

LORENTZIAN PARA-SASAKIAN MANIFOLD WITH QUARTER-SYMMETRIC NON-METRIC CONNECTION

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ABSTRACT. The object of the present paper is to study quarter symmetric non-metric connection on a LP-Sasakian manifold. In this paper, we consider some properties of the curvature tensor, projective curvature tensor, concircular curvature tensor, conformal curvature tensor with respect to quarter symmetric non-metric connection in a LP-Sasakian manifolds. Finally we consider submanifolds with respect to quarter symmetric non-metric connection.

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1. INTRODUCTION

In [5], H. A. Hayden introduced a metric connection with non-zero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semi-symmetric (symmetric) and non-metric connection have been studied by many authors ([1], [2], [4], [7], [17], [22]). The idea of quarter-symmetric linear connections in a differential manifold was introduced by S.Golab [4]. A linear connection is said

to be a quarter-symmetric connection if its torsion tensor $\bar{T}$ is of the form

$$(1) \quad \bar{T}(X, Y) = u(Y)\varphi X - u(X)\varphi Y,$$

for any vector fields X, Y on a manifold, where u is a 1-form and φ is a tensor of type $(1, 1)$. If $\varphi = I$, then the quarter-symmetric connection is reduced to a semi-symmetric connection. Hence quarter-symmetric connection can be viewed as a generalization of semi-symmetric connection.. The connection $\bar{\nabla}$ is said to be a metric connection if there is a Riemannian metric g in M such that $\bar{\nabla}g = 0$, otherwise it is non-metric. In [16], Sharfuddin and Hussian defined a semi-symmetric metric connection in an almost contact manifold, by setting

$$\bar{T}(X, Y) = \eta(Y)X - \eta(X)Y.$$

In [20] and [21] the authors studied the semi-symmetric metric connection and semi-symmetric non-metric connection in a Kenmotsu manifold, respectively. Some properties of a quarter-symmetric metric connection on Sasakian manifolds and quarter-symmetric non-metric connection on P-sasakian manifolds are investigated in [12] and [13].

On the other hand, in 1989, K. Matsumoto [8] introduced the notion of Lorentzian para-Sasakian manifolds. I. Mihai and R. Rosca [10] studied the same manifolds independently and they obtained several results on such manifolds. Lorentzian para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [9], I. Mihai, A.A. Shaikh and U. C. De [11]. S. K. Srivastava and R. P. Kushwaha studied Lorentzian para-Sasakian manifolds admitting a special semi symmetric recurrent metric connection [19]. In [14] S.Y. Perktas, E. Kilic and M, M, Tripathi investigated curvature tensors with respect to semi-symmetric connection in a Lorentzian para-Sasakian manifold. It is shown that a Lorentzian para-Sasakian manifold with semi-symmetric non-metric connection is an η -Einstein manifold [15].

In the present paper, we study quarter symmetric non-metric connection on a Lorentzian para-Sasakian manifold. Section 2 is devoted to preliminaries. In section 3, we introduce quarter symmetric non-metric connection on a Lorentzian para-Sasakian manifold. In section 4, we calculate curvature tensor and Ricci tensor of a Lorentzian para-Sasakian manifold with respect to quarter symmetric non-metric connection, respectively. We give an example and some characterizations. In section 5, it is shown that if a Lorentzian para-Sasakian manifold is ϕ - projectively flat

with respect to quarter symmetric non-metric connection, then the manifold is an η -Einstein manifold with respect to quarter symmetric non-metric connection. In section 6, we find an expression for concircular curvature tensor with respect to quarter symmetric non-metric connection. In section 7, we have proved that if a Lorentzian para-Sasakian manifold is conformally flat with respect to quarter symmetric non-metric connection, then the manifold is an Einstein manifold with respect to quarter symmetric non-metric connection. Finally, submanifolds of a Lorentzian para-Sasakian manifold with respect to quarter-symmetric non-metric connection are studied.

