

THE EXCHANGE VARIATIONS BETWEEN BEZIER DIRECTRIX CURVES OF TWO DEVELOPABLE RULED SURFACES

VAHIDE BULUT, ALI CALISKAN

¹DEPARTMENT OF MATHEMATICS, EGE UNIVERSITY, IZMIR, 35100, TURKEY

E-MAILS: VAHIDEBULUT@MAIL.EGE.EDU.TR, ALI.CALISKAN@EGE.EDU.TR

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ABSTRACT. Developable ruled surfaces and Bezier curves are widely used in Computer Aided Geometric Design (CAGD). A ruled surface is a surface generated by sweeping a line through space, or more specifically, sweeping a line along a directrix curve. A developable ruled surface is a surface where the surface normal at all points along a given ruling is constant. If the generators and directrix curve of a developable ruled surface are known, operations with these surfaces and designs of them are much simplified.

In this paper, we present obtaining the Bezier directrix curve of a developable ruled surface from the Bezier directrix curve of the other developable ruled surface and the exchange variations of the arc lengths of these directrix curves. Also, we expressed the curvature exchanges between these Bezier directrix curves of two developable ruled surfaces. Also, based on this variation, we have discussed some ideas about the variations of developable ruled surfaces.

AMS Classification: 53A04, 53A05, 49K05

Keywords: Developable ruled surface; Bezier curve; Arc length; Variation

1. INTRODUCTION

Developable ruled surfaces are widely used in sheet-metal and plate-metal based industries [3, 5], surfaces of marine craft and aircraft skins [8] and architectural

structures since these surfaces can be unfolded (developed) into a plane without stretching or tearing.

A developable ruled surface can be a plane, a conical surface, a cylindrical surface or tangential surface of a curve, [6]. Gaussian curvature of these surfaces is zero at all points of the surface, [7].

Bezier curves frequently used in computer graphics and related fields. Bezier curves are named after their inventor Dr. Pierre Bezier. He developed a curve formulation in the early 1960's to use this curve in shape design that would be intuitive enough for designers and artists to use, without requiring a background in mathematics.

In this paper, we present obtaining the Bezier directrix curve of a developable ruled surface from the Bezier directrix curve of the other developable ruled surface. Also, we expressed the exchange variations of the arc lengths of these directrix curves and the curvature exchanges between these Bezier directrix curves of two developable ruled surfaces.

2. RULED AND DEVELOPABLE RULED SURFACES

A surface generated by one parameter family of straight lines is called ruled surface. A ruled surface is given as;

$$(1) \quad P(s, t) = x(s) + td(s)$$

where, $x(s)$ is the base curve or the directrix and for constant s , the line $l_s(t) = x(s_0) + td(s_0)$ is the generator or the ruling of the ruled surface, as seen from Figure (1).

A ruled surface where the tangent plane is constant at all points along a given ruling is called developable ruled surface. A developable ruled surface is a surface where the surface normal at all points along a given ruling is constant.

A ruled surface is a developable ruled surface if and only if, [7]

$$(2) \quad \left(d(s), \frac{dx}{ds}, \frac{dd}{ds} \right) = 0.$$

A surface is developable if and only if $K=0$ everywhere on the surface, [7].

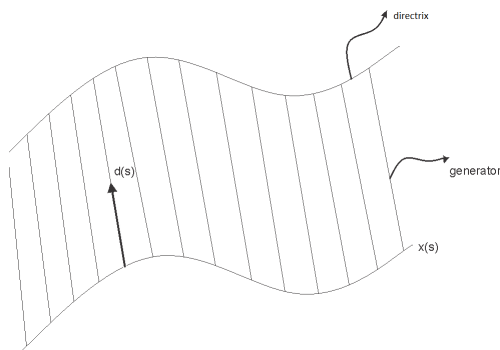


FIGURE 1. A developable ruled surface

3. A NEW BEZIER CURVE OBTAINED FROM THE OTHER BEZIER CURVE

A Bezier curve is a parametric curve. An n th-degree Bezier curve is defined as

$$(3) \quad x(u) = \sum_{i=0}^n B_{i,n}(u) P_i \quad 0 \leq u \leq 1$$

The basis functions $\{B_{i,n}(u)\}$ are the n th- degree Bernstein polynomials defined by

$$B_{i,n}(u) = \frac{n!}{i!(n-i)!} u^i (1-u)^{n-i}$$

where, $\{P_i\}$ are called control points, [9].

