

GENERALIZED LAG ANTI-SYNCHRONIZATION OF COUPLED CHAOTIC SYSTEMS

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ABSTRACT. This paper aims to study the generalized lag anti-synchronization of two dynamical systems via linear transformation. The method is based on the stability criterion of linear systems and the lag anti-synchronization of driving and response systems are ensured by that all roots of characteristic equations of the jacobian matrix have negative real parts. The linear generalized lag anti-synchronization can be robustly achieved, without calculating the Lyapunov exponents. In the proposed method the functional relation between driving system and response system are also achieved.

AMS Classification: 34C28

Keywords: Chaos synchronization; Generalized lag anti-synchronization (GLAS);
Rossler system

1. INTRODUCTION

Chaos synchronization [16] is of the fundamental importance in a variety of complex physical (Reyleigh-Benard convection), chemical (Belouov-Zhabotinski reaction) and electrical (Chua Matsumoto circuits) systems. Possible application areas of chaos synchronization are in secure communication, optimization of non-linear system performance, solar system, modelling brain activity and pattern recognition

phenomena. Different types synchronization have been formed in interacting chaotic systems. Complete, generalized, phase, lag, projective and anti-synchronization of chaotic dynamical systems described analytically and numerically. Complete synchronization is characterized by the convergence of two chaotic trajectories, $y(t) = x(t)$ [23]. Generalized synchronization is defined as the relation between the states of drive and driven systems, i.e., $y(t) = F(x(t))$ [17]. Phase synchronization means entrainment of phases of chaotic oscillations, $n\Phi_x - m\Phi_y = \text{constant}$ (n and m are integers), while their amplitudes remain chaotic and uncorrelated [19]. Lag synchronization for the first time was introduced by Rosenblum et. al. [20] under certain approximations in studying synchronization between bidirectionally coupled systems described by the ordinary differential equations with parameter mismatches $y(t) = x(t - \tau)$. Projective synchronization for the first time was introduced by Mainieri and Rahacek in 1999 [18]. He observed that projective synchronization in coupled partially linear chaotic systems where the driving and driven vectors synchronize up to a constant ratio α (scaling vector). Identical synchronization and anti-phase synchronization are the special cases of projective synchronization where $\alpha = 1$ and $\alpha = -1$ respectively. This proportional feature can be used to extend binary digital to M-nary digital communication for achieving fast communication. Anti-synchronization is a phenomenon that the state variables of the synchronized system have the same absolute values but opposite signs. We say that anti-synchronization of two systems S_1 and S_2 are achieved if $\lim_{t \rightarrow \infty} \|x_1(t) + x_2(t)\| = 0$, where $x_1(t), x_2(t)$ are the state vectors of the systems S_1 and S_2 respectively. It was well known that the first observation of synchronization of oscillators by Huygens in the seventeenth century was, in fact, anti-synchronization (AS) between pendulum clocks. Chil-Min Kim et al. [11] observed anti-synchronization phenomena in coupled identical chaotic oscillators. Zhang and Sun [28] studied anti-synchronization based on a suitable separation technique. In 2005 Hu et al. [4] observed adaptive control for anti-synchronization of Chua's chaotic system. Anti-synchronization of colpitts oscillators using active control was studied by Li et al. in 2005 [3]. An observer based anti-synchronization was investigated by Guo-Hui Li et al. [12]. Wenlin Li et al. [13] investigated anti-synchronization of two different chaotic systems in 2008. Mossa Al-Sawalha et al. [14] studied anti-synchronization of two hyperchaotic systems via nonlinear control. Recently

Khan et al. [10] have studied generalized anti-synchronization of different chaotic system via linear transformation. Therefore chaos synchronization has attracted the attention of many researchers. Basically, there are two types of synchronous chaotic systems. One is a unidirectional system. A unidirectional system consists of a driving system and a response system. Another type is coupled chaotic system where one system sends driving signals to other system and in the meantime, receives driving signals from other systems [26]. However, in the applications, it may be in such situations, information exchanged among systems are time delayed information. Time delay naturally raises one question: are the chaotic systems still synchronized when their driving signals are time delayed signals? Usually time delay will defy synchronization in coupled chaotic systems because time delay changes the structures of coupled systems and it is hard to get synchronization is non-identical subsystems. However when special structures of coupled systems are established, perfect synchronization can be obtained. He et al. [21] have studied time delayed chaotic systems and their synchronization. In 2004 Li et al. [2] introduced the synchronization in general complex dynamical networks with coupling delays. In 2006, Zhou et al. [8] have observed synchronization in general complex delayed dynamical network. Tarai et al. [26] have studied bidirectional coupled chaotic Chen's system with time lag in 2007. Shahverdiev et al. [24] have investigated chaos synchronization regimes in multiple-time-delay semiconductor lasers in 2008. In 2009 Huang et al. [5] have proposed synchronization of delayed chaotic system with parameter mismatches by using intermittent linear state feedback. Huys et al. [6] have discussed amplitude and phase effects on the synchronization of delay coupled oscillators in 2010. Illing et al. [7] have observed isochronal chaos synchronization of delay coupled optoelectronic oscillators in 2011. Recently in 2012, Ahn [1] have studied robust chaos synchronization based on adaptive fuzzy delayed feedback H_∞ control. In 1999, Yang and Chua [27] studied generalized synchronization of chaos via linear transformation. In this paper, they studied a special case of generalized synchronization in which the synchronization manifold is linear. But they cannot observed the generalized synchronization of identical chaotic systems with time lag. But in this paper we studied generalized lag anti-synchronization via linear transformation with time lag. The method is based on the stability criterion of linear systems and the synchronization of driving and response systems are ensured

