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# SOME CURVATURE IDENTITIES ON AN ALMOST CONFORMAL GRADIENT SHRINKING RICCI SOLITON

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Abstract. In this paper we have introduced the notion of almost conformal gradient shrinking Ricci soliton and established some curvature identities for almost conformal gradient shrinking Ricci soliton.

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**Keywords:** Ricci flow; Ricci soliton; almost Ricci soliton; conformal Ricci flow; gradient conformal Ricci soliton; curvature operator

## 1. Introduction

Hamilton introduced the study of Ricci flow [6] in the year 1982. This concept was developed to answer Thurston's geometric conjecture which says that each closed three manifold admits a geometric decomposition. Hamilton also [7] classified all compact manifolds with positive curvature operator in dimension four. After that the Ricci flow becomes a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman also did an excellent work on Ricci flow [8], [9].

The Ricci flow equation is given by

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \tag{1}$$

on a compact Riemannian manifold $M$ with Riemannian metric $g(t)$. A solution to the Ricci flow is called a Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. If the vector field which induces the diffeomorphism is in fact the gradient of a function, then it is called as gradient Ricci soliton [3]. We know, for compact Riemannian manifold Ricci soliton becomes gradient Ricci soliton.

A.E. Fischer developed the concept of conformal Ricci flow [4] during 2003-04. In classical Ricci flow equation the unit volume constraint plays an important role but in conformal Ricci flow equation scalar curvature $R$ is considered as constraint. The conformal Ricci flow equation on $M$ where $M$ is considered as a smooth closed connected oriented n-Manifold($n > 3$), is defined by the equation [4],

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} - g_{ij}(\frac{2}{n} + p), \tag{2}$$

and $R(g) = -1$,
where p is a scalar non-dynamical field(time dependent scalar field).

Now, for a gradient Ricci soliton on a Riemannian manifold $(M^n, g_{ij})$, the equation will be [3],

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}, \tag{3}$$

for some constant $\rho$ and for a smooth function $f$ on $M$. $f$ is called a potential function of the Ricci soliton and $\nabla$ is the levi-civita connection on $f$. When $\rho = 0$, it is called Steady soliton, when $\rho > 0$, the soliton is called shrinking and for $\rho < 0$, it is called expanding soliton. If $f$ is constant then the above equation becomes $R_{ij} = \rho g_{ij}$. Hence the manifold becomes Einstein Manifold. In particular a gradient shrinking Ricci soliton satisfies the equation,

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} = 0, \tag{4}$$

where $\tau = T - t$ and $T$ is the time when the soliton deforms into a point, and $f$ is the Ricci potential function.

Again for conformal Ricci soliton, if the vector field is the gradient of a function $f$, then we call it as a conformal gradient Ricci soliton [2]. For conformal gradient Ricci soliton the equation is,

$$R_{ij} + \nabla_i \nabla_j f = \left(\frac{1}{2\tau} - \frac{2}{n} - p\right) g_{ij}, \tag{5}$$

and for conformal gradient shrinking Ricci soliton $(\frac{1}{2\tau} - \frac{2}{n} - p) > 0$, which implies $n < \frac{4\tau}{1-2p\tau}$. Also we know that $n > 3$. So we have $3 < n < \frac{4\tau}{1-2p\tau}$. In 2014, A. Bhattacharyya and N. Basu have studied some curvature identities on gradient shrinking conformal Ricci soliton in [2].

The concept of Ricci almost soliton was first introduced by S. Pigola, M. Rigoli, M. Rimoldi, A. G. Setti in 2010 [11]. R. Sharma has also done excellent work in almost Ricci soliton [12]. A Riemannian manifold $(M^n, g)$ is an almost Ricci soliton [1], if there exist a complete vector field $X$ and a smooth soliton function $\lambda : M^n \to \mathbb{R}$ satisfying,

$$R_{ij} + \frac{1}{2}(X_{ij} + X_{ji}) = \lambda g_{ij},$$

where $R_{ij}$ and $X_{ij} + X_{ji}$ stand for the Ricci tensor and the Lie derivative $\pounds_X g$ in coordinates respectively. It will be called expanding, steady or shrinking, respectively, if $\lambda < 0, \lambda = 0$ or $\lambda > 0$.

We introduce the notion of almost gradient shrinking Ricci soliton by

$$R_{ij} + \nabla_i \nabla_j f - \lambda g_{ij} = 0, \tag{6}$$

where $\lambda : M^n \to \mathbb{R}$ is a smooth function.

