

QUATERNIONIC OSCULATING CURVES IN EUCLIDEAN AND SEMI-EUCLIDEAN SPACE

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(Received: 1 September 2015, Accepted: 12 December 2015)

ABSTRACT. In this study, the osculating curves in Euclidean space $\mathbb{E}^3$ and $\mathbb{E}^4$, well known in differential geometry, are studied through the instrumentality of quaternions. We inoculate sundry delineations for quaternionic osculating curves in the Euclidean space $\mathbb{E}^3$, then we portray the quaternionic osculating curve in $\mathbb{E}^4$ as a quaternionic curve whose position vector every time reclines in the orthogonal complement $\mathbf{N}_2^\perp$ (or $\mathbf{N}_3^\perp$) of its first binormal vector field $\mathbf{N}_2$ (or $\mathbf{N}_3$), where $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3\}$ be the Frenet instrumentations of the quaternionic curve in the Euclidean space $\mathbb{E}^4$. We feature quaternionic osculating curves from the point of view their curvature functions K, k and $(r - K)$ and serve the necessary and the sufficient conditions for arbitrary quaternionic curve in $\mathbb{E}^4$ to be a quaternionic osculating. Moreover, we gain an explicit equation of a quaternionic osculating curve in $\mathbb{E}^4$. In the last two section, we described quaternionic osculating curves in the semi-Euclidean space and some theorems are testified.

AMS Classification: 11R52, 53A04, 53C50

Keywords: Osculating curve; Frenet equations; Curvature; Real quaternion

1. INTRODUCTION

Hamilton established quaternion in 1843. Substantial pacesetters to this study contained Euler's four square consubstantialities and Rodrigues parameterizations of common rotations by four parameters, but neither of these authors performed the four parameter rotations such an algebra. Gauss had also discovered quaternions in 1819, but this labour was not produced until 1900. Hamilton considered that the complex numbers could be explicated as if points in a plane, and he was seeking about a method to do the identical for points in three dimensional space. Points in space can be symbolized by their coordinates, which are triples of numbers, and for ages Hamilton had recognized how to suffix and deduct triples of numbers. On the other hand, Hamilton had been already smitten with the difficulty of multiplication and division. He could not solved how to estimate the division of the coordinates of two points in space.

The big development in quaternions at last appear in Dublin, while Hamilton was toddling with his wife, the remarks abaft quaternions were distilling in his memory. When the response struck him, Hamilton could not withstand the impulse to gouge out the formula for the quaternions,

$$i^2 = j^2 = k^2 = ijk = -1$$

into the stone of Brougham Bridge as he suspend on it.

Hamilton entitled a tetrad with these taxiings of multiplication a quaternion, and he distinguished most of the arrears of his porch to operating and prelecting them. Hamilton's cure is further geometric than the groovy approximate, which keynotes quaternions' algebraic properties. He based a school of "quaternionists", and he proved tobring fame quaternions in various books. Elements of Quaternions is one of them. Tait hold on raising quaternions. By the way, quaternions were a imperative exam subject. Subjects in physics and geometry that would now be defined handling vectors, i.e. kinematics and Maxwell's equations, were declared thoroughly in the way of quaternions.

Quaternions started to be substituted by vector analysis, which had been advanced by Gibbs and the others. In addition to this, quaternions have had a invigoration for the tardy 20th Century, in the first place in so far as their benefit in depecting spatial rotations. The notations of rotations by quaternions are more

harden and sooner to calculate than the notations by matrices. Besides, different from Euler angles they are not delicate to gimbal lock. Therefore, quaternions are put account in computer graphics, computer vision, robotics, control theory, signal processing, attitude control, physics, bioinformatics, molecular dynamics, computer simulations, and orbital mechanics. For instance, it is general for the attitude-control systems of spacecraft to be controlled in the sense of quaternions.

The set of quaternions is indicated $\mathbb{H}$ and the quaternions are a unit sample of a upward plenary class of hypercomplex numbers. The quaternions are not commutative, but they are associative, and they model a group conversant as the quaternion group. By comparing with the complex numbers being offerable such a aggregate of real and imaginary sections, $a_01 + a_1i$, a quaternion can also be marked such a linear combination

$$a_01 + a_1i + a_2j + a_3k.$$

As a set, the quaternions $\mathbb{H}$ are coincide with four dimensional vector space over the real numbers. $\mathbb{H}$ has three processes: addition, scalar multiplication and quaternion multiplication. The sum of two elements of $\mathbb{H}$ is described to be their aggregate such elements of $\mathbb{E}^4$. In the same manner, the product of an element of $\mathbb{H}$ by a real number is qualified to be even such the product in $\mathbb{E}^4$. To determine the product of two elements in $\mathbb{H}$ needs a option of basis for $\mathbb{E}^4$. The members of this basis are consuetudinary showed as $1, i, j$ and k . All member of $\mathbb{H}$ can be uniquely written as a linear combination of these basis members, that is, as $a_01 + a_1i + a_2j + a_3k$, where a_0, a_1, a_2 , and a_3 are real numbers. The basis element 1 will be the identicalness member of $\mathbb{H}$, meaning that multiplication by 1 does nothing, and due to this reason, member of $\mathbb{H}$ are commonly written $a_01 + a_1i + a_2j + a_3k$, repressive the basis member 1 . Given this basis, associative quaternion multiplication is described by first defining the products of basis elements and then defining all other products using the distributive law. Quaternions are explained widely in [1].

Bharathi and Nagaraj treated a subject is named as quaternionic curve in $\mathbb{E}^3$ and $\mathbb{E}^4$ [2]. Sivridağ, Güneş and Keleş narrated quaternionic curve in $\mathbb{D}^3$ and $\mathbb{D}^4$, [3]. Çöken and Tuna [4] adapted this concept to semi-Euclidean space. After these performances, the purport quaternionic curve acquires influence in differential geometry. Then all of the geometrical areas are depastured this notation. Theory of curve is one of them. Moreover, the authors ask about "How to transfer this

curve quaternionic form?” Now, we serve samples of these studies: quaternionic Smarandache curves [5, 6], quaternionic inclined curves [4], quaternionic Bertrand curve [7, 8], quaternionic slant helices [9, 10], quaternionic rectifying curves [11, 12], quaternionic Mannheim curves [13], quaternionic Salkowski and similar curves [14], quaternionic involute-evolute curves [15, 16], quaternionic normal curves [17, 18].