2. PRELIMINARIES

A differentiable manifold of dimension n is called Lorentzian para-Sasakian (briefly, LP-Sasakian) [8, 10], if it admit a $(1, 1)$ tensor field ϕ , a contravariant vector field ξ , a 1-form η and Lorentzian metric g which satisfy

$$(2) \quad \eta(\xi) = -1,$$

$$(3) \quad \phi^2(X) = X + \eta(X)\xi,$$

$$(4) \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

$$(5) \quad g(X, \xi) = \eta(X),$$

$$(6) \quad \nabla_X \xi = \phi X,$$

$$(7) \quad (\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

where ∇ is Levi-Civita connection with respect to the Lorentzian metric g . It can be easily seen that in an LP-Sasakian manifold the following relations hold:

$$(8) \quad \phi\xi = 0, \eta(\phi X) = 0, \text{rank}\phi = n - 1,$$

If we write

$$(9) \quad \Phi(X, Y) = g(\phi X, Y),$$

for any vector field X and Y , then the tensor field $\Phi(X, Y)$ is a symmetric $(0, 2)$ tensor field [8]. Also, since the vector η is closed in an LP-Sasakian [8, 11] manifold, we have

$$(10) \quad (\nabla_X \eta)Y = \Phi(X, Y), \Phi(X, \xi) = 0,$$

for any vector field X and Y .

Let M be an n -dimensional LP-Sasakian manifold. Then the following relations hold [11, 9]:

$$(11) \quad g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y),$$

$$(12) \quad R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X,$$

$$(13) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$(14) \quad R(\xi, X)\xi = X + \eta(X)\xi,$$

$$(15) \quad S(X, \xi) = (n-1)\eta(X),$$

$$(16) \quad S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y)$$

for any vector fields X, Y and Z , where R and S are the curvature and Ricci tensors of M , respectively.

A LP-Sasakian manifold M is said to be η Einstein if its Ricci tensor S is of the form

$$(17) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

for any $X, Y \in \Gamma(TM)$, where a, b are scalar functions such that $b \neq 0$. If $b = 0$ then M is called Einstein manifold.

Let M' be a submanifold of a LP-Sasakian manifold. The Gauss and Weingarten formulas are given

$$(18) \quad \nabla_X Y = \nabla'_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM'),$$

$$(19) \quad \nabla_X N = -A_N X + \nabla_X^\perp N, \quad \forall N \in \Gamma(T^\perp M'),$$

where $\{\nabla_X Y, A_N X\}$ and $\{h(X, Y), \nabla_X^\perp N\}$ belong to $\Gamma(TM')$ and $\Gamma(T^\perp M')$, respectively.

3. QUARTER-SYMMETRIC NON-METRIC CONNECTION

Let M be an n -dimensional LP-Sasakian manifold with Levi-Civita connection ∇ . If we set

$$(20) \quad \overline{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X,$$

for any vector field X and Y , then $\bar{\nabla}$ is a linear connection on M . We know that the torsion tensor $\bar{T}$ with respect to connection $\bar{\nabla}$ is given

$$\bar{T}(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y].$$

From (20) we get

$$(21) \quad \bar{T}(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

Furthermore (20) we have

$$(22) \quad (\bar{\nabla}_X g)(Y, Z) = -\eta(Y)\Phi(X, Z) - \eta(Z)\Phi(X, Y),$$

for any vector field X and Y , which implies that $\bar{\nabla}$ is a quarter symmetric non-metric connection on M . Also by using (8), (9) and (22) we get

$$(\bar{\nabla}_\xi g)(Y, Z) = 0,$$

which means that the metric g is ξ -parallel with respect to quarter symmetric non-metric connection.

From (6) and (20) we have the following proposition:

Proposition 1. *Let M be a LP-Sasakian manifold, then the manifold M is ξ -parallel with respect to quarter symmetric non-metric connection $\bar{\nabla}$, that is $\bar{\nabla}_X \xi = 0$.*

Now, by using (3), (7), (8) and (20) we obtain

$$(23) \quad (\bar{\nabla}_X \phi)Y = g(X, Y)\xi + \eta(X)\eta(Y)\xi,$$

Thus we have

Proposition 2. *Let M be a LP-Sasakian manifold with quarter symmetric non-metric connection. We have the following relation:*

$$(24) \quad (\bar{\nabla}_X \phi)Y = (\bar{\nabla}_Y \phi)X, \quad X, Y \in \Gamma(TM).$$