We also introduce the notion of almost conformal gradient shrinking Ricci soliton by

$$R_{ij} + \nabla_i \nabla_j f = (\lambda - \frac{2}{n} - p) g_{ij} \tag{7}$$

where $\lambda : M^n \to \mathbb{R}$ is a smooth function.

Ricci soliton is a very important tool in general relativity. Just like Ricci soliton, almost Ricci soliton could be very helpful in the study of relativity. Peter Petersen and William Wylie [10], M. Fernandez [5] studied the rigidity of gradient Ricci soliton and shrinking Ricci solitons by using integral identities. So we shall try to establish the following identities and the theorems to study the rigidity of almost conformal gradient Ricci soliton in near future and to develop this concept we need to focus on the notion of almost conformal gradient shrinking Ricci soliton.

Inner multiplication by $g^{ij}$ to the equation (7) gives

$$R + \triangle f = n\lambda - 2 - pn$$

where $\triangle$ is the Laplacian.

For conformal Ricci flow $R = -1$, so we have

$$\triangle f = n(\lambda - p) - 1. \tag{8}$$

Also for a gradient shrinking Ricci soliton we have

$$R + |\nabla f|^2 = \frac{f - c}{\tau},$$

where $c$ is a constant.

So for almost gradient shrinking Ricci soliton we get

$$R + |\nabla f|^2 = 2\lambda(f - c).$$

Now for conformal Ricci flow $R = -1$, so we have

$$|\nabla f|^2 = 2\lambda(f - c) + 1. \tag{9}$$

Thus we have the following result:

**Result1.1**. : For an almost conformal gradient shrinking Ricci soliton,

$$\triangle f = n(\lambda - p) - 1,$$

and

$$|\nabla f|^2 = 2\lambda(f - c) + 1.$$

From [3] we have

$$(divRm)_{jkl} = \nabla_i R_{ijkl}$$

Now from Bianchi's second identity we get

$$R_{ijkl,i} + R_{ijli,k} + R_{ijik,l} = 0$$

which implies

$$\nabla_i R_{ijkl} = -\nabla_k R_{ijli} - \nabla_l R_{ijik}. \tag{10}$$

Again

$$R_{ijli} = g_{im} R^m_{jli}.$$

Putting $m = i$ in the above equation we get

$$R_{ijli} = g_{ii} R^i_{jli} = -R_{jl}.$$

Similarly

$$R_{ijik} = g_{im} R^m_{jik},$$

putting $m = i$ in the above equation we have

$$R_{ijik} = g_{ii} R^i_{jik} = R_{jk}.$$

So (10) reduces to

$$\nabla_i R_{ijkl} = \nabla_k R_{jl} - \nabla_l R_{jk}. \tag{11}$$

Taking covariant derivative on both sides of (7) we get

$$\nabla_k R_{ij} + \nabla_k f_{ij} = \lambda_k g_{ij},$$

where $f_{ij}$ is the covariant derivative of $f$ with respect to $j$ and $i$ respectively.

Or

$$\nabla_k R_{jl} + \nabla_k f_{jl} = \lambda_k g_{jl},$$

which implies

$$\nabla_k R_{jl} = \lambda_k g_{jl} - \nabla_k f_{jl} \tag{12}$$

and

$$\nabla_l R_{jk} = \lambda_l g_{jk} - \nabla_l f_{jk}. \tag{13}$$

Putting (11) and (12) in (13) we have

$$\nabla_i R_{ijkl} = \lambda_k g_{jl} - \nabla_k f_{jl} - \lambda_l g_{jk} + \nabla_l f_{jk}$$

$$= (\lambda_k g_{jl} - \lambda_l g_{jk}) + (\nabla_l f_{jk} - \nabla_k f_{jl})$$

$$= (\lambda_k g_{jl} - \lambda_l g_{jk}) + (\nabla_l \nabla_k f_j - \nabla_k \nabla_l f_j)$$

$$= (\lambda_k g_{jl} - \lambda_l g_{jk}) + R_{lkjp} f_p.$$

So

$$\nabla_i R_{ijkl} = (\lambda_k g_{jl} - \lambda_l g_{jk}) + R_{lkjp} f_p. \tag{14}$$