When we make a literature review, we fixate that there is no paper about quaternionic osculating curve, in $\mathbb{E}^3$ and $\mathbb{E}^4$. Because of this reason we decided to work on this matter. As is known literature review is very important for quaternionic curve. The concept quaternion was scrutinised in first part of introduction, and the study of quaternionic curve was boned upp, finally if we use the definition of osculating curve in $\mathbb{E}^3$ and $\mathbb{E}^4$. We have to give more information about osculating curve in $\mathbb{E}^3$ and $\mathbb{E}^4$ the curves for which the radius vector all the time recline in their osculating plane, are for primitiveness denominated osculating curves. Hence, the radius vector as far as certain opted origin, of a osculating curve $\mathbb{E}^3$, secures the statement

$$(1.1) \quad \alpha(s) = \lambda(s) \mathbf{t}(s) + \mu(s) \mathbf{n}(s),$$

where λ and μ are random differentiate functions in regard to the arc length parameter $s \in I \subset \mathbb{R}$. In [19], İlarıslan and Nesovic inoculate sundry delineations for osculating curves in the Euclidean space $\mathbb{E}^3$, then they labeled the osculating curve in $\mathbb{E}^4$ as a curve whose radius vector all the time recline in the orthogonal integrant $N_2^\perp$ (or $N_3^\perp$) of its first binormal vector field N_2 (or N_3). Herewith, $N_2^\perp$ is termed by

$$N_2^\perp = \{K \in \mathbb{E}^4, \langle K, N_2 \rangle = 0\}.$$

$N_2^\perp$ is a three dimensional subspace of $\mathbb{E}^4$, spanned by the tangent, the principal normal, and the second binormal vector fields T , N_1 and N_3 respectively, [19]. Therefore, the radius vector due to certain opted origin, of a osculating curve in $\mathbb{E}^4$, secures the statement

$$(1.2) \quad \alpha(s) = \lambda(s) \mathbf{T}(s) + \mu(s) \mathbf{N}_1(s) + \nu(s) \mathbf{N}_3(s)$$

for certain differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ in arc length function s , [19]. Along the same line, $N_3^\perp$ is bestowed by

$$N_3^\perp = \{K \in \mathbb{E}^4, \langle K, N_3 \rangle = 0\}.$$

$N_3^\perp$ is a three dimensional subspace of $\mathbb{E}^4$, spanned by the tangent, the principal normal, and the second binormal vector fields T , N_1 and N_2 respectively, [19]. Therefore, the radius vector due to certain opted origin, of a osculating curve in $\mathbb{E}^4$, secures the statement

$$(1.3) \quad \alpha(s) = \lambda(s) \mathbf{T}(s) + \mu(s) \mathbf{N}_1(s) + \nu(s) \mathbf{N}_2(s).$$

In this paper, we use the equation (1.3).

2. PRELIMINARIES

In this section we epitomise real quaternion. A real quaternion is labeled with $q = a_0 + a_1i + a_2j + a_3k$ (or $q = S_q + \mathbf{V}_q$, where the attributes $S_q = a_0$ and $\mathbf{V}_q = a_1i + a_2j + a_3k$ signify scalar and vector compartment of q such that

$$\begin{aligned} \text{i)} \quad & i^2 = j^2 = k^2 = ijk = -1 \\ \text{ii)} \quad & ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik, \end{aligned}$$

where (ijk) is an even permutation of (123) in the Euclidean space $\mathbb{E}^4$. The quaternion product of two real quaternions is served as follows:

$$(2.1) \quad p \times q = S_p S_q - \langle \mathbf{V}_p, \mathbf{V}_q \rangle + S_p \mathbf{V}_q + S_q \mathbf{V}_p + \mathbf{V}_p \wedge \mathbf{V}_q, \forall p, q \in H,$$

where we have used the dot and cross products in $\mathbb{E}^3$. $\bar{q}$ is called conjugate of q and described as noted below:

$$\bar{q} = S_q - \mathbf{V}_q = a_0 - a_1i - a_2j - a_3k.$$

This defines the symmetric real-valued, non-degenerate, bilinear form as follows:

$$\langle p, q \rangle : H \times H \rightarrow \mathbb{R}, (p, q) \rightarrow \langle p, q \rangle = \frac{1}{2} (p \times \bar{q} + q \times \bar{p}).$$

Hence it is called the quaternion inner product. The norm of a real quaternion q is

$$\|q\| = q \times \bar{q} + q \times \bar{q} = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

If $\|q\| = 1$, then q is called a unit quaternion. q is called a spatial quaternion whenever $q + \bar{q} = 0$, [2].

Definition 2.1. *Let v be a vector on a curve, in this position this vector can be decoupled uniquely as a sum of two vectors, one tangent to the curve, named the tangential component of the vector, and another one perpendicular to the curve, named the normal component of the vector. It means that, let given a curve γ :*

$I \subset \mathbb{R} \rightarrow \mathbb{E}^3$, and $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$ be the Frenet trihedron in the point $\gamma(s)$ of the curve γ . We can write $\gamma(s) = a \mathbf{t}(s) + b \mathbf{n}_1(s) + c \mathbf{n}_2(s)$, in this position $\langle \gamma(s), \mathbf{t}(s) \rangle = a$ and $\langle \gamma(s), \mathbf{n}_1(s) \rangle = b$ is the tangential component and the principal normal component of the radius vector of a unit speed curve $\gamma(s)$ in $\mathbb{E}^3$, respectively [19].

3. QUATERNIONIC OSCULATING CURVES IN E^3

$\mathbb{E}^3$ is consubstantiated the space of spatial real quaternion

$$\mathbb{H}_{pure} = \left\{ \gamma \in \mathbb{H} : \gamma + \bar{\gamma} = 0 \right\}.$$

Let $I = [0, 1]$ be an interval in $\mathbb{R}$ and $s \in I$ be the parameter along the smooth curve

$$\begin{aligned} \gamma : I \subset \mathbb{R} &\rightarrow \mathbb{H}_{pure} \\ s &\rightarrow \gamma(s) = \gamma_1(s)i + \gamma_2(s)j + \gamma_3(s)k, \end{aligned}$$

opted in fact the tangent $\mathbf{t} = \gamma'(s)$ has unit length $\|\mathbf{t}\| = 1$ for all s . This state provides;

$$\mathbf{t}' \times \alpha \mathbf{t} + \mathbf{t} \times \alpha \mathbf{t}' = 0.$$

The last equation observes that $\mathbf{t}'$ is orthogonal to $\mathbf{t}$ and $\mathbf{t}' \times \alpha \mathbf{t}$ is a spatial real quaternion. Let $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$ be the Frenet trihedron in the point $\gamma(s)$ of the curve γ . Then Frenet equations are

$$\begin{aligned} \mathbf{t}'(s) &= k \mathbf{n}_1(s) \\ \mathbf{n}_1'(s) &= -k \mathbf{t}(s) + r \mathbf{n}_2(s) \\ \mathbf{n}_2'(s) &= -r \mathbf{n}_1(s), \end{aligned} \tag{3.1}$$

where $\mathbf{t}$ is the unit tangent, $\mathbf{n}_1$ is the unit principal normal, $\mathbf{n}_2$ is the unit binormal, k is the principal curvature and r is the torsion of the real quaternionic curve γ , [2]. The radius vector of the spatial quaternionic osculating curve provides the following equation

$$\gamma(s) = \lambda(s) \mathbf{t}(s) + \mu(s) \mathbf{n}_1(s),$$

where λ and μ are arbitrary differentiable functions.

Theorem 3.1. *Let $\gamma(s)$ be a unit speed regular spatial quaternionic curve in $\mathbb{H}_{pure}$. Then γ is a spatial quaternionic osculating curve if and only if γ is a straight line (in terms of quaternionic) or planar curve (in terms of quaternionic).*

Proof. Suppose that $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{H}_{pure}$ is parameterized by the arc length function s . Let γ is a spatial quaternionic osculating curve. Then the radius vector $\gamma(s)$ of the curve γ provides the folloowing statement

$$(3.2) \quad \gamma(s) = \lambda(s) \mathbf{t}(s) + \mu(s) \mathbf{n}_1(s)$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ in arc length function s . By differentiating the radius vector (3.2) with respect to s and using the Frenet equation (3.1) we get

$$\mathbf{t} = (\lambda' - \mu k) \mathbf{t} + (\lambda k + \mu') \mathbf{n}_1 + \mu r \mathbf{n}_2.$$

It follows that

$$(3.3) \quad \begin{aligned} \lambda' - \mu k &= 1 \\ \lambda k + \mu' &= 0 \\ \mu r &= 0. \end{aligned}$$

From the last equation $\mu r = 0$, we have $\mu = 0$ or $r = 0$.