Let's give a Bezier curve defined by the equation (3). We can change a point $x(\hat{u})$ to a new location $\bar{x}$.

We need to find new control points $\{\hat{P}_i\}$ such that

$$(4) \quad \bar{x} = \sum_{i=0}^n B_{i,n}(\hat{u}) \hat{P}_i,$$

which can be viewed as one equation in $n-1$ unknowns. Let's suppose that P_0 and P_n should stay the same. Since this is undetermined linear system (for $n > 2$) with infinitely many solutions, we need to find the one that is closest to the original control points. Linear system can be written as follows

$$(5) \quad [B_{1,n}(\hat{u}) \dots B_{n-1,n}(\hat{u})] \begin{bmatrix} \hat{P}_1 \\ \cdot \\ \cdot \\ \cdot \\ \hat{P}_{n-1} \end{bmatrix} = \bar{x} - \hat{P}_0 B_{0,n}(\hat{u}) - \hat{P}_n B_{n,n}(\hat{u}) = m$$

or

$$(6) \quad A\hat{B} = m$$

where $A = [B_{1,n}(\hat{u}) \dots B_{n-1,n}(\hat{u})]$. The unknowns can be written as

$$(7) \quad \hat{B} = B + A^T c$$

where c is the unknown vector. Using a least squares approach the following equation

$$A\hat{B} = AB + AA^T c$$

is obtained. Thus,

$$c = \frac{1}{a} (\bar{x} - x)$$

and finally

$$\hat{B} = B + \frac{1}{a} A^T (\bar{x} - x)$$

where, $a = AA^T$.

All Bezier points $\hat{P}_1, \dots, \hat{P}_{n-1}$ move in the same direction $\bar{x} - x$, but with different displacement magnitudes, [4].

4. THE VARIATION OF NEAREST NEIGHBOR CURVE

Let the curve $x(s)$ be planar or spatial curve. Hence, the nearest neighbor curve $\bar{x}(s)$ can be defined by

$$(8) \quad \vec{\bar{x}} = \vec{x} + u\mathbf{T} + v\mathbf{N} + w\mathbf{B} = \vec{x} + \vec{y}$$

where $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ are the unit vectors of the S. Frenet frame and the quantities u, v, w are the functions that multiplied with the parameter ε of s as follows

$$(9) \quad u = \varepsilon \bar{u}(s), \quad v = \varepsilon \bar{v}(s), \quad w = \varepsilon \bar{w}(s), \quad \vec{y} = \varepsilon \vec{\bar{y}}(s).$$

If the condition ($\varepsilon \rightarrow 0$) occurs, then the nearest neighbor curve $\bar{x}(s)$ approaches to the curve $x(s)$.

Let's expand the series according to ε 's powers of the arc length

$$(10) \quad \bar{s} = \int_{s_1}^{s_2} \sqrt{\bar{x}'^2} ds$$

of the nearest neighbor curve. If this expansion is shown as a linear term δs according to ε , the following equation

$$(11) \quad \bar{s} = s + \delta s + \dots$$

can be written. Hence, δs is called as first variation of the arc length and defined by

$$(12) \quad \delta s = \varepsilon \cdot \lim_{\varepsilon \rightarrow 0} \frac{\bar{s} - s}{\varepsilon}.$$

If the curvature and the torsion of the curve ($\vec{x}$) can be shown as

$$\kappa = \frac{1}{\rho}, \quad \sigma = \frac{1}{\tau}$$

then, the S. Frenet formulas can be written as below

$$(13) \quad \mathbf{T}' = \kappa \mathbf{N}, \quad \mathbf{N}' = -\kappa \mathbf{T} + \sigma \mathbf{B}, \quad \mathbf{B}' = -\sigma \mathbf{N}.$$

After some computations,

$$\bar{s} = s + \int_{s_1}^{s_2} \vec{x}' \cdot \vec{y}' ds + \dots = s + [u]_{s_1}^{s_2} - \int_{s_1}^{s_2} \kappa v ds + \dots$$

and finally the first variation of the arc length

$$(14) \quad \delta s = [u]_{s_1}^{s_2} - \int_{s_1}^{s_2} \kappa v ds$$

is found.