Multiply by $e^{-f}$ on both sides of (14) yields

$$\nabla_i (R_{ijkl}) e^{-f} - e^{-f} f_p R_{lkjp} = (\lambda_k g_{jl} - \lambda_l g_{jk}) e^{-f}.$$

Replacing $i$ by $p$ in the first term of the above equation and using the property of Riemannian curvature tensor we get

$$\nabla_p (R_{klpj}) e^{-f} - e^{-f} f_p R_{lkjp} = (\lambda_k g_{jl} - \lambda_l g_{jk}) e^{-f},$$

which implies

$$\nabla_i (R_{ijkl} e^{-f}) = (\lambda_k g_{jl} - \lambda_l g_{jk}) e^{-f}. \tag{15}$$

Again from (15) we have

$$-\nabla_i (R_{jikl} e^{-f}) = (\lambda_k g_{jl} - \lambda_l g_{jk}) e^{-f}.$$

Contracting over $j$ and $l$ in the above equation we have

$$-\nabla_i(g^{jl}R_{jikl}e^{-f}) = (\lambda_k g^{jl}g_{jl} - \lambda_l g^{jl}g_{jk})e^{-f}$$

i.e.

$$\nabla_i(R_{ik}e^{-f}) = \lambda_k(1-n)e^{-f}. \tag{16}$$

Hence from (15) and (16) we state the following result:

**Result 1.2. :** For an almost conformal gradient shrinking Ricci soliton,

1. $\nabla_i(R_{ijkl}e^{-f}) = (\lambda_k g_{jl} - \lambda_l g_{jk})e^{-f}$
2. $\nabla_i(R_{ik}e^{-f}) = \lambda_k(1-n)e^{-f}$

## 2. IDENTITIES ON RIEMANNIAN CURVATURE UNDER ALMOST CONFORMAL GRADIENT SHRINKING RICCI SOLITON

In this section we derive some curvature identities on almost conformal gradient shrinking Ricci soliton.

**Lemma 2.1. :** On an almost compact conformal gradient shrinking Ricci soliton we have

$$\int |divRm|^2 e^{-f} = 2\int (\lambda_k g_{jl}R_{ijkl,i} - f_{ki}R_{lj}R_{ijkl})e^{-f} \tag{17}$$

*Proof*: From (11) and (14) we get

$$\int |divRm|^2 e^{-f}$$
$$= \int \{(\lambda_k g_{jl} - \lambda_l g_{jk}) + R_{lkjp} f_p\}(R_{jl,k} - R_{jk,l})e^{-f}$$
$$= \int (\lambda_k g_{jl} - \lambda_l g_{jk})(R_{jl,k} - R_{jk,l})e^{-f} + \int R_{lkjp} f_p (R_{jl,k} - R_{jk,l})e^{-f}. \tag{18}$$

Now simplifying the second term of (18) we obtain

$$\int R_{lkjp} f_p (R_{jl,k} - R_{jk,l})e^{-f}$$
$$= \int R_{lkjp} f_p R_{jl,k} e^{-f} - \int R_{lkjp} f_p R_{jk,l} e^{-f}.$$

Integrating by parts to the above equation we obtain

$$= -\int R_{lkjp} f_{pk} R_{jl} e^{-f} + \int R_{lkjp} f_{pl} R_{jk} e^{-f}$$
$$= -2\int R_{lkjp} f_{kp} R_{lj} e^{-f}.$$

So (18) becomes

$$= \int (\lambda_k g_{jl} R_{jl,k} - \lambda_k g_{jl} R_{jk,l} - \lambda_l g_{jk} R_{jl,k} + \lambda_l g_{jk} R_{jk,l})e^{-f} - 2\int R_{lkjp} R_{lj} f_{kp} e^{-f}.$$

Changing $l \to k$, $k \to l$ in third and fourth terms in first integral of (18) we have

$$\int |divRm|^2 e^{-f} = 2\int \lambda_k g_{jl}(R_{jl,k} - R_{jk,l})e^{-f} - 2\int R_{lkjp} R_{lj} f_{kp} e^{-f}, \tag{19}$$

which implies

$$\int |divRm|^2 e^{-f} = 2\int (\lambda_k g_{jl} R_{ijkl,i} - f_{ki} R_{lj} R_{ijkl})e^{-f}.$$

**Theorem 2.1. :** On an almost compact conformal gradient shrinking Ricci soliton

$$\int Rm(Rc,Rc)e^{-f} = \frac{1}{2}\int |divRm|^2 e^{-f} + \int |Rc|^2(\lambda - \frac{2}{n} - p)e^{-f} - \int \lambda_k g_{jl} R_{ijkl,i} e^{-f} \tag{20}$$