- (1) Suppose that $\mu = 0$, in this position we can write $\mu' = 0$, from the last equation we get $\lambda k = 0$ and $\lambda' = 1$. Thus $\lambda(s) \neq 0$ and $k = 0$. In this position γ is a straight line (in terms of quaternionic).
- (2) Suppose that $r = 0$, then γ is a planar curve (in terms of quaternionic).

Conversely, let γ be a straight line (in terms of quaternionic) or planar curve (in terms of quaternionic), then γ is a spatial quaternionic osculating curve. $\square$

Theorem 3.2. Let $\gamma(s)$ be a unit speed spatial quaternionic osculating in $\mathbb{H}_{pure}$ then tangential component a (i.e., $\langle \gamma(s), \mathbf{t}(s) \rangle = a$) and the principal normal component b (i.e., $\langle \gamma(s), \mathbf{n}_1(s) \rangle = b$) of the radius vector of the spatial quaternionic osculating curve secures the following equation

$$(3.4) \quad a^2(s) + b^2(s) = 2 \int a(s) ds.$$

Conversely, if the tangential component a (i.e., $\langle \gamma(s), \mathbf{t}(s) \rangle = a$) and the principal normal component b (i.e., $\langle \gamma(s), \mathbf{n}_1(s) \rangle = b$) of the position vector of a unit speed spatial quaternionic curve $\gamma(s)$ in $\mathbb{E}^3$ satisfy the equation (3.4) then γ is a spatial quaternionic osculating curve, or a spatial quaternionic rectifying curve in $\mathbb{E}^3$.

Proof. Let $\gamma(s)$ be a unit speed spatial quaternionic osculating in $\mathbb{H}_{pure}$. From (3.3) we can write,

$$\begin{aligned}\lambda\lambda' - \lambda\mu k &= \lambda \\ \mu\lambda k + \mu\mu' &= 0.\end{aligned}$$

It follows that

$$\lambda\lambda' + \mu\mu' = \lambda.$$

Because of the fact that $\gamma(s) = \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{n}_1(s)$ we get

$$\begin{aligned}a(s) = \langle \gamma(s), \mathbf{t}(s) \rangle &= \langle \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{n}_1(s), \mathbf{t}(s) \rangle \\ &= \lambda(s)\langle \mathbf{t}(s), \mathbf{t}(s) \rangle + \mu(s)\langle \mathbf{n}_1(s), \mathbf{t}(s) \rangle \\ &\Rightarrow a(s) = \lambda(s),\end{aligned}$$

and similarly $b(s) = \mu(s)$. Thus, we have

$$a(s)a'(s) + b(s)b'(s) = a(s).$$

In this way

$$(3.5) \quad a^2(s) + b^2(s) = 2 \int a(s) ds.$$

Conversely, we assume that the tangential component a (i.e., $\langle \gamma(s), \mathbf{t}(s) \rangle = a$) and the principal normal component b (i.e., $\langle \gamma(s), \mathbf{n}_1(s) \rangle = b$) of the position vector of a unit speed spatial quaternionic curve $\gamma(s)$ in $\mathbb{R}^3$ satisfy the equation (3.5). Then we get from (3.5)

$$(3.6) \quad a(s)a'(s) + b(s)b'(s) = a(s),$$

where $a'(s) = \frac{d}{ds} \langle \gamma(s), \mathbf{t}(s) \rangle$ and $b'(s) = \frac{d}{ds} \langle \gamma(s), \mathbf{n}_1(s) \rangle$. By using (3.1), we obtain from (3.6)

$$a(s)(1 + k(s)\langle \gamma(s), \mathbf{n}_1(s) \rangle) + b(s)(r(s)\langle \gamma(s), \mathbf{n}_2(s) \rangle - k(s)\langle \gamma(s), \mathbf{t}(s) \rangle) = a(s)$$

and from this equation, we can write

$$a(s) + a(s)k(s)\langle \gamma(s), \mathbf{n}_1(s) \rangle + b(s)r(s)\langle \gamma(s), \mathbf{n}_2(s) \rangle - b(s)k(s)\langle \gamma(s), \mathbf{t}(s) \rangle = a(s)$$

or

$$a(s)k(s)b(s) + r(s)\langle \gamma(s), \mathbf{n}_1(s) \rangle \langle \gamma(s), \mathbf{n}_2(s) \rangle - b(s)k(s)a(s)$$

$$(3.7) \quad r(s)\langle \gamma(s), \mathbf{n}_1(s) \rangle \langle \gamma(s), \mathbf{n}_2(s) \rangle = 0.$$

From (3.7), we have $r(s) = 0$, $\langle \gamma(s), \mathbf{n}_1(s) \rangle = 0$ or $\langle \gamma(s), \mathbf{n}_2(s) \rangle = 0$. If $r(s) = 0$, which means that $\gamma(s)$ is a planar curve (in terms of quaternionic) and using Theorem 3.1 we get $\gamma(s)$ is a spatial quaternionic osculating curve. If $\langle \gamma(s), \mathbf{n}_1(s) \rangle = 0$, then $\gamma(s)$ is a spatial quaternionic rectifying curve. If $\langle \gamma(s), \mathbf{n}_2(s) \rangle = 0$, then $\gamma(s)$ is a spatial quaternionic osculating curve. $\square$

4. QUATERNIONIC OSCULATING CURVES IN $\mathbb{E}^4$

The four dimensional Euclidean space $\mathbb{E}^4$ is assimilated with the space of unit real quaternion. Let

$$\begin{aligned} \beta : I \subset \mathbb{R} &\rightarrow \mathbb{H} \\ s &\rightarrow \beta(s) = \gamma_0(s) + \gamma_1(s)i + \gamma_2(s)j + \gamma_3(s)k \end{aligned}$$

be a smooth curve in $\mathbb{E}^4$ described over I . Let the parameter s be selected that $\mathbf{T} = \beta'(s)$ has unit magnitude. Let $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3\}$ be the Frenet elements of β . Then the Frenet equations are

$$\begin{aligned} \mathbf{T}'(s) &= K\mathbf{N}_1(s) \\ \mathbf{N}_1'(s) &= -K\mathbf{T}(s) + k\mathbf{N}_2(s) \\ \mathbf{N}_2'(s) &= -k\mathbf{N}_1(s) + (r - K)\mathbf{N}_3(s) \\ \mathbf{N}_3'(s) &= -(r - K)\mathbf{N}_2(s), \end{aligned} \tag{4.1}$$

where $\mathbf{N}_1 = \mathbf{t} \times \mathbf{T}$, $\mathbf{N}_2 = \mathbf{n}_1 \times \mathbf{T}$, $\mathbf{N}_3 = \mathbf{n}_2 \times \mathbf{T}$ and $K = \|\mathbf{T}'(s)\|$.

