

ON MIXED SUPER QUASI-EINSTEIN WARPED PRODUCTS

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(Received: 12 May 2015, Accepted: 19 July 2015)

ABSTRACT. In this paper, we study mixed super quasi-Einstein warped product manifolds for arbitrary dimension $n \geq 3$ and we give an example of mixed super quasi-Einstein manifold $(MS(QE)_n)$ to ensure the existence of such manifold. Also in the last section we also give an example of warped product on mixed super quasi-Einstein manifold.

AMS Classification: 53C25, 53B30, 53C15

Keywords: Einstein manifold; Quasi-Einstein manifold; Super quasi-Einstein manifold; Mixed super quasi-Einstein manifold; Warped product manifold

1. INTRODUCTION

A Riemannian manifold (M^n, g) , $(n > 2)$ is Einstein if its Ricci tensor S of type $(0,2)$ is of the form $S = \alpha g$, where α is smooth function, which turns into $S = \frac{r}{n}g$, r being the scalar curvature of the manifold. Let (M^n, g) , $(n > 2)$ be a Riemannian manifold and $U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$, then the manifold (M^n, g) is said to be quasi-Einstein manifold [3,7] if on $U_S \subset M$, we have

$$(1.1) \quad S - \alpha g = \beta A \otimes A,$$

where A is a 1-form on U_S and α and β some functions on U_S . It is clear that the 1-form A as well as the function β are nonzero at every point on U_S . From the above definition, it follows that every Einstein manifold is quasi-Einstein. In particular, every Ricci-flat manifold (e.g., Schwarzschild spacetime) is quasi-Einstein. The scalars α, β are known as the associated scalars of the manifold. Also, the 1-form A is called the associated 1-form of the manifold defined by $g(X, \rho) = A(X)$ for any vector field X , ρ being a unit vector field, called the generator of the manifold. Such an n -dimensional quasi-Einstein manifold is denoted by $(QE)_n$.

In [6], De and Gazi introduced nearly quasi-Einstein manifold, denoted by $N(QE)_n$ and gave an example of a 4-dimensional Riemannian nearly quasi Einstein manifold, where the Ricci tensor S of type (0,2) which is not identically zero satisfies the condition

$$(1.2) \quad S(X, Y) = lg(X, Y) + mD(X, Y),$$

where l and m are non-zero scalars and D is a non-zero symmetric tensor of type (0,2).

In [4], Chaki introduced super quasi-Einstein manifold, denoted by $S(QE)_n$ and gave an example of a 4-dimensional semi Riemannian super quasi Einstein manifold, where the Ricci tensor S of type (0,2) which is not identically zero satisfies the condition

$$(1.3) \quad \begin{aligned} S(X, Y) = & \alpha g(X, Y) + \beta A(X)A(Y) + \gamma[A(X)\eta(Y) + \\ & A(Y)\eta(X)] + \delta D(X, Y), \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are scalars such that β, γ, δ are nonzero and A, η are two nonzero 1-forms such that

$$g(X, \mu) = A(X) \text{ and } g(X, \rho) = \eta(X),$$

μ, ρ being unit vectors which are orthogonal, i.e., $g(\mu, \rho) = 0$ and D is a symmetric (0,2) tensor with zero trace which satisfies the condition

$$D(X, \mu) = 0, \quad \forall X \in \chi(M).$$

Here $\alpha, \beta, \gamma, \delta$ are called the associated scalars, and A, η are called the associated main and auxiliary 1-forms respectively, μ, ρ are called the main and the auxiliary

generators and D is called the associated tensor of the manifold.

In [1], A. Bhattacharyya, M. Tarafdar and D. Debnath introduced mixed super quasi-Einstein manifold, denoted by $MS(QE)_n$ and gave an example of mixed super quasi Einstein manifold, where the Ricci tensor S of type (0,2) which is not identically zero satisfies the condition

$$(1.4) \quad \begin{aligned} S(X, Y) = & \alpha g(X, Y) + \beta A(X)A(Y) + \gamma \eta(X)\eta(Y) + \delta[A(X)\eta(Y) + \\ & A(Y)\eta(X)] + \psi D(X, Y), \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \psi$ are scalars such that $\beta, \gamma, \delta, \psi$ are nonzero and A, η are two nonzero 1-forms such that

$$g(X, \mu) = A(X) \text{ and } g(X, \rho) = \eta(X),$$

μ, ρ being unit vectors which are orthogonal, i.e., $g(\mu, \rho) = 0$ and D is a symmetric (0,2) tensor with zero trace which satisfies the condition

$$D(X, \mu) = 0, \quad \forall X \in \chi(M).$$

Here $\alpha, \beta, \gamma, \delta, \psi$ are called the associated scalars, and A, η are called the associated main and auxiliary 1-forms respectively, μ, ρ are called the main and the auxiliary generators and D is called the associated tensor of the manifold.