4. CURVATURE TENSOR

Let M be an n -dimensional LP-Sasakian manifold. The curvature tensor $\bar{R}$ of the quarter symmetric non-metric connection $\bar{\nabla}$ on M is defined by

$$(25) \quad \bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]} Z,$$

From (10), (20) and (25) we have

$$(26) \quad \begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + \Phi(X, Z)\phi Y - \Phi(Y, Z)\phi X \\ &\quad + \eta(Z)\{\eta(Y)X - \eta(X)Y\}, \end{aligned}$$

where

$$(27) \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

is the curvature tensor with respect to the Levi-Civita connection ∇ . Using (26) and the first Bianchi identity we have the following proposition

Proposition 3. *Let M be an n -dimensional LP-Sasakian manifold with quarter symmetric non-metric connection. Then the first Bianchi identity of the quarter-symmetric nonmetric connection $\bar{\nabla}$ on M is provided.*

From (13) and (26) we have

$$(28) \quad \begin{aligned} \bar{K}(X, Y, Z, U) &= K(X, Y, Z, U) + \Phi(X, Z)g(\phi Y, U) - \Phi(Y, Z)g(\phi X, U) \\ &\quad + \eta(Z)K(X, Y, \xi, U) \end{aligned}$$

where K and $\bar{K}$ are given by

$$K(X, Y, Z, U) = g(R(X, Y)Z, U)$$

and

$$\bar{K}(X, Y, Z, U) = g(\bar{R}(X, Y)Z, U).$$

Theorem 4. *Let M be an n -dimensional LP-Sasakian manifold with quarter symmetric non-metric connection. Then we have the following equations:*

$$\begin{aligned}
 (29) \quad & \bar{K}(X, Y, Z, U) + \bar{K}(Y, X, Z, U) = 0, \\
 & \bar{K}(X, Y, Z, U) + \bar{K}(X, Y, U, Z) = \eta(Z)K(X, Y, \xi, U) \\
 (30) \quad & + \eta(U)K(X, Y, \xi, Z), \\
 & \bar{K}(X, Y, Z, U) - \bar{K}(Z, U, X, Y) = \eta(Y)\eta(Z)g(X, U) \\
 (31) \quad & - \eta(X)\eta(U)g(Y, Z),
 \end{aligned}$$

for any $X, Y, Z, U \in \Gamma(TM)$.

Proof. From (28) we obtain (29). By using (13) and (28) we get (30). Again from (28) we have (31). $\square$

Proposition 5. *Let M be an n -dimensional LP-Sasakian manifold with quarter symmetric non-metric connection. Then we have the following equations:*

$$\begin{aligned}
 & g(\bar{R}(X, Y)Z, \xi) = \eta(\bar{R}(X, Y)Z) \\
 (32) \quad & = g(R(X, Y)Z, \xi), \\
 (33) \quad & \bar{R}(X, Y)\xi = 0, \\
 (34) \quad & \bar{R}(\xi, X)Y = g(X, Y)\xi + \eta(X)\eta(Y)\xi,
 \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. From (5), (8) and (26) we obtain (32). By using (2), (9), (13) and (26) we get (33). Again from (2), (9), (12) and (26) we have (34). $\square$

The Ricci tensor $\bar{S}$ of an LP-Sasakian manifold M with respect to quarter symmetric non-metric connection $\bar{\nabla}$ is given by

$$\bar{S}(X, Y) = \sum_{i=1}^n \varepsilon_i g(\bar{R}(e_i, X)Y, e_i).$$

The scalar curvature of M with respect to quarter symmetric non-metric connection $\bar{\nabla}$ is defined by

$$\bar{r} = \sum_{i=1}^n \varepsilon_i \bar{S}(e_i, e_i)$$

where $X, Y \in \Gamma(TM)$, $\{e_1, e_2, \dots, e_n\}$ is an orthonormal frame and $\varepsilon_i = g(e_i, e_i)$.