On the other hand, the curvature $\bar{\kappa} = 1 : \bar{\rho}$ of the curve $(\vec{x})$ is [2]

$$(15) \quad \bar{\kappa} = \frac{\sqrt{\vec{x}'^2 \vec{x}''^2 - (\vec{x}' \vec{x}'')^2}}{\left(\sqrt{\vec{x}'^2}\right)^3}.$$

5. ARC LENGTH PARAMETERIZATION

The parameterization of curves is one of the important problems in CAGD. For a general parametric curve $x(u)$, an arc length parametrization $x(s)$ is the length l between two points on the curve $x(s_0)$ and $x(s_1)$ is $l = s_1 - s_0$. Any linear relationship between l and s is called an arc-length parametrization. For most formulations used in curve design, the length of the curve is not linearly related to the values of the parameter. In many applications of parametric curves it is essential to be able to relate easily the parametric values to the arc length, and reciprocally. An example can be given as in computer assisted animation systems where the animator defines a flexible object (such as a shoe-lace, a rope, etc..) whose length is to be kept constant. Arc length parametrization is also needed to compute the speed along curves, such as motion paths used in animation, [10].

Many methods for parameterization try to be close to arc length one. Arc length parameterization is usually very difficult. Even if the arc length parameterization could be obtained, they are generally not of the desirable Bezier form. Hence, approximate arc length parameterization will be a good alternative way, [1]. Therefore, several algorithms and solution methods have been used in the literature to get approximate arc length parameterization.

6. THE EXCHANGE VARIATIONS BETWEEN BEZIER DIRECTRIX CURVES OF TWO DEVELOPABLE RULED SURFACE

Let $P(s, t) = x(s) + t\mathbf{T}(s)$, $x'(s) = \mathbf{T}$ be a developable ruled surface. As mentioned in Section 2, the curve $x(s)$ is the directrix curve of this surface. Since, the Bezier curve, which is given with the equation (3), can be reparametrized with arc

length parameter by using several methods in the literature, this Bezier curve can be assumed as the directrix curve of the developable ruled surface $P(s, t)$. Let's suppose the Bezier curve $\bar{x}(s)$ in (4) corresponds to the nearest neighbor curve in (8). Hence, the nearest neighbor Bezier curve can be expressed by using arc length parameter as

$$(16) \quad \bar{x} = \sum_{i=0}^n B_{i,n}(\hat{u}) \hat{P}_i = \vec{x} = \vec{x} + u\mathbf{T} + v\mathbf{N} + w\mathbf{B} = \vec{x} + \vec{y}.$$

On the other hand, we can write a developable ruled surface whose directrix curve is the nearest neighbor Bezier curve $\bar{x}(s)$ of the directrix curve $x(s)$ of the developable ruled surface $P(s, t)$ as

$$(17) \quad \bar{P}(s, t) = \bar{x}(s) + t\bar{\mathbf{T}}(s), \quad \bar{x}'(s) = \bar{\mathbf{T}}.$$

Since,

$$(18) \quad \bar{x}'(s) = \bar{\mathbf{T}} = \vec{x}' + \vec{y}' = \vec{x}' = \mathbf{T} + \varepsilon \vec{y}'$$

and

$$(19) \quad \vec{y}' = \varepsilon \vec{y}' = (u' - \kappa v) \mathbf{T} + (v' + \kappa u - \sigma w) \mathbf{N} + (w' + \sigma v) \mathbf{B},$$

we can obtain the unit tangent vector of the Bezier directrix curve $\bar{x}(s)$ as

$$(20) \quad \bar{x}'(s) = \bar{\mathbf{T}} = (1 + u' - \kappa v) \mathbf{T} + (v' + \kappa u - \sigma w) \mathbf{N} + (w' + \sigma v) \mathbf{B}.$$

Therefore, by using the equations (17) and (20), the developable ruled surface $\bar{P}(s, t)$ can be written with the following equation

$$(21) \quad \bar{P}(s, t) = x(s) + t \left[(1 + u + u' - v\kappa) \mathbf{T} + (u\kappa + v + v' - w\sigma) \mathbf{N} + (v\sigma + w + w') \mathbf{B} \right]$$

If the Bezier directrix curves $(\bar{x})$ and $(\vec{x})$ have common boundary points $[u(s_1) = 0, u(s_2) = 0]$, then first variation of the arc length of these curves is

$$(22) \quad \delta s = - \int_{s_1}^{s_2} \kappa v ds.$$

If we write δn instead of v ,

$$(23) \quad \delta s = - \int_{s_1}^{s_2} \kappa \delta n ds$$

is obtained.