*Proof:* We have from lemma (2.1)

$$\int |divRm|^2 e^{-f} = 2\int (\lambda_k g_{jl} R_{ijkl,i} - f_{kp} R_{lj} R_{lkjp})e^{-f}.$$

Also from (7) we have

$$f_{kp} = (\lambda - \frac{2}{n} - p)g_{kp} - R_{kp}. \tag{21}$$

Putting the value of $f_{kp}$ from (21) in (17) we get

$$\int |divRm|^2 e^{-f} = 2\int \lambda_k g_{jl} R_{ijkl,i} e^{-f} - 2\int R_{lkjp} R_{lj}[(\lambda - \frac{2}{n} - p)g_{kp} - R_{kp}]e^{-f}$$
$$= 2\int \lambda_k g_{jl} R_{ijkl,i} e^{-f} - 2\int |Rc|^2(\lambda - \frac{2}{n} - p)e^{-f} + 2\int Rm(Rc,Rc)e^{-f},$$

which implies

$$\int Rm(Rc,Rc)e^{-f} = \frac{1}{2}\int |divRm|^2 e^{-f} + \int |Rc|^2(\lambda - \frac{2}{n} - p)e^{-f} - \int \lambda_k g_{jl} R_{ijkl,i} e^{-f}.$$

**Lemma 2.2. :**

$$\nabla_i \nabla_j R_{ik} - \nabla_j \nabla_i R_{ik} = R_{jm} R_{mk} - R_{ijmk} R_{im} \tag{22}$$

*Proof:* Similar to the proof given in lemma 2.2 of [3].

**Lemma 2.3. :** On an almost compact conformal gradient shrinking Ricci soliton

$$\int \nabla_k R_{jl} \nabla_l R_{jk} e^{-f} = \int Rm(Rc, Rc) e^{-f} - \int |Rc|^2 (\lambda - \frac{2}{n} - p) e^{-f} \tag{23}$$

*Proof:* We start with the integration

$$\begin{aligned} & -2 \int \nabla_k R_{jl} \nabla_l R_{jk} e^{-f} \\ &= 2 \int R_{jk} (\nabla_i \nabla_j R_{ik} - \nabla_j R_{ik} f_i) e^{-f} \\ &= 2 \int R_{jk} (\nabla_i \nabla_j R_{ik}) e^{-f} - 2 \int R_{jk} \nabla_j R_{ik} f_i e^{-f} \end{aligned}$$

$$= 2 \int R_{jk} (\nabla_j \nabla_i R_{ik} + R_{mj} R_{mk} - R_{ijmk} R_{im}) e^{-f} + 2 \int R_{jk} R_{ik} f_{ij} e^{-f}. \tag{24}$$

Changing $m \to i$ in the first term of (24) we have

$$= 2 \int R_{jk} R_{ki} (f_{ij} + R_{ij}) e^{-f} - 2 \int R_{ijmk} R_{im} R_{jk} e^{-f}. \tag{25}$$

Putting (7) in (25) we get

$$= 2 \int R_{jk} R_{ki} [(\lambda - \frac{2}{n} - p) g_{ij} - R_{ij} + R_{ij}] e^{-f} - 2 \int R_{ijmk} R_{im} R_{jk} e^{-f} \tag{26}$$

i.e.

$$-2 \int \nabla_k R_{jl} \nabla_l R_{jk} e^{-f} = 2 \int |Rc|^2 (\lambda - \frac{2}{n} - p) e^{-f} - 2 \int Rm(Rc, Rc) e^{-f}.$$

Thus we obtain the following result:

For an almost compact conformal gradient shrinking Ricci soliton,

$$\int \nabla_k R_{jl} \nabla_l R_{jk} e^{-f} = \int Rm(Rc, Rc) e^{-f} - \int |Rc|^2 (\lambda - \frac{2}{n} - p) e^{-f}.$$

**Theorem 2.2. :** On an almost compact conformal gradient shrinking Ricci soliton we have

$$\int Rm(Rc, Rc) e^{-f} = \int |\nabla Rc|^2 e^{-f} + (\lambda - \frac{2}{n} - p) \int |Rc|^2 e^{-f} - \frac{1}{2} \int |divRm|^2 e^{-f} \tag{27}$$