From (28) we have

$$(35) \quad \begin{aligned} \bar{S}(X, Y) &= \sum_{i=1}^n \varepsilon_i \{g(R(e_i, X)Y, e_i) + g(\phi X, \phi Y) - \Phi(X, Y)g(\phi e_i, e_i)\} \\ &- \eta(Y)\eta(X) + n\eta(X)\eta(Y). \end{aligned}$$

Since the Ricci tensor of the Levi-Civita connection ∇ is given by

$$S(X, Y) = \sum_{i=1}^n \varepsilon_i g(R(e_i, X)Y, e_i),$$

then by using (4) and (35) we get

$$(36) \quad \bar{S}(X, Y) = S(X, Y) + g(X, Y) - \text{trace}(\Phi)\Phi(X, Y) + n\eta(X)\eta(Y).$$

Ricci tensor of the Levi-Civita connection is symmetric, thus from (36) we have the following theorem:

Theorem 6. *Let M be an n -dimensional LP-Sasakian manifold. Then the Ricci tensor $\bar{S}$ of quarter symmetric non-metric connection $\bar{\nabla}$ is symmetric.*

Using (36) we get the following theorem

Theorem 7. *Let M be an n -dimensional LP-Sasakian manifold. Then we have*

$$(37) \quad \bar{r} = r,$$

where r is scalar curvature of Levi-Civita connection.

Lemma 8. *Let M be an n -dimensional LP-Sasakian manifold. Then we have.*

$$(38) \quad \bar{S}(X, \xi) = 0,$$

$$(39) \quad \bar{S}(\phi X, \phi Y) = \bar{S}(X, Y),$$

for any $X, Y \in \Gamma(TM)$.

Proof. From (36) we get

$$\bar{S}(X, \xi) = S(X, \xi) + g(X, \xi) - \text{trace}(\Phi)\Phi(X, \xi) + n\eta(X)\eta(\xi)$$

Using (2), (8) and (15) we have (38). From (36) we get

$$\bar{S}(\phi X, \phi Y) = S(\phi X, \phi Y) + g(\phi X, \phi Y) - \text{trace}(\Phi)\Phi(\phi X, \phi Y) + n\eta(\phi X)\eta(\phi Y)$$

By using (4), (8) and (16) we obtain (39). □

Example 9. We consider a 3-dimensional manifold $M = \{(x, y, z) \in R^3 : z > 0\}$, where (x, y, z) are the standard coordinates in R^3 . Let E_1, E_2, E_3 be a linearly independent global frame on M given by

$$(40) \quad E_1 = e^z \frac{\partial}{\partial x}, \quad E_2 = e^{z-ax} \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z},$$

where a is a nonzero constant such that $a \neq 1$. Let g be the Lorentzian metric defined by

$$g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0, \quad g(E_1, E_1) = g(E_2, E_2) = 1, \quad g(E_3, E_3) = -1,$$

Let η be the 1-form defined by $\eta(U) = g(U, E_3)$, for any $U \in TM$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi E_1 = -E_1, \phi E_2 = -E_2$ and $\phi E_3 = 0$. Then, using the linearity of ϕ and g we have $\eta(E_3) = -1$, $\phi^2 U = U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$ for any $U, W \in TM$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M .

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g . Then, we have

$$(41) \quad [E_1, E_2] = -ae^z E_2, \quad [E_1, E_3] = -E_1, \quad [E_2, E_3] = -E_2,$$

Using Koszul formula for the Lorentzian metric g , we can easily calculate

$$(42) \quad \begin{aligned} \nabla_{E_1} E_1 &= -E_3, & \nabla_{E_1} E_2 &= 0, & \nabla_{E_1} E_3 &= -E_1, \\ \nabla_{E_2} E_1 &= ae^z E_2, & \nabla_{E_2} E_2 &= -ae^z E_1 - E_3, & \nabla_{E_2} E_3 &= -E_2, \\ \nabla_{E_3} E_1 &= 0, & \nabla_{E_3} E_2 &= 0, & \nabla_{E_3} E_3 &= 0. \end{aligned}$$