Also, the curvature relationship between the directrix curves $(\vec{x})$ and $(\vec{\bar{x}})$ can be expressed with

$$(24) \quad \bar{\kappa} = \kappa + \varepsilon \left[\left(\mathbf{N} \vec{y}'' \right) - 2\kappa \left(\mathbf{T} \vec{y}' \right) \right] + \dots$$

Therefore, from the equations (8), (9) and (19), the curvature variation of the directrix curves $(\vec{x})$ and $(\vec{\bar{x}})$ can be written as

$$(25) \quad \delta \kappa = \kappa^2 v - \kappa u' + \frac{d}{ds} \left(v' + \kappa u - \sigma w \right) - \left(w' + \sigma v \right) \sigma.$$

7. EXAMPLE

Let's given the Bezier curve $x(u)$ with the arc length parameterization, which is mentioned in Section 5, as $x(s)$ with the following equation.

$$x(s) = \left(2 \sin \left(\frac{s}{\sqrt{29}} \right), 2 \cos \left(\frac{s}{\sqrt{29}} \right), \frac{5s}{\sqrt{29}} \right).$$

We can define the nearest neighbor curve $\bar{x}(s)$ using the equations (8) and (9) as

$$\begin{aligned} \bar{x}(s) = & \left(\frac{19-s}{10} \sin \left(\frac{s}{\sqrt{29}} \right) + \frac{12s+15}{10\sqrt{29}} \cos \left(\frac{s}{\sqrt{29}} \right), \right. \\ & \frac{19-s}{10} \cos \left(\frac{s}{\sqrt{29}} \right) - \frac{12s+15}{10\sqrt{29}} \sin \left(\frac{s}{\sqrt{29}} \right), \\ & \left. \frac{51s-6}{10\sqrt{29}} \right). \end{aligned}$$

Thus, we can express the developable ruled surfaces which have these curves $x(s)$ and $\bar{x}(s)$ as the directrix curves with the following equations.

$$P(s, t) = \left(2 \sin \left(\frac{s}{\sqrt{29}} \right) + \frac{2t}{\sqrt{29}} \cos \left(\frac{s}{\sqrt{29}} \right), 2 \cos \left(\frac{s}{\sqrt{29}} \right) - \frac{2t}{\sqrt{29}} \sin \left(\frac{s}{\sqrt{29}} \right), \frac{5}{\sqrt{29}} (s + t) \right)$$

and

$$\begin{aligned} \bar{P}(s, t) = & \left(\frac{19-s}{10} \sin \left(\frac{s}{\sqrt{29}} \right) + \frac{12s+15}{10\sqrt{29}} \cos \left(\frac{s}{\sqrt{29}} \right) \right. \\ & + t \left(\left(-\frac{1}{10} \right) \sin \left(\frac{s}{\sqrt{29}} \right) + \frac{12}{10\sqrt{29}} \cos \left(\frac{s}{\sqrt{29}} \right) - \frac{12s+15}{10 \cdot 29} \cos \left(\frac{s}{\sqrt{29}} \right) \right), \\ & \frac{19-s}{10} \cos \left(\frac{s}{\sqrt{29}} \right) - \frac{12s+15}{10\sqrt{29}} \sin \left(\frac{s}{\sqrt{29}} \right) \\ & + t \left(\left(-\frac{1}{10} \right) \cos \left(\frac{s}{\sqrt{29}} \right) - \frac{19-s}{10} \sin \left(\frac{s}{\sqrt{29}} \right) - \frac{12}{10\sqrt{29}} \sin \left(\frac{s}{\sqrt{29}} \right) - \frac{12s+15}{10 \cdot 29} \cos \left(\frac{s}{\sqrt{29}} \right) \right), \\ & \left. \frac{51s-6}{10\sqrt{29}} + t \frac{51}{10\sqrt{29}} \right). \end{aligned}$$

We can see the directrix curves $x(s)$ and $\bar{x}(s)$ and also the developable ruled surfaces $P(s, t)$ and $\bar{P}(s, t)$ and the relations between them from the Figures (2), (3) and (4), and (5), (6) and (7), respectively.