It is gained the Serret-Frenet formulae in [2] and the elements for β by using of the Serret-Frenet formulae for γ in $\mathbb{E}^3$. Even if, there are connection with curvatures of β and γ . These connections can be elucidated that the torsion of β is the principal curvature of γ . Separately, the bitorsion of β is $(r - K)$, where r is the torsion of γ and K is the principal curvature of β . These connections are only specified for quaternions, [2].

Now, we depict the quaternionic osculating curve in $\mathbb{H}$ as far as curvatures. Let $\beta = \beta(s)$ be unit speed quaternionic osculating curve in $\mathbb{H}$, with nonzero curvatures $K(s)$, $k(s)$ and $(r(s) - K(s))$. Then, the radius vector $\beta(s)$ of the curve β secures the following equation

$$\beta(s) = \lambda(s)\mathbf{T}(s) + \mu(s)\mathbf{N}_1(s) + \nu(s)\mathbf{N}_3(s) \tag{4.2}$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$. Therefore, we put up the following theorems of quaternionic osculating curve β .

Theorem 4.1. *Let $\beta(s)$ be a unit speed quaternionic osculating curve in $\mathbb{H}$, with non-zero curvatures $K(s)$, $k(s)$ and $(r(s) - K(s))$. Then β is equal to a quaternionic osculating curve if and only if*

$$(4.3) \quad \left(\frac{1}{K} \left(\frac{r-K}{k} \right)' \right)' + \left(\frac{r-K}{k} \right) K = -\frac{1}{c}, \quad c \in \mathbb{R}.$$

Proof. Let $\beta = \beta(s)$ be a unit speed quaternionic osculating curve and $K(s)$, $k(s)$ and $(r(s) - K(s))$ be non-zero curvatures of β . Then, the position vector $\beta(s)$ of the curve β secures the following equation

$$\beta(s) = \lambda(s) \mathbf{T}(s) + \mu(s) \mathbf{N}_1(s) + \nu(s) \mathbf{N}_3(s)$$

where $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are certain differentiable functions. calculate a derivative this balance in terms of the arc length function s and handling the Frenet equations (4.1), we rout out that,

$$\mathbf{T} = (\lambda' - \mu K) \mathbf{T} + (\lambda K + \mu') \mathbf{N}_1 + (\mu k - \nu(r - K)) \mathbf{N}_2 + \nu' \mathbf{N}_3.$$

It pursues that

$$(4.4) \quad \begin{aligned} \lambda' - \mu K &= 1 \\ \lambda K + \mu' &= 0 \\ \mu k - \nu(r - K) &= 0 \\ \nu' &= 0 \end{aligned}$$

and herewith

$$(4.5) \quad \begin{aligned} \lambda(s) &= -c \frac{1}{K} \left(\frac{r-K}{k} \right)' \\ \mu(s) &= c \left(\frac{r-K}{k} \right) \\ \nu(s) &= c, \end{aligned}$$

where $c \in \mathbb{R}$. For that matter, the functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are denoted to the degree the curvature functions $K(s)$, $k(s)$ and $(r(s) - K(s))$ of the quaternionic osculating curve β . Even, by impositioning the first balance in (4.4) and connection (4.5) we rout out that curvatures $K(s)$, $k(s)$ and $(r(s) - K(s))$ sprovide the equation (4.3).

Conversely, supposing that the curvatures $K(s)$, $k(s)$ and $(r(s) - K(s))$, of an random unit speed quaternionic osculating curve β in $\mathbb{H}$, provide the eq. (4.3).

Let's say the vector $X \in \mathbb{H}$ endowed by

$$X(s) = \beta(s) + c \frac{1}{K} \left(\frac{r-K}{k} \right)' \mathbf{T}(s) - c \left(\frac{r-K}{k} \right) \mathbf{N}_1(s) - c \mathbf{N}_3(s).$$

By imposing the connections (4.1) and (4.3), we rout out $X'(s) = 0$, which means that X is a constant vector. This comprises that β is equal to a quaternionic osculating curve. $\square$

Theorem 4.2. *Let $\beta(s)$ be a unit speed quaternionic osculating curve in H , with non-zero curvatures $K(s)$, $k(s)$ and $(r(s) - K(s))$. Then the following statements insure:*

(i). *The curvatures $K(s)$, $k(s)$ and $(r(s) - K(s))$ secures the following equation*

$$(4.6) \quad r(s) = K(s) + k(s) \left\{ \left[\frac{1}{c} \int \sin \left(\int K(s) ds \right) ds + c_1 \right] \cos \left(\int K(s) ds \right) - \left[\frac{1}{c} \int \cos \left(\int K(s) ds \right) ds + c_2 \right] \sin \left(\int K(s) ds \right) \right\}.$$

(ii). *The tangential component and the principal normal component of the radius vector of the quaternionic osculating curves are, respectively bestowed on by*

$$(4.7) \quad \begin{aligned} \langle \beta(s), \mathbf{T}(s) \rangle &= -c \frac{1}{K} \left(\frac{r-K}{k} \right)' \\ \langle \beta(s), \mathbf{N}_1(s) \rangle &= c \left(\frac{r-K}{k} \right) \end{aligned}$$

(iii). *The second binormal component of the radius vector of the quaternionic osculating curve is non-zero constant.*

Conversely, if $\beta(s)$ is a unit speed quaternionic osculating curves in $\mathbb{H}$ with non-zero curvatures $K(s)$, $k(s)$ and $(r(s) - K(s))$ and one of the equations (i), (ii) or (iii) holds, then β is a quaternionic osculating curve or equal to a quaternionic osculating curve.

Proof. Let us first suppose that $\beta(s)$ be a unit speed quaternionic osculating curve in H , with non-zero curvatures $K(s)$, $k(s)$ and $(r(s) - K(s))$. The position vector of the curve satisfies the equation (4.3). If we express $y(s) = \frac{r(s)-K(s)}{k(s)}$ and $p(s) = \frac{1}{K(s)}$, then equation (4.3) can be set down as

$$\frac{d}{ds} \left(p(s) \frac{dy}{ds} \right) + \frac{y(s)}{p(s)} = -\frac{1}{c}, \quad c \in \mathbb{R}.$$

If we shift variances in overhead statement as $t(s) = \int \frac{1}{p(s)} ds$, then we meet

$$\frac{d^2 y}{dt^2} + y = -\frac{1}{cK}, \quad c \in \mathbb{R},$$

the settlement of this differential equation is

$$y = \left(\frac{1}{c} \int \frac{\sin t}{K} dt + c_1 \right) \cos t + \left(-\frac{1}{c} \int \frac{\cos t}{K} dt + c_2 \right) \sin t$$

where c, c_1 and $c_2 \in \mathbb{R}$. Let's say that, $y(s) = \frac{r(s)-K(s)}{k(s)}$ and $dt = K(s)ds$, we get

$$\begin{aligned} r(s) = & K(s) + k(s) \left\{ \left[\frac{1}{c} \int \sin \left(\int K(s) ds \right) ds + c_1 \right] \cos \left(\int K(s) ds \right) \right. \\ & \left. - \left[\frac{1}{c} \int \cos \left(\int K(s) ds \right) ds + c_2 \right] \sin \left(\int K(s) ds \right) \right\}. \end{aligned}$$

Thus we prove the statement (i). By exercising the connections (4.2) and (4.5), we can write the radius vector of the curve as pursue:

$$(4.8) \quad \beta(s) = -c \frac{1}{K} \left(\frac{r-K}{k} \right)' \mathbf{T}(s) + c \left(\frac{r-K}{k} \right) \mathbf{N}_1(s) + c \mathbf{N}_3(s).$$