From the above computations, it can be easily seen that for $E_3 = \xi$, (ϕ, ξ, η, g) is an LP-Sasakian structure on M [6]. Thus from (20) quarter symmetric non-metric connection on M is given by

$$(43) \quad \begin{aligned} \bar{\nabla}_{E_1} E_1 &= -E_3, & \bar{\nabla}_{E_1} E_2 &= 0, & \bar{\nabla}_{E_1} E_3 &= 0, \\ \bar{\nabla}_{E_2} E_1 &= ae^z E_2, & \bar{\nabla}_{E_2} E_2 &= -ae^z E_1 - E_3, & \bar{\nabla}_{E_2} E_3 &= 0, \\ \bar{\nabla}_{E_3} E_1 &= 0, & \bar{\nabla}_{E_3} E_2 &= 0, & \bar{\nabla}_{E_3} E_3 &= 0. \end{aligned}$$

Using the above relations, we can calculate the non-vanishing components of the curvature tensor with respect to quarter symmetric non-metric connection as follows:

$$(44) \quad \begin{aligned} \bar{R}(E_1, E_2)E_1 &= a^2 e^{2z} E_2, \quad \bar{R}(E_1, E_2)E_2 = -a^2 e^{2z} E_1, \\ \bar{R}(E_1, E_3)E_1 &= -E_3, \quad \bar{R}(E_2, E_3)E_2 = -E_3. \end{aligned}$$

Hence, we obtain the equations (33) and (34). From the equations (44), we can easily calculate the non-vanishing components of the ricci tensor with respect to quarter symmetric non-metric connection as follows:

$$(45) \quad \bar{S}(E_1, E_1) = -1 - a^2 e^{2z}, \quad \bar{S}(E_2, E_2) = -1 - a^2 e^{2z}.$$

then we get the equations (38) and (39).

Theorem 10. Let M be an n - dimensional LP -Sasakian manifold. Then

$$\bar{R}(\xi, Y) \cdot \bar{S} = 0,$$

for any $Y \in \Gamma(TM)$.

Proof. Using (4) and (34) we obtain

$$\bar{S}(\bar{R}(\xi, Y)Z, W) + \bar{S}(Z, \bar{R}(\xi, Y)W) = \{g(\phi Y, \phi Z)\bar{S}(\xi, W) + g(\phi Y, \phi W)\bar{S}(\xi, Z)\}.$$

From (38) proof is completed. $\square$

5. PROJECTIVE CURVATURE TENSOR

Let M be an n - dimensional LP -Sasakian manifold. the projective curvature tensor $\bar{P}$ of type $(1, 3)$ of M with respect to quarter symmetric non-metric connection $\bar{\nabla}$ is defined by

$$(47) \quad \bar{P}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{n-1} \{\bar{S}(Y, Z)X - \bar{S}(X, Z)Y\}.$$

From (26) and (36), using (47) we get

$$(48) \quad \begin{aligned} \bar{P}(X, Y)Z &= P(X, Y)Z - \frac{1}{n-1} \{g(Y, Z)X - g(X, Z)Y + \text{trace}(\Phi)\Phi(X, Z)Y \\ &\quad - \text{trace}(\Phi)\Phi(Y, Z)X + \eta(Z)\eta(Y)X - \eta(Z)\eta(X)Y\} + \Phi(X, Z)\phi Y \\ &\quad - \Phi(Y, Z)\phi X. \end{aligned}$$

where

$$(49) \quad P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}\{S(Y, Z)X - S(X, Z)Y\}.$$

is the projective curvature tensor of M with respect to Levi-Civita connection ∇ . Using (48) we get

$$(50) \quad \bar{P}(X, Y)Z + \bar{P}(Y, X)Z = 0,$$

and

$$(51) \quad \bar{P}(X, Y)Z + \bar{P}(Y, Z)X + \bar{P}(Z, X)Y = 0.$$

Definition 11. *Let M be an n - dimensional LP -Sasakian manifold. Then M is said to be ξ - projectively flat with respect to quarter symmetric non-metric connection $\bar{\nabla}$ if $\bar{P}(X, Y)\xi = 0$ on M .*