Also in [1], A. Bhattacharyya, M. Tarafdar and D. Debnath introduced the notion of a Riemannian manifold (M, g) of a mixed super quasi-constant sectional curvature

as a Riemannian manifold with the curvature tensor satisfies the condition

(1.5)

$$\begin{aligned}
 R(X, Y, Z, W) = & a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + b[g(Y, Z)A(X)A(W) \\
 & - g(X, Z)A(Y)A(W) + g(X, W)A(Y)A(Z) \\
 & - g(Y, W)A(X)A(Z)] + c[g(Y, Z)\eta(X)\eta(W) - \\
 & g(X, Z)\eta(Y)\eta(W) + g(X, W)\eta(Y)\eta(Z) \\
 & - g(Y, W)\eta(X)\eta(Z)] + d[g(Y, Z)\{A(X)\eta(W) + A(W)\eta(X)\} \\
 & - g(X, Z)\{A(Y)\eta(W) + A(W)\eta(Y)\} \\
 & + g(X, W)\{A(Y)\eta(Z) + A(Z)\eta(Y)\} \\
 & - g(Y, W)\{A(X)\eta(Z) + A(Z)\eta(X)\}] + \varrho[g(Y, Z)D(X, W) \\
 & - g(X, Z)D(Y, W) + g(X, W)D(Y, Z) - g(Y, W)D(X, Z)],
 \end{aligned}$$

where a, b, c, d, ϱ are scalar functions with $b \neq 0, c \neq 0, d \neq 0, \varrho \neq 0$ where A and η are non zero 1-forms such that

$$g(X, E) = A(X), \quad g(X, \dot{E}) = \eta(X) \quad \forall X, \quad g(E, \dot{E}) = 0$$

$E, \dot{E}$ being orthogonal unit vector fields, D is symmetric (0,2) tensor with zero trace which satisfies the condition

$$D(X, E) = 0 \quad \forall X.$$

Let M be an m -dimensional, $m \geq 3$, Riemannian manifold and $p \in M$. Denote by $K(\pi)$ or $K(u \wedge v)$ the sectional curvature of M associated with a plane section $\pi \subset T_p M$, where $\{u, v\}$ is an orthonormal basis of π . For any n -dimensional subspace $L \subseteq T_p M$, $2 \leq n \leq m$, its scalar curvature $\tau(L)$ is denoted in [5] by $\tau(L) = 2\sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$, where $\{e_1, e_2, \dots, e_n\}$ is any orthonormal basis of L [11]. When $L = T_p M$, the scalar curvature $\tau(L)$ is just the scalar curvature $\tau(p)$ of M at p .

2. WARPED PRODUCT MANIFOLDS

The notion of warped product generalizes that of a surface of revolution. It was introduced in [2] for studying manifolds of negative curvature. Let (B, g_B) and (F, g_F) be two Riemannian manifolds and f is a positive differentiable function on B . Consider the product manifold $B \times F$ with its projections $\pi : B \times F \rightarrow B$ and $\sigma : B \times F \rightarrow F$. The warped product $B \times_f F$ is the manifold $B \times F$ with the

Riemannian structure such that $\|X\|^2 = \|\pi^*(X)\|^2 + f^2(\pi(p))\|\sigma^*(X)\|^2$, for any vector field X on M . Thus we have

$$(2.1) \quad g = g_B + f^2 g_F$$

holds on M . The function f is called the warping function of the warped product [9].

Since $B \times_f F$ is a warped product, then we have $\nabla_X Z = \nabla_Z X = (X \ln f)Z$ for unit vector fields X, Z on B and F , respectively. Hence, we find $K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = (1/f)\{(\nabla_X X_f - X^2 f)\}$. If we chose a local orthonormal frame $e_1, \dots, e_n$ such that $e_1, \dots, e_{n_1}$ are tangent to B and $e_{n_1+1}, \dots, e_n$ are tangent to F , then we have

$$(2.2) \quad \frac{\Delta f}{f} = \sum_{i=1}^n K(e_i \wedge e_j),$$

for each $s = n_1 + 1, \dots, n$ [9].

