)\eta(R(W,X)Z) \\ &= \frac{1}{2n}A(W)[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] \\ &- B(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &+ \frac{1}{2n}A(X)[S(W,Z)\eta(Y) - S(Y,Z)\eta(W)] \\ &- B(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)] \\ &+ \frac{1}{2n}A(Y)[S(X,Z)\eta(W) - S(W,Z)\eta(X)] \\ &- B(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)]. \end{aligned} \tag{46}$$

Now using (2.8)in (4.8), we obtain

$$\begin{aligned}aA(W)\alpha^2[g(Y,Z)\eta(X) &- g(X,Z)\eta(Y)] + aA(X)\alpha^2[g(W,Z)\eta(Y) \\ &- g(Y,Z)\eta(W)] + aA(Y)\alpha^2[g(X,Z)\eta(W) \\ &- g(W,Z)\eta(X)] = \frac{1}{2n}A(W)[S(Y,Z)\eta(X) \\ &- S(X,Z)\eta(Y)] - B(W)[g(Y,Z)\eta(X) \\ &- g(X,Z)\eta(Y)] + \frac{1}{2n}A(X)[S(W,Z)\eta(Y) \\ &- S(Y,Z)\eta(W)] - B(X)[g(W,Z)\eta(Y) \\ &- g(Y,Z)\eta(W)] + \frac{1}{2n}A(Y)[S(X,Z)\eta(W) \\ &- S(W,Z)\eta(X)] - B(Y)[g(X,Z)\eta(W) \\ &- g(W,Z)\eta(X)]. \end{aligned} \tag{47}$$

Putting $Y = Z = e_i$ in (4.9) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$\begin{aligned} &a) \quad A(W)\eta(X) = A(X)\eta(W) \\ &b) \quad B(W)\eta(X) = B(X)\eta(W) \end{aligned} \tag{48}$$

for all vector fields X, W. Putting $X = \xi$ in (4.10), we obtain

$$\begin{aligned} &a) \quad \eta(W)\eta(\rho_1) + A(W) = 0 \\ &b) \quad \eta(W)\eta(\rho_1) + B(W) = 0 \end{aligned} \tag{49}$$

for all vector field W, $A(W) = g(W, \rho_1)$ and $B(W) = g(W, \rho_2)$ where ρ_1 and ρ_2 are vector fields associated with 1-forms A and B respectively.
Thus we can state the following theorem:

Theorem 4.2. *In a generalized projective ϕ-recurrent Lorentzian α-Sasakian manifold (M^{2n+1}) $(n \geq 1)$, the characteristic vector field ξ and the vector fields ρ_1 and ρ_2 associated to the 1-form A and B are co-directional and the 1-form A and B are given by* (4.11).

5. **Quasi-projectively flat Lorentzian α-Sasakian Manifold**

A Lorentzian α-Sasakian manifold M^n is said to be quasi-projectively flat if

$$g(P(\phi X, Y)Z, \phi W) = 0 \tag{50}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

In view of the equation (1.3), we have

$$\begin{aligned} g(P(X,Y)Z,W) &= g(R(X,Y)Z,W) - \frac{1}{2n}[S(Y,Z)g(X,W) \\ &- S(X,Z)g(Y,W)]. \end{aligned} \tag{51}$$

Putting $X = \phi X$ and $W = \phi W$ in the above equation, we get

$$\begin{aligned} g(P(\phi X,Y)Z,\phi W) &= g(R(\phi X,Y)Z,\phi W) - \frac{1}{2n}[S(Y,Z)g(\phi X,\phi W) \\ &- S(\phi X,Z)g(Y,\phi W)]. \end{aligned} \tag{52}$$

Now assume that M^n is quasi-projectively flat. Then by virtue of equations (5.1) and (5.3), we have

$$(53) g(R(\phi X,Y)Z,\phi W) = \frac{1}{n-1}[S(Y,Z)g(\phi X,\phi W) - S(\phi X,Z)g(Y,\phi W)].$$

Using equations (2.3) and (2.12) in the above equations, we get

$$\begin{aligned} g(R(\phi X,Y)Z,\phi W) &= \frac{1}{2n}[2n\alpha^2 g(Y,Z)\{g(X,W)+\eta(X)\eta(W)\} \\ &- S(\phi X,Z)g(Y,\phi W)]. \end{aligned} \tag{54}$$

Let $\{e_1, e_2, ..., e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M^n. Then $\{\phi e_1, \phi e_2, ..., \phi e_{2n}, \xi\}$ is also a local orthonormal basis of M^n. Putting $X = W = e_i$ in (5.5) and taking summation over i, $1 \leq i \leq 2n$, we get

$$\begin{aligned} \sum_{i=1}^{2n} g(R(\phi e_i,Y)Z,\phi e_i) &= \frac{1}{2n}[2n\alpha^2 g(Y,Z)\sum_{i=1}^{2n}\{g(e_i,e_i)+\eta(e_i)\eta(e_i)\} \\ &- \sum_{i=1}^{2n} S(\phi e_i,Z)g(Y,\phi e_i)]. \end{aligned} \tag{55}$$