Using (33) and (47) we have

$$\bar{P}(X, Y)\xi = -\frac{1}{n-1}\{\bar{S}(Y, \xi)X - \bar{S}(X, \xi)Y\}.$$

From the last equation, using (38) we obtain

$$(52) \quad \bar{P}(X, Y)\xi = 0.$$

Then we have the following theorem

Theorem 12. *Let M be an n - dimensional LP -Sasakian manifold with quarter symmetric non-metric connection. Then M is ξ - projectively flat with respect to quarter symmetric non-metric connection.*

Definition 13. *Let M be an n - dimensional LP -Sasakian manifold. Then M is said to be ϕ - projectively flat with respect to quarter symmetric non-metric connection if $g(\bar{P}(\phi X, \phi Y)\phi Z, \phi U) = 0$ on M .*

From (47)

$$(53) \quad \bar{P}(\phi X, \phi Y)\phi Z = \bar{R}(\phi X, \phi Y)\phi Z - \frac{1}{n-1}\{\bar{S}(\phi Y, \phi Z)X - \bar{S}(\phi X, \phi Z)\phi Y\}.$$

Using (39) and (53), ϕ - projectively flatness implies

$$(54) \quad \bar{K}(\phi X, \phi Y, \phi Z, \phi U) = \frac{1}{n-1}\{\bar{S}(Y, Z)g(\phi X, \phi U) + \bar{S}(X, Z)g(\phi Y, \phi U)\}.$$

Let $\{e_1, e_2, \dots, e_{n-1}, \xi\}$ be a local orthogonal basis of the vector fields in M and using the fact that $\{\phi e_1, \phi e_2, \dots, \phi e_{n-1}, \xi\}$ is also a local orthogonal basis, putting $X = U = e_i$ and summing up with respect to $i = 1, 2, \dots, n-1$ we obtain

$$(55) \quad \sum_{i=1}^{n-1} \bar{K}(\phi e_i, \phi Y, \phi Z, \phi e_i) = \frac{1}{n-1} \left\{ \sum_{i=1}^{n-1} \bar{S}(Y, Z)g(\phi e_i, \phi e_i) + \bar{S}(e_i, Z)g(\phi Y, \phi e_i) \right\}.$$

We know that

$$(56) \quad \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n-1,$$

and from (39)

$$(57) \quad \sum_{i=1}^{n-1} \bar{S}(e_i, Z)g(\phi Y, \phi e_i) = \bar{S}(Y, Z).$$

Thus from (56) and (57), the equation (55) becomes

$$(58) \quad \sum_{i=1}^{n-1} \bar{K}(\phi e_i, \phi Y, \phi Z, \phi e_i) = \frac{n-2}{n-1} \bar{S}(Y, Z).$$

Moreover we have

$$(59) \quad \sum_{i=1}^{n-1} \bar{K}(\phi e_i, \phi Y, \phi Z, \phi e_i) = \bar{S}(\phi Y, \phi Z) - g(\bar{R}(\xi, \phi Y)\phi Z, \xi).$$

Using (11), (32), (39) and (59) we get

$$(60) \quad \sum_{i=1}^{n-1} \bar{K}(\phi e_i, \phi Y, \phi Z, \phi e_i) = \bar{S}(Y, Z) + g(\phi Y, \phi Z)$$

Therefore by using (58) and (60) we obtain

$$(61) \quad \bar{S}(Y, Z) = (1-n)\{g(Y, Z) + \eta(Y)\eta(Z)\}$$

Hence we have the following theorem

Theorem 14. *If a LP-Sasakian manifold is ϕ - projectively flat with respect to quarter symmetric non-metric connection, then the manifold is an η - Einstein manifold with respect to quarter symmetric non-metric connection.*

Theorem 15. *Let M be an n - dimensional LP-Sasakian manifold. Then we have*

$$\bar{P}(\xi, Y) \cdot \bar{S} = 0,$$

for any $Y \in \Gamma(TM)$.