We need the following two lemmas from [9], for later use:

Lemma 2.1. *Let $M = B \times_f F$ be a warped product, with Riemannian curvature tensor R_M . Given field X, Y, Z on B and U, V, W on F , then:*

- (1) $R_M(X, Y)Z = R_B(X, Y)Z$,
- (2) $R_M(V, X)Y = -(H^f(X, Y)/f)V$, where H^f is the Hessian of f ,
- (3) $R_M(X, Y)V = R_M(V, W)X = 0$,
- (4) $R_M(X, V)W = -(g(V, W)/f)\nabla_X(\text{grad } f)$,
- (5) $R_M(V, W)U = R_F(V, W)U + (\|\text{grad } f\|^2 / f^2)\{g(V, U)W - g(W, U)V\}$.

Lemma 2.2. *Let $M = B \times_f F$ be a warped product, with Ricci tensor S_M . Given fields X, Y on B and V, W on F , then:*

- (1) $S_M(X, Y) = S_B(X, Y) - \frac{d}{f}H^f(X, Y)$, where $d = \dim F$
- (2) $S_M(X, V) = 0$,
- (3) $S_M(V, W) = S_F(V, W) - g(V, W)f^\#$, $f^\# = \frac{\Delta f}{f} + \frac{d-1}{f^2}\|\text{grad } f\|^2$, where Δf is the Laplacian of f on B .

Moreover, the scalar curvature τ_M of the manifold M satisfies the condition

$$(2.3) \quad \tau_M = \tau_B + \frac{\tau_F}{f^2} - 2d \frac{\Delta f}{f} - d(d-1) \frac{|\nabla f|^2}{f^2},$$

where τ_B and τ_F are the scalar curvatures of B and F , respectively.

In [8], GEBAROWSKI studied Einstein warped product manifolds and proved the following three theorems:

Theorem 2.1. *Let (M, g) be a warped product $I \times_f F$, $\dim I = 1$, $\dim F = n-1$ ($n \geq 3$). Then (M, g) is an Einstein manifold if and only if F is Einstein with constant scalar curvature τ_F in the case $n = 3$ and f is given by one of the following formulae, for any real number b ,*

$$f^2(t) = \begin{cases} \frac{4}{a} K \sinh^2 \frac{\sqrt{a}(t+b)}{2}, & a > 0 \\ K(t+b)^2, & a = 0 \\ -\frac{4}{a} K \sin^2 \frac{\sqrt{-a}(t+b)}{2}, & a < 0 \end{cases}$$

for $K > 0$, $f^2(t) = b \exp(at)$ ($a \neq 0$), for $K = 0$, $f^2(t) = -\frac{4}{a} K \cosh^2 \frac{\sqrt{a}(t+b)}{2}$, ($a > 0$), for $K < 0$, where a is the constant appearing after first integration of the equation $q'' e^q + 2K = 0$ and $K = \frac{\tau_F}{(n-1)(n-2)}$.

Theorem 2.2. *Let (M, g) be a warped product $B \times_f F$ of a complete connected r -dimensional ($1 < r < n$) Riemannian manifold B and $(n-r)$ -dimensional Riemannian manifold F . If (M, g) is a space of constant sectional curvature $K > 0$, then B is a sphere of radius $\frac{1}{\sqrt{K}}$.*

Theorem 2.3. *Let (M, g) be a warped product $B \times_f F$ of a complete connected $n-1$ -dimensional Riemannian manifold B and one-dimensional Riemannian manifold I . If (M, g) is an Einstein manifold with scalar curvature $\tau_M > 0$ and the Hessian of f is proportional to the metric tensor g_B , then*

- (i) (B, g_B) is an $(n-1)$ -dimensional sphere of radius $\rho = \left(\frac{\tau_B}{(n-1)(n-2)}\right)^{-\frac{1}{2}}$.
- (ii) (M, g) is a space of constant sectional curvature $K = \frac{\tau_M}{n(n-1)}$.

Motivated by the above study by GEBAROWSKI and the paper by S. Sular and C. Ozgur [12], in the present paper our aim is to generalize Theorem 2.1, Theorem

2.2 and Theorem 2.3 for mixed super quasi-Einstein manifolds. Next we established an example of mixed super quasi-Einstein manifold to ensure the existence of such manifold. Also in the last section we also give an example of warped product on mixed super quasi-Einstein manifold.