From here, we get $\langle \beta(s), \mathbf{T}(s) \rangle = -c \frac{1}{K} \left(\frac{r-K}{k} \right)'$, $\langle \beta(s), \mathbf{N}_1(s) \rangle = c \left(\frac{r-K}{k} \right)$ and $\langle \beta(s), \mathbf{N}_3(s) \rangle = c$, $c \in \mathbb{R}$. Thus we proved the statements (ii) and (iii). Conversely, consider that statement (i) holds. Then the curvature functions $k(s)$, $K(s)$ and $(r(s) - K(s))$ provides the equation

$$\begin{aligned} r(s) = & K(s) + k(s) \left\{ \left[\frac{1}{c} \int \sin \left(\int K(s) ds \right) ds + c_1 \right] \cos \left(\int K(s) ds \right) \right. \\ & \left. - \left[\frac{1}{c} \int \cos \left(\int K(s) ds \right) ds + c_2 \right] \sin \left(\int K(s) ds \right) \right\}. \end{aligned}$$

By differentiating the last equation two times in terms of s we get,

$$\left(\frac{1}{K} \left(\frac{r-K}{k} \right)' \right)' + \left(\frac{r-K}{k} \right) K = -\frac{1}{c}, \quad c \in \mathbb{R}.$$

In this position β is equal to a quaternionic osculating curve. Next let's suppose statements (ii) holds. By taking derivative of $\langle \beta(s), \mathbf{N}_1(s) \rangle = c \left(\frac{r-K}{k} \right)$ with respect to s and using (4.1) we get,

$$-Kh(\beta(s), \mathbf{T}(s)) + kh(\beta(s), \mathbf{N}_2(s)) = c \left(\frac{r-K}{k} \right)'.$$

By using $\langle \beta(s), \mathbf{T}(s) \rangle = -c \frac{1}{K} \left(\frac{r-K}{k} \right)'$ and $k \neq 0$, we get $\langle \beta(s), \mathbf{N}_2(s) \rangle = 0$, which means that β is equal osculating curve.

If statement (iii) holds, then we have $\langle \beta(s), \mathbf{N}_3(s) \rangle = c$, $c \in \mathbb{R}$. By differentiating the last equation in terms of s and wielding (4.3), we get

$$-(r-K) \langle \beta(s), \mathbf{N}_2(s) \rangle = 0$$

It pursues that $\langle \beta(s), \mathbf{N}_2(s) \rangle = 0$ and hence the curve is a osculating curve. This proves the theorem. $\square$

5. QUATERNIONIC OSCULATING CURVES IN $\mathbb{E}_1^3$ AND $\mathbb{E}_2^4$

A semi real quaternion is described by $q = ae_1 + be_2 + ce_3 + de_4$ such that

$$(5.1) \quad \begin{aligned} e_i \times e_i &= -\varepsilon(e_i), \quad 1 \leq i \leq 3 \\ e_i \times e_j &= \varepsilon(e_i) \varepsilon(e_j) e_k, \text{ in } \mathbb{E}_1^3 \\ e_i \times e_j &= -\varepsilon(e_i) \varepsilon(e_j) e_k, \text{ in } \mathbb{E}_2^3, \end{aligned}$$

where (ijk) is an even permutation of (123) . Remark here that we denote the set of all semi real quaternions by $\mathbb{H}_{semi}$ where *semi* demonstrates index = 1, 2.

$$\mathbb{H}_{semi} = \left\{ \begin{array}{l} q = ae_1 + be_2 + ce_3 + de_4; \quad a, b, c, d \in \mathbb{R}, \quad e_1, e_2, e_3 \in \mathbb{E}_1^3 \\ \langle e_i, e_i \rangle_{semi} = \varepsilon(e_i), \quad 1 \leq i \leq 3 \end{array} \right\}$$

If e_i is a spacelike or timelike vector, then $\varepsilon(e_i) = +1$ or -1 respectively. For $p = S_p + \mathbf{V}_p$ and $q = S_q + \mathbf{V}_q$ the multiplication of two semi real quaternions p and q is defined as follows

$$p \times q = S_p S_q - \langle \mathbf{V}_p, \mathbf{V}_q \rangle + S_p \mathbf{V}_q + S_q \mathbf{V}_p + \mathbf{V}_p \wedge \mathbf{V}_q, \quad \forall p, q \in \mathbb{H}_{semi},$$

where we have rided the scalar and cross products in E_1^3 . The conjugate of a quaternion, $\bar{q} = -ae_1 - be_2 - ce_3 + d$ for every $q \in \mathbb{H}_{semi}$. This helps to define the symmetric, non-degenerate, bilinear form $\langle \cdot, \cdot \rangle_{semi}$ as follows:

$$\langle \cdot, \cdot \rangle_{semi} : \mathbb{H}_{semi} \times \mathbb{H}_{semi} \rightarrow \mathbb{R}$$

$$\langle p, q \rangle_{pure-semi} = \frac{1}{2} [\varepsilon(p) \varepsilon(\bar{q}) (p \times \bar{q}) + \varepsilon(q) \varepsilon(\bar{p}) (q \times \bar{p})] \varepsilon(p) \varepsilon(\bar{q}) (p \times \bar{q}) \quad \text{for } \mathbb{E}_1^3$$

$$\langle p, q \rangle_{semi} = -\frac{1}{2} [\varepsilon(p) \varepsilon(\bar{q}) (p \times \bar{q}) + \varepsilon(q) \varepsilon(\bar{p}) (q \times \bar{p})] \varepsilon(p) \varepsilon(\bar{q}) (p \times \bar{q}) \quad \text{for } \mathbb{E}_2^4.$$

The norm of semi real quaternion q is denoted by $\|q\|_{semi} = |\langle q, q \rangle_{semi}| = |\bar{p}(q \times \bar{q})| = |-a^2 - b^2 + c^2 + d^2|$ for $p, q \in H_v$ where if $\langle p, q \rangle_{semi} = 0$ then p and q are called

orthogonal. q is called a spatial semi real quaternion whenever $q + \bar{q} = 0$ [5]. The Serret-Frenet formulae for semi real quaternionic curves in $\mathbb{E}_1^3$ and $\mathbb{E}_2^4$ are as follows:

The three dimensional semi-Euclidean space $\mathbb{E}_1^3$ is assimilate into the space of spatial quaternions $\mathbb{H}_{pure-semi} = \{\gamma \in \mathbb{H}_{semi} \mid \gamma \times \bar{\gamma} = 0\}$. Let $I = [0, 1]$ be an interval in $\mathbb{R}$ and

$$\begin{aligned} \Gamma : I \subset \mathbb{R} &\rightarrow \mathbb{H}_{pure-semi} \\ s &\rightarrow \Gamma(s) = \sum_{i=1}^3 \gamma_i(s) e_i, \quad (1 \leq i \leq 3) \end{aligned}$$

be an arc lengthed curve with nonzero curvatures. Let $\{k, r\}$ and $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$ denotes the Frenet frame of γ . Then Serret-Frenet formulae are endowed by

$$\begin{aligned} (5.2) \quad \mathbf{t}'(s) &= \varepsilon_{\mathbf{n}_1} k \mathbf{n}_1(s) \\ \mathbf{n}_1'(s) &= -\varepsilon_{\mathbf{t}} k \mathbf{t}(s) + \varepsilon_{\mathbf{n}_1} r \mathbf{n}_2(s) \\ \mathbf{n}_2'(s) &= -\varepsilon_{\mathbf{n}_2} r \mathbf{n}_1(s), \end{aligned}$$

where k is the principal curvature, r is torsion of a γ and

$$\langle \mathbf{t}, \mathbf{t} \rangle_{semi} = \varepsilon_{\mathbf{t}}, \langle \mathbf{n}_1, \mathbf{n}_1 \rangle_{semi} = \varepsilon_{\mathbf{n}_1}, \langle \mathbf{n}_2, \mathbf{n}_2 \rangle_{semi} = \varepsilon_{\mathbf{n}_2} [4].$$