Proof. Using (4), (34), (38) and (47) we obtain

$$(62) \quad \overline{S}(\overline{P}(\xi, Y)Z, W) + \overline{S}(Z, \overline{P}(\xi, Y)W) = 0.$$

From (62) the proof is complete. $\square$

6. CONCIRCULAR CURVATURE TENSOR

Let M be an n -dimensional LP -Sasakian manifold. The concircular curvature tensor of M with respect to quarter symmetric non-metric connection $\overline{\nabla}$ is defined by

$$(63) \quad \overline{C}^*(X, Y)Z = \overline{R}(X, Y)Z - \frac{\overline{r}}{n(n-1)}\{g(Y, Z)X - g(X, Z)Y\}$$

By using (26) and (38) we get from (63)

$$(64) \quad \begin{aligned} \overline{C}^*(X, Y)Z &= C^*(X, Y)Z + \Phi(X, Z)\phi Y - \Phi(Y, Z)\phi X + \eta(Z)\{\eta(Y)X - \eta(X)Y\} \\ &\quad + \frac{r}{n(n-1)}\{g(Y, Z)X - g(X, Z)Y\}. \end{aligned}$$

Using (29) and (63) we get

$$(65) \quad \overline{C}^*(X, Y)Z + \overline{C}^*(Y, X)Z = 0,$$

and

$$(66) \quad \overline{C}^*(X, Y)Z + \overline{C}^*(Y, Z)X + \overline{C}^*(Z, X)Y = 0.$$

If we consider $\overline{C}^* = C^*$, substituting Y by ξ in (64), from (2) and (5) we have

$$(67) \quad g(X, Z) = -\eta(X)\eta(Z).$$

Using equations (3), (4), (9) and (67) we obtain

$$(68) \quad \Phi(X, Z) = 0.$$

From (36), (67) and (68) we have the following theorem

Theorem 16. *In a LP -Sasakian manifold, if concircular curvature tensor is invariant under quarter symmetric non-metric connection then we have*

$$\overline{S}(X, Y) = S(X, Y) + (n-1)\eta(X)\eta(Y),$$

for any $X, Y \in \Gamma(TM)$.

7. CONFORMAL CURVATURE TENSOR

Let M be an n - dimensional LP-Sasakian manifold. The conformal curvature tensor of M with respect to quarter symmetric non-metric connection $\bar{\nabla}$ is given

$$\begin{aligned}
 \bar{C}(X, Y, Z, U) &= \bar{K}(X, Y, Z, U) \\
 &\quad - \frac{1}{n-2} \{g(Y, Z)\bar{S}(X, U) - g(X, Z)\bar{S}(Y, U) \\
 &\quad + \bar{S}(Y, Z)g(X, U) - \bar{S}(X, Z)g(Y, U)\} \\
 (69) \quad &\quad + \frac{\bar{r}}{(n-1)(n-2)} \{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}
 \end{aligned}$$

using (28), (36) and (38) we obtain

$$\begin{aligned}
 \bar{C}(X, Y, Z, U) &= C(X, Y, Z, U) \\
 &\quad - \frac{1}{n-2} \{g(X, Z)[-2g(Y, U) + \text{trace}\Phi\Phi(Y, U) - n\eta(Y)\eta(U)] \\
 &\quad + g(Y, Z)[2g(X, U) - \text{trace}\Phi\Phi(X, U) + n\eta(X)\eta(U)] \\
 &\quad + g(X, U)[- \text{trace}\Phi\Phi(Y, Z) + n\eta(Y)\eta(Z)] \\
 &\quad + g(Y, U)[\text{trace}\Phi\Phi(X, Z) - n\eta(X)\eta(Z)]\} \\
 &\quad + \Phi(X, Z)\Phi(Y, U) - \Phi(Y, Z)\Phi(X, U) \\
 (70) \quad &\quad + \eta(Z)\{\eta(Y)g(X, U) - \eta(X)g(Y, U)\},
 \end{aligned}$$

where C is the conformal curvature tensor of M with respect to Levi Civita connection.