3. MIXED SUPER QUASI-EINSTEIN WARPED PRODUCTS

In this section, we consider mixed super quasi-Einstein warped product manifolds and prove some results concerning these type of manifolds.

Theorem 3.1. *Let (M, g) be a warped product manifold $I \times_f F$, where $\dim I = 1$ and $\dim F = n - 1$ ($n \geq 3$). If (M, g) is mixed super quasi-Einstein manifold with associated scalars $\alpha, \beta, \gamma, \delta$ and ψ , then F is also a mixed super quasi-Einstein manifold.*

Proof. Let us consider $(dt)^2$ be the metric on I . Taking $f = \exp\{\frac{q}{2}\}$ and making use of the Lemma 2.2, we can write

$$(3.1) \quad S_M\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -\frac{n-1}{4}[2q'' + (q')^2]$$

and

$$(3.2) \quad S_M(V, W) = S_F(V, W) - \frac{1}{4}e^q[2q'' + (n-1)(q')^2]g_F(V, W),$$

for all vector fields V, W on F .

Since M is super quasi-Einstein, from (1.4) we have

$$(3.3) \quad \begin{aligned} S_M\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = & \alpha g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + \beta A\left(\frac{\partial}{\partial t}\right)A\left(\frac{\partial}{\partial t}\right) + \gamma \eta\left(\frac{\partial}{\partial t}\right)\eta\left(\frac{\partial}{\partial t}\right) + \delta\left[A\left(\frac{\partial}{\partial t}\right)\eta\left(\frac{\partial}{\partial t}\right) \right. \\ & \left. + A\left(\frac{\partial}{\partial t}\right)\eta\left(\frac{\partial}{\partial t}\right)\right] + \psi D\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right), \end{aligned}$$

and

$$(3.4) \quad \begin{aligned} S_M(V, W) = & \alpha g(V, W) + \beta A(V)A(W) + \gamma \eta(V)\eta(W) + \delta[A(V)\eta(W) + A(W)\eta(V)] \\ & + \psi D(V, W). \end{aligned}$$

Now let $U, \dot{U} \in \chi(M)$. Decomposing the vector fields U and $\dot{U}$ uniquely into its components U_I, U_F and $\dot{U}_I, \dot{U}_F$ on I and F , respectively, we can write $U = U_I + U_F$

and $\dot{U} = \dot{U}_I + \dot{U}_F$. Since $\dim I = 1$, we can take $U_I = \eta_1 \frac{\partial}{\partial t}$ which gives us $U = \eta_1 \frac{\partial}{\partial t} + U_F$ and $\dot{U}_I = \eta_2 \frac{\partial}{\partial t}$ which gives us $\dot{U} = \eta_2 \frac{\partial}{\partial t} + U_F$, where η_1 and η_2 are functions on M . Then we can write

$$(3.5) \quad \begin{aligned} A\left(\frac{\partial}{\partial t}\right) &= g\left(\frac{\partial}{\partial t}, U\right) = \eta_1, \\ \eta\left(\frac{\partial}{\partial t}\right) &= g\left(\frac{\partial}{\partial t}, \dot{U}\right) = \eta_2. \end{aligned}$$

On the other hand, by the use of (2.1) and (3.5), the equations (3.3) and (3.4) reduce to

$$(3.6) \quad S_M\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \alpha + \beta\eta_1^2 + \gamma\eta_2^2 + 2\delta\eta_1\eta_2 + \psi D\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)$$

and

$$(3.7) \quad \begin{aligned} S_M(V, W) &= \alpha e^q g_F(V, W) + \beta A(V)A(W) + \gamma\eta(V)\eta(W) + \\ &\quad \delta[A(V)\eta(W) + A(W)\eta(V)] + \psi D(V, W). \end{aligned}$$

Comparing the right hand side of the equations (3.1) and (3.6) we get

$$(3.8) \quad \alpha + \beta\eta_1^2 + \gamma\eta_2^2 + 2\delta\eta_1\eta_2 + \psi D\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -\frac{n-1}{4}[2q'' + (q')^2].$$

Similarly, comparing the right hand sides of (3.2) and (3.7) we obtain

$$(3.9) \quad \begin{aligned} S_F(V, W) &= \frac{1}{4}e^q[2q'' + (n-1)(q')^2 + 4\alpha]g_F(V, W) + \beta A(V)A(W) + \gamma\eta(V)\eta(W) \\ &\quad + \delta[A(V)\eta(W) + A(W)\eta(V)] + \psi D(V, W). \end{aligned}$$

Which implies that F is a mixed super quasi-Einstein manifold. This completes the proof of the theorem.