The four dimensional semi-Euclidean space $\mathbb{E}_2^4$ is assimilate into the space of unit quaternions. Let $I = [0, 1]$ be an unit interval in the real line $\mathbb{R}$ and

$$\begin{aligned} \beta : I \subset \mathbb{R} &\rightarrow H_{semi} \\ s &\rightarrow \beta(s) = \sum_{i=1}^4 \gamma_i(s) e_i, e_4 = +1. \end{aligned}$$

be a smooth curve in $\mathbb{E}_2^4$ with nonzero curvatures Let $\{K, k, r - K\}$ and the Frenet frame of the curve β is $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3\}$. Then Serret-Frenet formulae are endowed by

$$\begin{aligned} (5.3) \quad \mathbf{T}'(s) &= \varepsilon_{\mathbf{N}_1} K \mathbf{N}_1(s) \\ \mathbf{N}_1'(s) &= -\varepsilon_{\mathbf{N}_1} \varepsilon_{\mathbf{t}} K \mathbf{T}(s) + \varepsilon_{\mathbf{n}_1} k \mathbf{N}_2(s) \\ \mathbf{N}_2'(s) &= -\varepsilon_{\mathbf{t}} k \mathbf{N}_1(s) + \varepsilon_{\mathbf{n}_1} (r - K \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}) \mathbf{N}_3(s) \\ \mathbf{N}_3'(s) &= -\varepsilon_{\mathbf{n}_2} (r - K \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}) \mathbf{N}_2(s), \end{aligned}$$

where $\langle \mathbf{T}, \mathbf{T} \rangle_{semi} = \varepsilon_{\mathbf{T}}, \langle \mathbf{N}_1, \mathbf{N}_1 \rangle_{semi} = \varepsilon_{\mathbf{N}_1}, \langle \mathbf{N}_2, \mathbf{N}_2 \rangle_{semi} = \varepsilon_{\mathbf{N}_2}$ and $\mathbf{T}(s)$ the tangent vector of β and $K = \varepsilon_{\mathbf{N}_1} \left\| \mathbf{T}'(s) \right\|_{semi}$, [4].

Theorem 5.1. *Let $\gamma : I \subset \mathbb{R} \rightarrow H_{pure-semi}$ be a unit speed regular spatial semi real quaternionic curve. Then γ is a spatial semi real quaternionic osculating curve if and only if γ is a straight line (in terms of spatial semi real quaternionic) or planar curve (in terms of spatial semi quaternionic).*

Theorem 5.2. Let $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{H}_{\text{pure-semi}}$ be a unit speed spatial semi real quaternionic curve, then tangential component a (i.e., $\langle \gamma(s), \mathbf{t}(s) \rangle_{\text{pure-semi}} = a$) and the principal normal component b (i.e., $\langle \gamma(s), \mathbf{n}_1(s) \rangle_{\text{pure-semi}} = b$) of the radius vector of the spatial semi real quaternionic curve secures the following equation:

$$a^2(s) + b^2(s) = 2 \int a(s) ds.$$

Conversely, if the tangential component a (i.e., $\langle \gamma(s), \mathbf{t}(s) \rangle = a$) and the principal normal component b (i.e., $\langle \gamma(s), \mathbf{n}_1(s) \rangle = b$) of the position vector of a unit speed spatial semi real quaternionic curve $\gamma(s)$ in $\mathbb{E}_1^3$ provides the last equation then γ is a spatial semi real quaternionic osculating curve, or a spatial semi real quaternionic rectifying curve.

Theorem 5.3. Let $\beta : I \subset \mathbb{R} \rightarrow H_{\text{semi}}$ be a unit speed semi real quaternionic osculating curve, with non-zero curvatures $k(s)$, $K(s)$ and $(r(s) - K(s))$. Then β is equal to a semi real quaternionic osculating curve if and only if

$$(5.4) \quad \left(\frac{1}{K} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right)' \right)' + \frac{r \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}^2 K - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}}^2 \varepsilon_{\mathbf{N}_1}^3 K^2}{k} = - \frac{\varepsilon_{\mathbf{n}_1} \varepsilon_{\mathbf{N}_1}}{c \varepsilon_{\mathbf{n}_2}}, \quad c \in \mathbb{R}.$$

Proof. Let $\beta : I \subset \mathbb{R} \rightarrow \mathbb{H}_{\text{semi}}$ be a unit speed semi real quaternionic osculating curve and $k(s)$, $K(s)$ and $(r(s) - K(s))$ be non-zero curvatures of β . Then, the radius vector $\beta(s)$ of β satisfies the equation

$$\beta(s) = \lambda(s) \mathbf{T}(s) + \mu(s) \mathbf{N}_1(s) + \nu(s) \mathbf{N}_3(s),$$

where $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are some differentiable functions. By differentiating this equation in terms of the arc length function s and using the Frenet equations (5.3), we find that,

$$\begin{aligned} \mathbf{T} = & \left(\lambda' - \mu \varepsilon_{\mathbf{N}_1} \varepsilon_{\mathbf{t}} K \right) \mathbf{T} + \left(\lambda \varepsilon_{\mathbf{N}_1} K + \mu' \right) \mathbf{N}_1 \\ & + (\mu k \varepsilon_{\mathbf{n}_1} - \nu \varepsilon_{\mathbf{n}_2} (r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K)) \mathbf{N}_2 + \nu' \mathbf{N}_3. \end{aligned}$$

It follows that

$$(5.5) \quad \begin{aligned} \lambda' - \mu \varepsilon_{\mathbf{N}_1} \varepsilon_{\mathbf{t}} K &= 1 \\ \lambda \varepsilon_{\mathbf{N}_1} K + \mu' &= 0 \\ \mu k \varepsilon_{\mathbf{n}_1} - \nu \varepsilon_{\mathbf{n}_2} (r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K) &= 0 \\ \nu' &= 0 \end{aligned}$$

and therefore

$$(5.6) \quad \begin{aligned} \lambda(s) &= -c \frac{\varepsilon_{\mathbf{n}_2}}{\varepsilon_{\mathbf{n}_1} \varepsilon_{\mathbf{N}_1} K} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right)' \\ \mu(s) &= c \frac{\varepsilon_{\mathbf{n}_2}}{\varepsilon_{\mathbf{n}_1}} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right) \\ \nu(s) &= c. \end{aligned}$$

where $c \in \mathbb{R}$. In this way, the functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are expressed in terms of the curvature functions $k(s)$, $K(s)$ and $(r(s) - K(s))$ of the semi real quaternionic osculating curve β . Moreover, by using the first equation in (5.5) and relation (5.6) we find that curvatures $k(s)$, $K(s)$ and $(r(s) - K(s))$ provides the equation (5.4).