Suppose that LP-Sasakian manifold is conformally flat with respect to quarter symmetric non-metric connection, that is $\bar{C}(X, Y, Z, U) = 0$. By using (69) we get

$$\begin{aligned}
 \bar{K}(X, Y, Z, U) &= \frac{1}{n-2} \{g(Y, Z)\bar{S}(X, U) - g(X, Z)\bar{S}(Y, U) \\
 &\quad + \bar{S}(Y, Z)g(X, U) - \bar{S}(X, Z)g(Y, U)\} \\
 (71) \quad &\quad - \frac{\bar{r}}{(n-1)(n-2)} \{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}.
 \end{aligned}$$

Putting $Y = Z = \xi$ in (71) and using (33), (37) and (38) we obtain

$$(72) \quad \bar{S}(X, U) = \frac{r}{n-1} g(X, U).$$

from (72) we have the following theorem

Theorem 17. *If a LP-Sasakian manifold is conformally flat with respect to quarter symmetric non-metric connection, then the manifold is Einstein manifold with respect to quarter symmetric non-metric connection.*

8. SUBMANIFOLDS OF AN LP-SASAKIAN MANIFOLD WITH RESPECT TO QUARTER-SYMMETRIC NON-METRIC CONNECTION

Let M' be a submanifold of a LP-Sasakian manifold M with quarter symmetric non metric connection $\tilde{\nabla}$. For any $X \in \Gamma(TM')$ and $N \in \Gamma(T^\perp M')$ we can write

$$(73) \quad \phi X = PX + QX, \quad PX \in \Gamma(TM'), \quad QX \in \Gamma(T^\perp M'),$$

$$(74) \quad \phi N = tN + sN, \quad tN \in \Gamma(TM'), \quad sN \in \Gamma(T^\perp M').$$

The Gauss and weingarten formulas with respect to $\bar{\nabla}$ are given, respectively,

$$(75) \quad \bar{\nabla}_X Y = \bar{\nabla}'_X Y + \bar{h}(X, Y)$$

$$(76) \quad \bar{\nabla}_X N = -\bar{A}_N X + \bar{\nabla}_X^\perp N$$

for any $X, Y \in \Gamma(TM')$, where $\bar{\nabla}'_X Y, \bar{A}_N X \in \Gamma(TM')$. Here, $\bar{\nabla}'$, $\bar{h}$ and $\bar{A}_N$ are called the induced connection on M , the second fundamental form and the Weingarten mapping with respect to $\bar{\nabla}$. Using (18), (19), (20), (73), (74), (75) and (76) we obtain

$$(77) \quad \bar{\nabla}'_X Y = \nabla'_X Y + \eta(Y)PX,$$

$$(78) \quad \bar{h}(X, Y) = h(X, Y) + \eta(Y)QX,$$

$$(79) \quad \bar{A}_N X = A_N X - \eta(N)PX,$$

$$(80) \quad \bar{\nabla}_X^\perp N = \nabla_X^\perp N + \eta(N)QX.$$

for any $X, Y \in \Gamma(TM')$. From (77) we get

$$(81) \quad (\bar{\nabla}'_X g)(Y, Z) = -\eta(Y)g(PX, Z) - \eta(Z)g(PX, Y)$$

On the other hand, the torsion tensor of the induced connection $\bar{\nabla}'$ is

$$(82) \quad T'(X, Y) = \eta(Y)PX - \eta(X)PY,$$

From the last two equation we have the following proposition.

Proposition 18. *Let M' be a submanifold of a LP-Sasakian manifold M with quarter symmetric non metric connection $\tilde{\nabla}$. Then the induced connection $\tilde{\nabla}'$ is a quarter symmetric non-metric connection on M' .*

From (77), if the submanifold is ϕ - anti-invariant we have the following proposition.

Proposition 19. *Let M' be a ϕ - anti-invariant submanifold of a LP-Sasakian manifold M with quarter symmetric non metric connection $\tilde{\nabla}$. Then the induced quarter symmetric non-metric connection and induced Riemannian connection are equivalent.*

Let the submanifold be ϕ - invariant, from (73) and (78) we have the following proposition.

Proposition 20. *Let M' be a ϕ - invariant submanifold of a LP-Sasakian manifold M with quarter symmetric non metric connection $\tilde{\nabla}$. Then the second fundamental form with respect to quarter symmetric non-metric connection and second fundamental form with respect to Riemannian connection are equivalent.*

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