Theorem 3.2. *Let (M, g) be a warped product $B \times_f F$ of a complete connected r -dimensional ($1 < r < n$) Riemannian manifold B and $(n-r)$ -dimensional Riemannian manifold F .*

(i) *If (M, g) is a space of mixed super quasi-constant sectional curvature, the Hessian of f is proportional to the metric tensor g_B and the associated vector fields E and $\dot{E}$ are the general vector field on M or $E, \dot{E} \in \chi(B)$, then B is isometric to the sphere of radius $\frac{1}{\sqrt{k}}$ in the $(r+1)$ dimensional Euclidean space. For $r = 2$, B is a 2-dimensional nearly quasi-Einstein manifold.*

(ii) *If (M, g) is a space of mixed super quasi-constant sectional curvature and the associated vector fields $E, \dot{E} \in \chi(F)$, then B is a nearly quasi-Einstein manifold.*

Proof. Assume that M is a space of super quasi-constant sectional curvature. Then from equation (1.5), we can write

$$\begin{aligned}
 (3.10) \quad R(X, Y, Z, W) = & a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + \\
 & b[g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W) + \\
 & g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)] + c[g(Y, Z) \\
 & \eta(X)\eta(W) - g(X, Z)\eta(Y)\eta(W) + g(X, W)\eta(Y)\eta(Z) \\
 & - g(Y, W)\eta(X)\eta(Z)] + d[g(Y, Z)\{A(X)\eta(W) + A(W)\eta(X)\} \\
 & - g(X, Z)\{A(Y)\eta(W) + A(W)\eta(Y)\} + g(X, W)\{A(Y)\eta(Z) + \\
 & A(Z)\eta(Y)\} - g(Y, W)\{A(X)\eta(Z) \\
 & + A(Z)\eta(X)\}] + \varrho[g(Y, Z)D(X, W) - g(X, Z)D(Y, W) \\
 & + g(X, W)D(Y, Z) - g(Y, W)D(X, Z)],
 \end{aligned}$$

for all vector fields X, Y, Z, W on B .

Decomposing the vector fields E and $\acute{E}$ uniquely into its components E_B, E_F and $\acute{E}_B, \acute{E}_F$ on B and F , respectively, we can write $E = E_B + E_F$ and $\acute{E} = \acute{E}_B + \acute{E}_F$. Then we can write

$$\begin{aligned}
 (3.11) \quad A(X) &= g(X, E) = g(X, E_B) = g_B(X, E_B), \\
 \eta(X) &= g(X, \acute{E}) = g(X, \acute{E}_B) = g_B(X, \acute{E}_B).
 \end{aligned}$$

In view of Lemma 2.1 and by using (2.1) and (3.11) in equation (3.10) and then after a contraction over X and W (we put $X = W = e_i$), we get

$$\begin{aligned}
 (3.12) \quad S_B(Y, Z) = & [a(r-1) + bg_B(E_B, E_B) + cg_B(\acute{E}_B, \acute{E}_B) + \varrho D_B(e_i, e_i)]g_B(Y, Z) \\
 & + b(r-2)A(Y)A(Z) + c(r-2)\eta(Y)\eta(Z) + d(r-2)[A(Y)\eta(Z) \\
 & + \eta(Y)A(Z)] + r\varrho D_B(Y, Z),
 \end{aligned}$$

which shows us B is a mixed super quasi-Einstein manifold. Contracting from (3.12) over Y and Z , we can write

$$(3.13) \quad \tau_B = (r-1)[ar + 2bg_B(E_B, E_B) + 2cg_B(\acute{E}_B, \acute{E}_B)] + 2r\varrho D_B(e_i, e_i).$$

Since M is a space of mixed super quasi-constant sectional curvature, in view of (2.2) and (3.10) we get

$$(3.14) \quad \frac{\Delta f}{f} = \frac{ar + bg_B(E_B, E_B) + cg_B(\dot{E}_B, \dot{E}_B) + r\varrho D_B(e_i, e_i)}{2}.$$