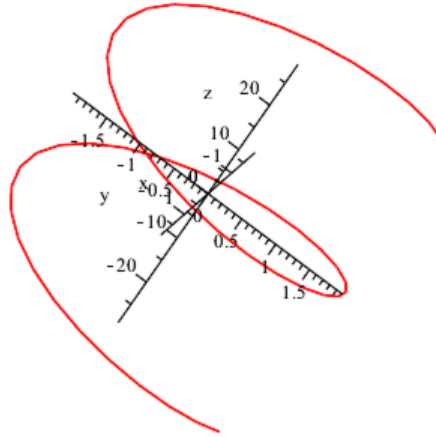


FIGURE 2. Directrix curve $x(s)$ of the developable ruled surface $P(s, t)$

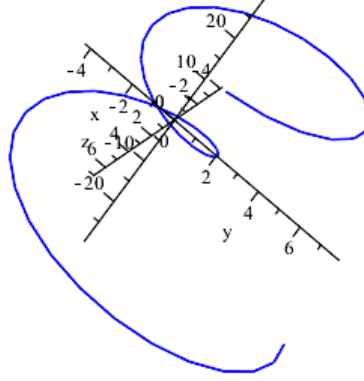


FIGURE 3. The nearest neighbor curve $\bar{x}(s)$ which is directrix curve of the developable ruled surface $\bar{P}(s, t)$

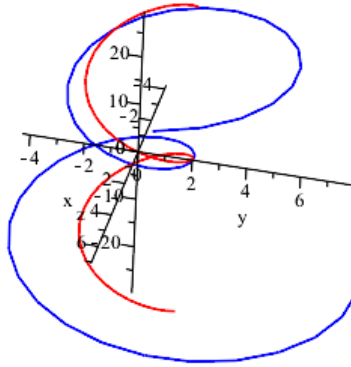
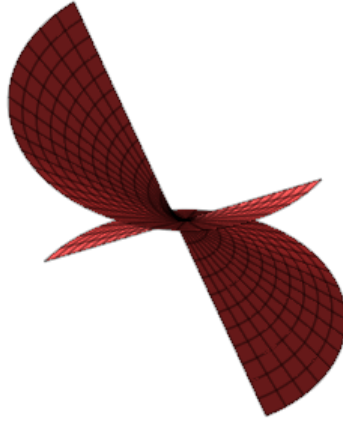
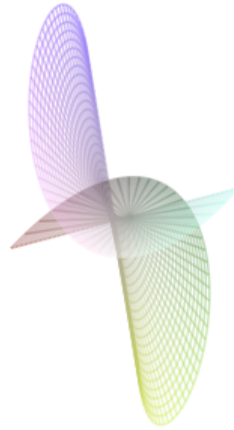


FIGURE 4. Directrix curve $x(s)$ and the nearest neighbor curve $\bar{x}(s)$

As seen from figures, we can regulate the curves and whereat the developable ruled surfaces only using the value ε . Hence, we can obtain the optimum Bezier curve and the optimum developable ruled surface that user wants.

8. CONCLUSION

In this paper, we expressed how to obtain the nearest neighbor Bezier directrix curve of a developable ruled surface from the Bezier directrix curve of the other

FIGURE 5. The developable ruled surface $P(s, t)$ FIGURE 6. The nearest neighbor developable ruled surface $\bar{P}(s, t)$

developable ruled surface. On the other hand, we presented the exchange variations of the arc lengths and the curvature exchanges of these directrix curves. Also, based on this variation, we gave some relationships between these two developable ruled surfaces. Therefore, the user can define optimum Bezier curve and the optimum

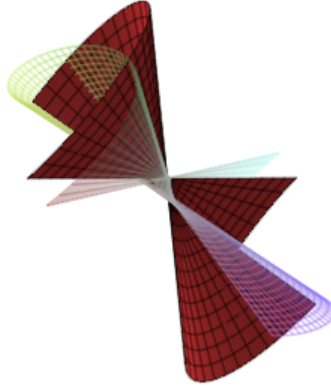


FIGURE 7. The developable ruled surfaces $P(s, t)$ and $\bar{P}(s, t)$

developable ruled surface which has this curve as directrix curve for designing the particular shape.

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