Conversely, let us suppose that the curvatures $k(s)$, $K(s)$ and $[r(s) - K(s)]$, of an arbitrary unit speed semi real quaternionic osculating curve β in H_{semi} , offers the eq. (5.4). Let's say that the vector $X \in \mathbb{H}$ imputed by

$$\begin{aligned} X(s) &= \beta(s) + c \frac{\varepsilon_{\mathbf{n}_2}}{\varepsilon_{\mathbf{n}_1} \varepsilon_{\mathbf{N}_1} K} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right)' \mathbf{T}(s) \\ &\quad - c \frac{\varepsilon_{\mathbf{n}_2}}{\varepsilon_{\mathbf{n}_1}} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right) \mathbf{N}_1(s) - c \mathbf{N}_3(s). \end{aligned}$$

By using the connections with (5.3) and (5.4), we rout out $X'(s) = 0$, which means that X is a constant vector. This implies that β is equal to a semi real quaternionic osculating curve. $\square$

Theorem 5.4. *Let $\beta : I \subset \mathbb{R} \rightarrow \mathbb{H}_{semi}$ be a unit speed semi real quaternionic osculating curve, with non-zero curvatures $k(s)$, $K(s)$ and $(r(s) - K(s))$. Then the following statements hold:*

(i). *The curvatures $k(s)$, $K(s)$ and $(r(s) - K(s))$ purveys the following statement*

$$(5.7) \quad \begin{aligned} r(s) &= \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K(s) \\ &+ k(s) \left\{ \left[\frac{\varepsilon_{\mathbf{n}_1}}{c \sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{n}_2}} K} \int \sin \left(\int \sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}} K(s) ds \right) ds + c_1 \right] \cos \left(\int \sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}} K(s) ds \right) \right. \\ &\quad \left. - \left[\frac{\varepsilon_{\mathbf{n}_1}}{c \sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{n}_2}} K} \int \cos \left(\int \sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}} K(s) ds \right) ds + c_2 \right] \sin \left(\int \sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}} K(s) ds \right) \right\}. \end{aligned}$$

(ii). *The tangential component and the principal normal component of the radius vector of the semi real quaternionic osculating curves are, respectively imputed by*

$$(5.8) \quad \begin{aligned} \langle \beta(s), \mathbf{T}(s) \rangle_{semi} &= -c \frac{\varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{T}}}{\varepsilon_{\mathbf{n}_1} \varepsilon_{\mathbf{N}_1} K} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right)' \\ \langle \beta(s), \mathbf{N}_1(s) \rangle_{semi} &= c \frac{\varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{N}_1}}{\varepsilon_{\mathbf{n}_1}} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right). \end{aligned}$$

(iii). *The second binormal component of the radius vector of the semi real quaternionic osculating curve is non zero constant. Conversely, if $\beta(s)$ is a unit speed semi*

real quaternionic osculating curves in H_{semi} with non-zero curvatures $k(s)$, $K(s)$ and $(r(s) - K(s))$ and one of the statements (i), (ii) or (iii) holds, then β is a semi real quaternionic osculating curve or equal to a semi real quaternionic osculating curve.

Proof. Let's suppose that $\beta(s)$ be a unit speed semi real quaternionic osculating curve in $\mathbb{H}_{semi}$, with non-zero curvatures $k(s)$, $K(s)$ and $(r(s) - K(s))$. The radius vector of the curve provides the statement (5.4). If we state $y(s) = \frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k}$ and $p(s) = \frac{1}{K(s)}$, then balance (5.4) can be written as

$$\frac{d}{ds} \left(p(s) \frac{dy}{ds} \right) + \frac{y(s)}{p(s)} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}^2 = - \frac{\varepsilon_{\mathbf{n}_1} \varepsilon_{\mathbf{N}_1}}{c \varepsilon_{\mathbf{n}_2}}$$

and we express $q(s) = \frac{1}{\sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}} K(s)}$ and, then the last equation can be written as

$$\frac{d}{ds} \left(q(s) \frac{dy}{ds} \right) + \frac{y(s)}{q(s)} = - \frac{\varepsilon_{\mathbf{n}_1}}{c \sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{n}_2}}}.$$

If we shift variances in overhead statement as $t(s) = \int \frac{1}{q(s)} ds$, then we have

$$\frac{d^2 y}{dt^2} + y = - \frac{\varepsilon_{\mathbf{n}_1}}{c \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{N}_1} K}, \quad c \in \mathbb{R},$$

the solution of this differential equation is

$$y = \left(\frac{\varepsilon_{\mathbf{n}_1}}{c \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{N}_1} K} \int \frac{\sin t}{K} dt + c_1 \right) \cos t + \left(- \frac{\varepsilon_{\mathbf{n}_1}}{c \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{N}_1} K} \int \frac{\cos t}{K} dt + c_2 \right) \sin t$$

where c, c_1 and $c_2 \in \mathbb{R}$. Let's say that, $y(s) = \frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k}$ and $dt = \sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}} K(s) ds$, we get

$$(5.9) \quad \begin{aligned} r(s) = & \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K(s) + k(s) \left\{ \left[\frac{\varepsilon_{\mathbf{n}_1}}{c \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{N}_1} K} \int \sin \left(\int \sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}} K(s) ds \right) \sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}} ds + c_1 \right] \right. \\ & \left. - \left[- \frac{\varepsilon_{\mathbf{n}_1}}{c \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{N}_1} K} \int \cos \left(\int \sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}} K(s) ds \right) \sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}} ds + c_2 \right] \sin \left(\int \sqrt{\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}} K(s) ds \right) \right\}. \end{aligned}$$

Thus we prove the statement (i). By using the relations (5.2) and (5.6), we can write the radius vector of the curve as follows:

$$\begin{aligned} \beta(s) = & -c \frac{\varepsilon_{\mathbf{n}_2}}{\varepsilon_{\mathbf{n}_1} \varepsilon_{\mathbf{N}_1} K} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right)' \mathbf{T}(s) + c \frac{\varepsilon_{\mathbf{n}_2}}{\varepsilon_{\mathbf{n}_1}} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right) \mathbf{N}_1(s) + \\ & c \mathbf{N}_3(s). \end{aligned}$$

From this equation, we have

$$\begin{aligned}\langle \beta(s), \mathbf{T}(s) \rangle_{semi} &= -c \frac{\varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{T}}}{\varepsilon_{\mathbf{n}_1} \varepsilon_{\mathbf{N}_1} K} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right)', \\ \langle \beta(s), \mathbf{N}_1(s) \rangle_{semi} &= c \frac{\varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{N}_1}}{\varepsilon_{\mathbf{n}_1}} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right)\end{aligned}$$

and

$$\langle \beta(s), \mathbf{N}_3(s) \rangle_{semi} = c \varepsilon_{\mathbf{N}_3}.$$

Thus we proved the recitals (ii) and (iii). Conversely, assume that recital (i) possess.