*Proof:*

$$\begin{aligned} &\int |divRm|^2 e^{-f} \\ &= \int |\nabla_k R_{jl} - \nabla_l R_{jk}|^2 e^{-f} \\ &= 2 \int |\nabla Rc|^2 e^{-f} - 2 \int \nabla_k R_{jl} \lambda_l R_{jk} e^{-f}. \end{aligned} \tag{28}$$

From lemma(2.3) we have

$$\begin{aligned} &-2 \int \nabla_k R_{jl} \nabla_l R_{jk} e^{-f} \\ &= 2 \int |Rc|^2 (\lambda - \frac{2}{n} - p) e^{-f} - 2 \int Rm(Rc, Rc) e^{-f}. \end{aligned}$$

So,

$$\begin{aligned} &\int |divRm|^2 e^{-f} \\ &= 2 \int |\nabla Rc|^2 e^{-f} + 2 \int |Rc|^2 (\lambda - \frac{2}{n} - p) e^{-f} - 2 \int Rm(Rc, Rc) e^{-f}, \end{aligned}$$

which implies

$$2 \int Rm(Rc, Rc) e^{-f} = \int |\nabla Rc|^2 e^{-f} + \int |Rc|^2 (\lambda - \frac{2}{n} - p) e^{-f} - \frac{1}{2} \int |divRm|^2 e^{-f}.$$

## 3. IDENTITIES ON RICCI CURVATURE UNDER ALMOST COMPACT CONFORMAL GRADIENT SHRINKING RICCI SOLITON

**Theorem 3.1. :** On an almost compact conformal gradient shrinking Ricci soliton

$$\frac{1}{2}\int g^{li}|Rc|^2\triangle e^{-f} = \int g^{li}|\nabla Rc|^2 e^{-f} + \int H g^{li} e^{-f} - \int g^{li}\nabla R_{ijkl} f_l R_{jk} e^{-f}$$
$$-\int R_{ijkl}R_{jk}(n\lambda - 2 - pn)e^{-f} + \int g^{li}R_{li}R_{ijkl}R_{jk}e^{-f} + \int \lambda_{ji}R_{jl}e^{-f} -$$
$$\int g^{li}\lambda_{ii}g_{jk}R_{jk}e^{-f}, \tag{29}$$

where $H = \nabla_i\nabla_j R_{ik}R_{jk}e^{-f} - \nabla_j\nabla_i R_{ik}R_{jk}e^{-f}$.

*Proof:*

$$\triangle R_{jk} = \nabla_i(\nabla_i R_{jk}).$$

From (11) and (15) we get

$$= \nabla_i(\nabla_j R_{ik} - R_{ijkl}f_l - (\lambda_i g_{jk} - \lambda_j g_{ik}))$$
$$= \nabla_i\nabla_j R_{ik} - (\nabla_i R_{ijkl})f_l - R_{ijkl}f_{li} - (\lambda_{ii}g_{jk} - \lambda_{ji}g_{ik}). \tag{30}$$

Again

$$< \triangle Rc, Rc >$$
$$= \nabla_i\nabla_i R_{jk}R_{jk}$$
$$= \nabla_i(\nabla_i R_{jk})R_{jk}$$
$$= \nabla_i(\nabla_j R_{ik} - R_{ijkl}f_l - (\lambda_i g_{jk} - \lambda_j g_{ik}))R_{jk}$$
$$= \nabla_i\nabla_j R_{ik}R_{jk} - (\nabla_i R_{ijkl})f_l R_{jk} - R_{ijkl}f_{li}R_{jk} - (\lambda_{ii}g_{jk} - \lambda_{ji}g_{ik})R_{jk}. \tag{31}$$

Again

$$\begin{aligned}
&\frac{1}{2}\triangle|Rc|^2\\
&=\frac{1}{2}\triangle(R_{jk}R_{jk})\\
&=\nabla_i(\nabla_i R_{jk}R_{jk})\\
&=(\triangle R_{jk}R_{jk})+|\nabla Rc|^2. \qquad (32)
\end{aligned}$$

Furthermore we have

$$\frac{1}{2}\int\triangle|Rc|^2e^{-f}=\frac{1}{2}\int|Rc|^2\triangle e^{-f}.$$

So from (32) we get

$$\frac{1}{2}\int|Rc|^2\triangle e^{-f}=\int(\triangle Rc,Rc)e^{-f}+\int|\nabla Rc|^2e^{-f}. \qquad (33)$$