On the other hand, since Hesssian of f is proportional to the metric tensor g_B , it can be written as follows

$$(3.15) \quad H^f(X, Y) = \frac{\Delta f}{r} g_B(X, Y).$$

Then by use of (3.13) and (3.14) in (3.15) we obtain

$$H^f + Kf g_B(X, Y) = 0,$$

$$\text{where } K = \frac{r(3-r)\varrho D_B(e_i, e_i) + (r-1)[bg_B(E_B, E_B) + cg_B(\dot{E}_B, \dot{E}_B)] - \tau_B}{2r(r-1)}$$

holds on B . So by OBATA's theorem [10], B is isometric to the sphere of radius $\frac{1}{\sqrt{K}}$ in the $(r+1)$ -dimensional Euclidean space. When $r = 2$ and since $b \neq 0$, $c \neq 0$ and $d \neq 0$, B becomes a 2-dimensional nearly quasi-Einstein manifold.

Assume that the associated vector fields $E, \dot{E} \in \chi(B)$. Then in view of Lemma 2.1 and by making use of (2.1) and (3.10) and after a contraction over X and W we obtain

$$(3.16) \quad \begin{aligned} S_B(Y, Z) = & [a(r-1) + b + c + \varrho D_B(e_i, e_i)]g_B(Y, Z) + b(r-2)A(Y)A(Z) \\ & + c(r-2)\eta(Y)\eta(Z) + d(r-2)[A(Y)\eta(Z) + \eta(Y)A(Z)] + r\varrho D_B(Y, Z), \end{aligned}$$

which gives us B is a mixed super quasi-Einstein manifold. By a contraction from (3.16) over Y and Z , we get

$$(3.17) \quad \tau_B = (r-1)[ar + 2b + 2c] + 2r\varrho D_B(e_i, e_i).$$

Since M is a space of mixed super quasi-constant sectional curvature, in view of (2.2) and (3.10) (for the case of $E, \dot{E} \in \chi(B)$) we get

$$(3.18) \quad \frac{\Delta f}{f} = \frac{ar + b + c + r\varrho D_B(e_i, e_i)}{2}.$$

On the other hand, since Hesssian of f is proportional to the metric tensor g_B , it can be written as follows

$$(3.19) \quad H^f(X, Y) = \frac{\Delta f}{r} g_B(X, Y).$$

Then by use of (3.17) and (3.18) in (3.19) we obtain

$$H^f(X, Y) + Kfg_B(X, Y) = 0, \text{ where } K = \frac{r(3-r)\varrho D_B(e_i, e_i) + (r-1)[b+c] - \tau_B}{2r(r-1)}$$

holds on B . So by OBATA's theorem [10], B is isometric to the sphere of radius $\frac{1}{\sqrt{K}}$ in the $(r+1)$ -dimensional Euclidean space. For $r=2$ and as $b \neq 0$, $c \neq 0$ and $d \neq 0$, B is a 2-dimensional nearly quasi-Einstein manifold.

Assume that the associated vector fields $E, \dot{E} \in \chi(F)$, then the equation (3.10) reduces to

$$(3.20) \quad \begin{aligned} R(X, Y, Z, W) = & a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + \varrho[g(Y, Z)D(X, W) \\ & - g(X, Z)D(Y, W) + g(X, W)D(Y, Z) - g(Y, W)D(X, Z)]. \end{aligned}$$

In view of Lemma 2.1 and by making use of (2.1), the above equation can written as

$$(3.21) \quad \begin{aligned} R(X, Y, Z, W) = & a[g_B(Y, Z)g_B(X, W) - g_B(X, Z)g_B(Y, W)] + \varrho[g_B(Y, Z)D_B(X, W) \\ & - g_B(X, Z)D_B(Y, W) + g_B(X, W)D_B(Y, Z) \\ & - g_B(Y, W)D_B(X, Z)]. \end{aligned}$$

Again a contraction of the above equation over X and W , we get

$$(3.22) \quad S_B(Y, Z) = [a(r-1) + \varrho D_B(e_i, e_i)]g_B(Y, Z) + r\varrho D_B(Y, Z),$$

which implies that B is a nearly quasi-Einstein manifold with scalar curvature $\tau_B = ar(r-1) + 2r\varrho D_B(e_i, e_i)$. Hence the proof of the theorem is completed.