Also,

$$\sum_{i=1}^{2n} g(R(\phi e_i,Y)Z,\phi e_i) = S(Y,Z)+g(Y,Z), \tag{56}$$

$$\sum_{i=1}^{2n} S(\phi e_i,Z)g(\phi e_i,Y) = S(Y,Z). \tag{57}$$

Hence, by virtue of equations (5.6), (5.7) and (5.8), the equation (5.5) takes the form

$$S(Y,Z) = ag(Y,Z), \tag{58}$$

where $a = \frac{\{\alpha^2(2n-1)-1\}2n}{2n+1}$. Hence we can state the following

Theorem 5.1. *A quasi-projectively flat Lorentzian α-Sasakian manifold is an Einstein manifold.*

6. **Example of 3-dimensional Lorentzian α-Sasakian manifold**

We consider a 3-dimensional manifold $M = \{(x, y, u) \in R^3\}$, where (x, y, u) are the standard coordinates of R^3. Let e_1, e_2, e_3 be the vector fields on M^3 given by

$$e_1 = e^{-u}\frac{\partial}{\partial x},\ e_2 = e^{-u}\frac{\partial}{\partial y},\ e_3 = e^{u}\frac{\partial}{\partial u}.$$

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vectors for each point of M and

hence a basis of $\chi(M)$. The Lorentzian metric g is defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0,$$
$$g(e_1, e_1) = 1,\ g(e_2, e_2) = 1,\ g(e_3, e_3) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$ and the $(1,1)$ tensor field ϕ is defined by

$$\phi e_1 = e_1,\ \phi e_2 = e_2,\ \phi e_3 = 0.$$

From the linearity of ϕ and g, we have

$$\eta(e_3) = -1,$$
$$\phi^2 X = X + \eta(X)e_3$$

and

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any $X \in \chi(M)$. Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = 0,\ [e_1, e_3] = e_1 e^u,\ [e_2, e_3] = e_2 e^u.$$

Koszul's formula is defined by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$$
$$-g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Then from above formula we can calculate the followings:

$$\nabla_{e_1} e_1 = e_3 e^u,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = e_1 e^u,$$
$$\nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = e_3 e^u,\ \nabla_{e_2} e_3 = e_2 e^u,$$
$$\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0.$$

From the above calculations, we see that the manifold under consideration satisfies $\eta(\xi) = -1$ and $\nabla_X \xi = -\alpha\phi X$ for $\alpha = -e^u$.
Hence the structure (ϕ, ξ, η, g) is a Lorentzian α-Sasakian manifold.

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References

[1] D.E. Blair, Inversion Theory and Conformal Mapping, Student Mathematical Library 9, American Mathematical Society, 2000.

[2] U.C. De, A.A. Shaikh, S.Biswas, On ϕ-recurrent Sasakian Manifolds, Novi Sad J.Math., 33, 13-48 (2003).

[3] U.C. De and G. Pathak, On 3-dimensional kenmotsu manifolds, Indian J.Pure Appl.Math., 35, 2, 159-165 (2004).

[4] W. Kuhnel, Conformal transformations between Einstein spaces, Conformal geometry(Bonn, 1985/1986), 105146, Aspects Math., E12, Vieweg, Braunschweig, 1988.

[5] R.S. Mishra, Structure on a differentiable manifold and their applications, Chandrama Prakashan, 50 A, Balrampur House, Allahabad, India (1984).

[6] B. Prasad, A pseudo projective curvature tensor on a Riemannian manifolds, Bull. Cal. Math. Soc. 94, 3, 163-166 (2002).

[7] D.G. Prakashs, C.S. Bagewadi, N.S.Basavarajappa, On pseudosymmetric Lorentzian α-Sasakian manifolds, IJPAM, 48, 1, 57-65 (2008).

[8] T. Takahashi, Sasakian symmetric spaces, Tohoku Math. J., 29, 91-113 (1977).

[9] S. Tanno, The Automorphism Groups of Almost Contact Riemannian Manifolds,, Tohoku Mathematical Journal, 21, 1, 21-38(1969).

[10] Venkatesha, C.S. Bagewadi, On 3-dimensional trans-Sasakian manifolds, AMSE, 42, 5, 63-73 (2005).

[11] A. Yildiz, C. Murathan, On Lorentzian α-Sasakian manifolds, Kyungpook Math. J., 45, 95-103 (2005).

[12] S. Yadav, D.L. Suthar, Certain derivation on Lorentzian α-Sasakian manifolds, Mathematics and Decision Science, 12, (2012).

[13] K. Yano, S. Bochner, Curvature and Betti numbers, Annals of Mathematics Studies, 32 (Princeton university press), 1953.

[14] A. Yildiz, M. Turan, B.F. Acet, On three dimensional Lorentzian α-Sasakian manifolds, Bulletin of Mathematical Analysis and Applications, 1, 3, 90-98 (2009).