Putting the value of (31) in (33) we obtain

$$\begin{aligned}
&=\int|\nabla Rc|^2e^{-f}+\int(\nabla_i\nabla_jR_{ik}R_{jk}e^{-f}-(\nabla_iR_{ijkl})f_lR_{jk}e^{-f}-R_{ijkl}R_{jk}f_{li}e^{-f}\\
&\qquad -\lambda_{ii}g_{jk}R_{jk}e^{-f}+\lambda_{ji}g_{ik}R_{jk}e^{-f})\\
&=\int|\nabla Rc|^2e^{-f}+\int(\nabla_i\nabla_jR_{ik}R_{jk}e^{-f}-\nabla_j\nabla_iR_{ik}R_{jk}e^{-f})+\int\nabla_j\nabla_iR_{ik}R_{jk}e^{-f}\\
&\qquad -\int\nabla_iR_{ijkl}f_lR_{jk}e^{-f}-\int R_{ijkl}R_{jk}f_{li}e^{-f}\\
&\qquad +\int\lambda_{ji}g_{ik}R_{jk}e^{-f}-\int\lambda_{ii}g_{jk}R_{jk}e^{-f}\\
&=\int|\nabla Rc|^2e^{-f}+\int He^{-f}+\int\nabla_j\nabla_iR_{ik}R_{jk}e^{-f}-\int\nabla_iR_{ijkl}f_lR_{jk}e^{-f}\\
&\qquad -\int R_{ijkl}R_{jk}f_{li}e^{-f}+\int\lambda_{ji}g_{ik}R_{jk}e^{-f}-\int\lambda_{ii}g_{jk}R_{jk}e^{-f},
\end{aligned}$$

where $H=\nabla_i\nabla_jR_{ik}R_{jk}e^{-f}-\nabla_j\nabla_iR_{ik}R_{jk}e^{-f}$.

$$=\int|\nabla Rc|^2e^{-f}+\int He^{-f}-\int\nabla_iR_{ijkl}f_lR_{jk}e^{-f}-\int R_{ijkl}R_{jk}f_{li}e^{-f}$$

$$+\int \lambda_{ji} g_{ik} R_{jk} e^{-f} - \int \lambda_{ii} g_{jk} R_{jk} e^{-f}. \tag{34}$$

Now for almost conformal gradient Ricci soliton we have

$$f_{li} = (\lambda - \frac{2}{n} - p) g_{li} - R_{li}.$$

So the above equation gives

$$\frac{1}{2}\int |Rc|^2 \triangle e^{-f}$$
$$= \int |\nabla Rc|^2 e^{-f} + \int H e^{-f} - \int \nabla_i R_{ijkl} f_l R_{jk} e^{-f} - \int R_{ijkl} R_{jk} [(\lambda - \frac{2}{n} - p) g_{li}$$
$$-R_{li}] e^{-f} + \int \lambda_{ji} g_{ik} R_{jk} e^{-f} - \int \lambda_{ii} g_{jk} R_{jk} e^{-f}. \tag{35}$$

Inner multiplication by $g^{li}$ in (35) yields

$$\frac{1}{2}\int g^{li} |Rc|^2 \triangle e^{-f}$$
$$= \int g^{li} |\nabla Rc|^2 e^{-f} + \int H g^{li} e^{-f} - \int g^{li} \nabla R_{ijkl} f_l R_{jk} e^{-f}$$
$$- \int g^{li} R_{ijkl} R_{jk} g_{li} (\lambda - \frac{2}{n} - p) e^{-f} + \int g^{li} R_{li} R_{ijkl} R_{jk} e^{-f}$$
$$+ \int g^{li} \lambda_{ji} g_{ik} R_{jk} e^{-f} - \int g^{li} \lambda_{ii} g_{jk} R_{jk} e^{-f}$$
$$= \int g^{li} |\nabla Rc|^2 e^{-f} + \int H g^{li} e^{-f} - \int g^{li} \nabla R_{ijkl} f_l R_{jk} e^{-f} - \int R_{ijkl} R_{jk} (n\lambda - 2 - pn) e^{-f}$$
$$+ \int g^{li} R_{li} R_{ijkl} R_{jk} e^{-f} + \int \lambda_{ji} R_{jl} e^{-f} - \int g^{li} \lambda_{ii} g_{jk} R_{jk} e^{-f}.$$

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