Theorem 3.3. *Let (M, g) be a warped product $B \times_f I$ of a complete connected $(n-1)$ -dimensional Riemannian manifold B and one-dimensional Riemannian manifold I . If (M, g) is a mixed super quasi-Einstein manifold with constant associated scalars $\alpha, \beta, \gamma, \delta$ and ψ and the Hessian of f is proportional to the metric tensor g_B , then (B, g_B) is an $(n-1)$ -dimensional sphere of radius $\frac{n-1}{\sqrt{\tau_B + \alpha}}$.*

Proof. Assume that M is a warped product manifold. Then by use of Lemma

2.2 we can write

$$(3.23) \quad S_B(X, Y) = S_M(X, Y) + \frac{1}{f}H^f(X, Y)$$

for any vector fields X, Y on B . On the other hand, since M is a mixed super quasi-Einstein manifold we have

$$(3.24) \quad \begin{aligned} S_M(X, Y) = & \alpha g(X, Y) + \beta A(X)A(Y) + \gamma \eta(X)\eta(Y) + \delta[A(X)\eta(Y) \\ & + A(Y)\eta(X)] + \psi D(X, Y). \end{aligned}$$

When $U, \dot{U} \in \chi(M)$, decomposing the vector fields U and $\dot{U}$ uniquely into its components U_I, U_F and $\dot{U}_I, \dot{U}_F$ on B and I , respectively, we can write

$$(3.25) \quad \begin{aligned} U &= U_B + U_I \\ \dot{U} &= \dot{U}_B + \dot{U}_I \end{aligned}$$

In view of (2.1), (3.24) and (3.25) the equation (3.23) can be written as

$$(3.26) \quad \begin{aligned} S_B(X, Y) = & \alpha g_B(X, Y) + \beta g_B(X, U_B)g_B(Y, U_B) + \gamma g_B(X, \dot{U}_B)g_B(Y, \dot{U}_B) \\ & + \delta[g_B(X, U_B)g_B(Y, \dot{U}_B) + g_B(Y, U_B)g_B(X, \dot{U}_B)] + \psi D_B(X, Y) \\ & + \frac{1}{f}H^f(X, Y). \end{aligned}$$

By contraction from the above equation over X, Y , we find

$$(3.27) \quad \tau_B = \alpha(n-1) + \beta g_B(U_B, U_B) + \gamma g_B(\dot{U}_B, \dot{U}_B) + \psi D_B(e_i, e_i) + \frac{\Delta f}{f}.$$

On the other hand, we know from the equation (3.24) that

$$(3.28) \quad \tau_M = \alpha n + \beta g_B(U_B, U_B) + \gamma g_B(\dot{U}_B, \dot{U}_B) + \psi D_B(e_i, e_i).$$

By use of (3.28) in (3.27) we get $\tau_B = \tau_M - \alpha + \frac{\Delta f}{f}$. In view of Lemma 2.2 we also know that

$$(3.29) \quad -\frac{\tau_M}{n} = \frac{\Delta f}{f}.$$

The last two equations give us $\tau_B = \frac{n-1}{n}\tau_M - \alpha$. On the other hand, since Hessian of f is proportional to the metric tensor g_B , it can be written as follows $H^f(X, Y) = \frac{\Delta f}{n-1}g_B(X, Y)$. As the consequence of the equation (3.29) we have $\frac{\Delta f}{n-1} = -\frac{1}{n(n-1)}\tau_M f$, which implies that

$$H^f(X, Y) + \frac{\tau_B + \alpha}{(n-1)^2}f g_B(X, Y) = 0.$$

So, by OBATA's theorem B is isometric to the $(n-1)$ -dimensional sphere of radius $\frac{n-1}{\sqrt{\tau_B+\alpha}}$. Hence the theorem proved.

4. EXAMPLES OF 4-DIMENSIONAL MIXED SUPER QUASI-EINSTEIN MANIFOLD

Here we construct a nontrivial concrete example of a mixed super quasi-Einstein manifold. Let us consider a Riemannian metric g on the 4-dimensional real number space M^4 by

$$ds^2 = g_{ij}dx^i dx^j = e^{2x^1}(dx^1)^2 + \sin^2 x^1[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4 and $0 < x^1 < \frac{\pi}{2}$. Therefore for the above metric only the nonvanishing components of the christoffel symbols, the curvature tensor and the Ricci tensor are

$$[11, 1] = 1, [12, 2] = [13, 3] = [14, 4] = \cot x^1,$$

$$[22, 1] = [33, 1] = [44, 1] = -\frac{\sin 2x^1}{2e^{2x^1}},$$

$$R_{2332} = \frac{\sin^2 x^1 \cos^2 x^1}{e^{2x^1}} = R_{2442} = R_{3443}$$

$$R_{1221} = -\sin^2 x^1(1 + \cot x^1) = R_{1331} = R_{1441},$$

$$R_{22} = \frac{2 \cos^2 x^1 - \sin^2 x^1(1 + \cot x^1)}{e^{2x^1}} = R_{33} = R_{44}$$