Then the curvature functions $k(s)$, $K(s)$ and $(r(s) - K(s))$ provides the recital

$$\begin{aligned}r(s) &= \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K(s) + k(s) \left\{ \left[\frac{\varepsilon_{\mathbf{n}_1}}{c \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{N}_1} K} \int \sin \left(\int \sqrt{\varepsilon_{\mathbf{t}}} \varepsilon_{\mathbf{N}_1} K(s) ds \right) \sqrt{\varepsilon_{\mathbf{t}}} \varepsilon_{\mathbf{N}_1} ds + c_1 \right] \right. \\ &\quad \left. \cos \left(\int \sqrt{\varepsilon_{\mathbf{t}}} \varepsilon_{\mathbf{N}_1} K(s) ds \right) \right. \\ &\quad \left. - \left[-\frac{\varepsilon_{\mathbf{n}_1}}{c \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{N}_1} K} \int \cos \left(\int \sqrt{\varepsilon_{\mathbf{t}}} \varepsilon_{\mathbf{N}_1} K(s) ds \right) \sqrt{\varepsilon_{\mathbf{t}}} \varepsilon_{\mathbf{N}_1} ds + c_2 \right] \right. \\ &\quad \left. \sin \left(\int \sqrt{\varepsilon_{\mathbf{t}}} \varepsilon_{\mathbf{N}_1} K(s) ds \right) \right\}.\end{aligned}$$

By differentiating the previous equation two times in regard to s we get,

$$\left(\frac{1}{K} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right)' \right)' + \frac{r \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}^2 K - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}}^2 \varepsilon_{\mathbf{N}_1}^3 K^2}{k} = -\frac{\varepsilon_{\mathbf{n}_1} \varepsilon_{\mathbf{N}_1}}{c \varepsilon_{\mathbf{n}_2}}, \quad c \in \mathbb{R}.$$

In this situation β is equal to a semi real quaternionic osculating curve. Next assume that recitals (ii) possess. By taking derivative of $\langle \beta(s), \mathbf{N}_1(s) \rangle_{semi} = c \frac{\varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{N}_1}}{\varepsilon_{\mathbf{n}_1}} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right)$ in as much as s and using (5.3) we get,

$$-\varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K \langle \beta(s), \mathbf{T}(s) \rangle_{semi} + \varepsilon_{\mathbf{n}_1} k \langle \beta(s), \mathbf{N}_2(s) \rangle_{semi} = c \frac{\varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{N}_1}}{\varepsilon_{\mathbf{n}_1}} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right)'.$$

By disposing $\langle \beta(s), \mathbf{T}(s) \rangle_{semi} = -c \frac{\varepsilon_{\mathbf{n}_2} \varepsilon_{\mathbf{T}}}{\varepsilon_{\mathbf{n}_1} \varepsilon_{\mathbf{N}_1} K} \left(\frac{r - \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1} K}{k} \right)'$ and $k \neq 0$, we get $\langle \beta(s), \mathbf{N}_2(s) \rangle_{semi} = 0$, which means that β is a quaternionic osculating curve. If recital (iii) possess, then we get $\langle \beta(s), \mathbf{N}_3(s) \rangle_{semi} = c \varepsilon_{\mathbf{N}_3}$. If we take a derivative of the last equation pursuant to s and disposing (5.1), we strike

$$-\varepsilon_{\mathbf{n}_2} (r - K \varepsilon_{\mathbf{T}} \varepsilon_{\mathbf{t}} \varepsilon_{\mathbf{N}_1}) \langle \beta(s), \mathbf{N}_2(s) \rangle_{semi} = 0.$$

It takes out that $\langle \beta(s), \mathbf{N}_2(s) \rangle_{semi} = 0$ and accordingly the curve is osculating curve. This proves the theorem. $\square$

REFERENCES

- [1] J.P. Ward, Quaternions and Cayley Numbers, Kluwer Academic Publishers, Boston/London, 1997.
- [2] K. Bharathi, M. Nagaraj, Quaternion Valued Function of a Real Variable Serret-Frenet Formulae, Indian J. Pure Math., 18, 6, 507–511 (1987).
- [3] A.İ. Sivridağ, R. Güneş, S. Keleş, The Serret-Frenet formulae for dual quaternion-valued functions of a single real valuable, Mechanism and Machine Theory, 29, 5, 749-754 (1994).
- [4] A.C. Çöken, A. Tuna, On the quaternionic inclined curves in the semi-Euclidean space $\mathbb{E}_2^4$, Appl. Math. Comput., 155, 373–389 (2004).
- [5] M. Çetin, H. Kocayigit, On the quaternionic Smarandache curves in Euclidean 3-Space, Int. J. Contemp. Math. Sciences, 8, 3, 139 – 150 (2013).
- [6] S. Şenyurt, A. Çalıskan, An application according to spatial quaternionic Smarandache curve, Applied Mathematical Sciences, 9, 5, 219-228 (2015).
- [7] İ. Gök, F.K. Aksoyak, Characterizations of the quaternionic bertrand curve in Euclidean Space $\mathbb{E}^4$, arXiv:1302.2809v1 [math.DG] (2013).
- [8] O. Keçilioğlu, K. İlarslan, Quaternionic Bertrand curve in Euclidean 4- Space, Bulletin of Mathematical Analysis and Applications, 5, 3, 27 – 38 (2013).
- [9] İ. Gök, O.Z. Okuyucu, F. Kahraman, H.H. Hacısalihoğlu, On the quaternionic slant helices in the Euclidean space, Adv. Appl. Clifford Alg., 21, 707–719 (2011).
- [10] F. Kahraman, I. Gök, H.H. Hacısalihoğlu, On the Quaternionic slant helices in the semi Euclidean space, Applied Mathematics and Computation, 218, 6391–6400 (2012).
- [11] M.A. Güngör, M. Tosun, Some characterizations of quaternionic rectifying curves, Differential Geometry-Dynamical Systems, 13, 89-100 (2011).
- [12] T. Erişir, M.A. Güngör, Some characterizations of quaternionic rectifying curves in the semi-Euclidean Space $\mathbb{E}_2^4$, Honam Mathematical Journal, 36, 67 - 83 (2013).
- [13] O.Z. Okuyucu, Characterizations of the quaternionic Mannheim curves in $\mathbb{E}^4$, International J. Math. Combin., 2 44-53 (2013).
- [14] M. Önder, Quaternionic Salkowski curves and similar curves, arXiv:1205.1368v1 [math.DG], 07 May 2012.
- [15] T. Soyfidan, M.A. Güngör, On the quaternionic involute evolute curve couples, IX. National Geometry Symposium Ondokuz Mayıs University Samsun, 07-10 June, 2011.
- [16] T. Soyfidan, M.A. Güngör, On the quaternionic involute evolute curves in the semi-Euclidean space $\mathbb{E}_2^4$, X, National Geometry Symposium, Balıkesir University , Balıkesir, 13-16 June 2012.
- [17] Ö.G. Yıldız, S. Özkaldı Karakuş, On the quaternionic normal curves in Euclidean space, IECMSA 2013, 2nd International Eurasian Conference On Mathematical Sciences and Applications, August 26 -29, 2013, Sarajevo, Bosnia Herzegovina.
- [18] G. Yıldız Ö., S. Özkaldı Karakuş, On the quaternionic normal curves in the Semi Euclidean space $\mathbb{E}_2^4$, X. National Geometry Symposium, Balıkesir University , Balıkesir, 13-16 June 2012.

- [19] K. İlarıslan, E. Nesovic, Some characterizations of osculating curves in the Euclidean spaces, *Demonstratio Mathematica*, XLI, 4 (2008).