$$R_{11} = -3(1 + \cot x^1), \quad r = \frac{6(\cot^2 x^1 - \cot x^1 - 1)}{e^{2x^1}} \neq 0,$$

provided $(\cot^2 x^1 - \cot x^1 - 1) \neq 0$ and the components which can be obtained from these by symmetric properties. From the above it can be says that M^4 is a Riemannian manifold of nonvanishing scalar curvature. We shall now show that this manifold is $MS(QE)_4$. Let us now define

$$\alpha = -\frac{(1 + \cot x^1)}{e^{2x^1}}, \quad \beta = \frac{1}{e^{2x^1}} = \gamma, \quad \delta = -\frac{1 + \cot x^1}{2 \cos^2 x^1}, \quad \psi = 1$$

the 1-forms are given by

$$A_i(x) = \begin{cases} \cos x^1, & i = 2, 3, 4 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad B_i(x) = \begin{cases} \cos x^1, & i = 2, 3, 4 \\ 0, & \text{otherwise,} \end{cases}$$

the associated tensor

$$D_{ij}(x) = \begin{cases} -3(1 + \cot x^1), & i = j = 1 \\ (1 + \cot x^1), & i = j = 2, 3, 4 \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

- (i) $R_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + 2\delta A_1 B_1 + \psi D_{11},$
- (ii) $R_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2 + 2\delta A_2 B_2 + \psi D_{22},$
- (iii) $R_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + 2\delta A_3 B_3 + \psi D_{33},$
- (iv) $R_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma B_4 B_4 + 2\delta A_4 B_4 + \psi D_{44}.$

Since all the cases other than (i) - (iv) are trivial, we can say

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j + \delta(A_i B_j + A_j B_i) + \psi D_{ij}, \text{ for } i, j = 1, 2, 3, 4.$$

So, we can say that the manifold under consideration is a mixed super quasi-Einstein manifold. Therefore we have

Example 4.1. Let (M^4, g) be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij} dx^i dx^j = e^{2x^1} (dx^1)^2 + \sin^2 x^1 [(dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4 . Then it is a mixed super quasi-Einstein manifold with nonvanishing scalar curvature.

Next we consider the Lorentzian metric g on M^4 by

$$ds^2 = g_{ij}dx^i dx^j = -e^{2x^1}(dx^1)^2 + \sin^2 x^1[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4 . Now, similar way after some construction of associated scalars and associated 1-forms, we can say that the manifold is a mixed super Quasi-Einstein manifold. Therefore we get another example of $MS(QE)_4$.

Example 4.2. Let (M^4, g) be a 4-dimensional Lorentzian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = -e^{2x^1}(dx^1)^2 + \sin^2 x^1[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4 . Then it is a mixed super quasi-Einstein manifold with nonvanishing scalar curvature.

5. EXAMPLE OF WARPED PRODUCT ON MIXED SUPER QUASI-EINSTEIN MANIFOLD

Here we consider the example 4.1, a 4-dimensional example of mixed super quasi-Einstein manifold. Let (M_4, g) be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = e^{2x^1}(dx^1)^2 + \sin^2 x^1[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M_4 . Then we have shown that it is mixed generalized quasi-Einstein manifold with nonzero and nonconstant scalar curvature.

To define warped product on $MS(QE)_4$, we consider the warping function $f : \mathbf{R} \rightarrow (0, \infty)$ by $f(x^1) = \sin x^1$, here we observe that $f = \sin x^1 > 0$ for $0 < x^1 < \frac{\pi}{2}$ and is a smooth function. And the line element defined on $\mathbf{R} \times \mathbf{R}^3$ which is of the form $I \times_f F$, where $I = \mathbf{R}$ is the base and $F = \mathbf{R}^3$ is the fibre.

Therefore the metric $ds_M^2 = ds_B^2 + f^2 ds_F^2$ that is

$$ds^2 = g_{ij}dx^i dx^j = e^{2x^1}(dx^1)^2 + \sin^2 x^1[(dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

is the example of Riemannian warped product on $MS(QE)_4$.

ACKNOWLEDGMENTS

The author is supported by Dr. DSK PDF No. F.4-2/2006(BSR)/MA/13-14/0002 of India.

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