by that all roots of characteristic equations of jacobian matrix have negative real parts. The linear GLAS can be robustly achieved without calculating the Lyapunov exponents. The new method can provide satisfactory GLAS for autonomous and non-autonomous systems. Here the autonomous Rossler system is treated as an example. The main goal of this paper is the predictability. Knowing the behaviour of driving system we able to predict the behaviour of the response system. Here the relationship is linear. If the transformation H is invertible then the driving system also be predictable from the behaviour of the response system [15].

2. LINEAR GLAS OF COUPLED CHAOTIC SYSTEMS

Any dynamical systems can be split into two parts, one is linear and another is nonlinear:

$$(1) \quad \dot{x} = Ax + \psi(x),$$

where A is an $n \times n$ any constant matrix and $\psi : R^n \rightarrow R^n$. We assume that the driving system transmit the signal $\psi(x)$ to the response system, then the unidirectional coupling scheme [9] is as follow:

$$(2) \quad \begin{aligned} \dot{x} &= Ax + \psi(x), \\ \dot{y} &= Ay - \Lambda\psi(x(t - \tau)), \end{aligned}$$

where Λ is $n \times n$ constant matrix.

Theorem

If the matrix Λ commutes with A , then the two dynamical systems in (2) are in a state of GLAS via the following transformation

$$(3) \quad y(\infty) = L(x(t - \tau)) = -\Lambda x(t - \tau),$$

if and only if all roots of characteristic equations A have negative real parts.

Proof: Let $z = y + \Lambda x(t - \tau)$, then the GLAS between the two chaotic systems in (2) via GLAS transformation $y = L(x(t - \tau)) = -\Lambda x(t - \tau)$ is equivalent to that of the origin of the following system

$$(4) \quad \begin{aligned} \dot{z} &= \dot{y} + \Lambda \dot{x}(t - \tau), \\ &= A(y + \Lambda x(t - \tau)), \\ &= Az. \end{aligned}$$

Therefore $z = 0$ is asymptotically stable if and only if all roots of characteristic equations of A have negative real parts. The matrices X which commutes with an $n \times n$ matrix A must be an $n \times n$ matrix which satisfies

$$(5) \quad AX = XA.$$

The above equation has many solutions, therefore we can construct various methods of linear GLAS between two chaotic dynamical systems. In this paper we discuss following three types of solutions of the equation (5).

I. Scaler multiplication of any diagonal matrix Λ of same order commutes with A i.e., $\Lambda A = A\Lambda$.

II. Any square matrix A commutes with itself i.e., $A.A = A.A$.

III. If A^{-1} exists, then $A^{-1}.A = A.A^{-1}$.

3. LINEAR GLAS OF COUPLED ROSSLER SYSTEMS

In section-3 we study linear GLAS of coupled Rossler systems. Otto Rossler(1976) [22], proposed the following system which is known as Rossler system.

$$(6) \quad \begin{aligned} \dot{x}_1 &= -x_2 - x_3, \\ \dot{x}_2 &= x_1 + ax_2, \\ \dot{x}_3 &= b + x_3(x_1 - c), \end{aligned}$$

where a, b and c are three positive parameters. This system contains only one non-linear term x_1x_3 and is even simpler than Lorenz system which has two non-linear term. In the culinary spirit of the pastry map and the baker's map, Otto Rossler(1976) found inspiration in a tuffy-putting machine. By pondering its action, he found the above set of three differential equations. The Rossler system can be classified into two parts, one linear and other non-linear as

$$(7) \quad \dot{x} = Ax + \psi(x),$$

where

$$(8) \quad A = \begin{pmatrix} -2 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -c \end{pmatrix},$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } \psi(x) = \begin{pmatrix} 2x_1 - x_3 \\ x_1 + (a+1)x_2 \\ b + x_1x_3 \end{pmatrix}.$$

where A is negative definite for $c > 0$. According to our scheme the driven system is

$$(9) \quad \dot{y} = Ay - \Lambda\psi(x(t-\tau)),$$

where the matrix Λ commutes with A , then the system (7) and (9) are in a state of GLAS.

4. RESULTS AND DISCUSSIONS

In this section, three simulation results are presented. The parameters are chosen as follows: $a = b = \frac{8}{3}$ and $c = 5.7$. The fourth order Runge-Kutta method is used for simulating the coupled dynamical systems with coupling delay $\tau = 0.5$. The initial conditions for the driving and driven systems are given by $(x_1(0), x_2(0), x_3(0)) = (.01, .01, .01)$ and $(y_1(0), y_2(0), y_3(0)) = (.02, .02, .02)$.

Simulation I

In Simulation-1, we consider

$$(10) \quad \Lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

where $\lambda \neq 0$. Clearly $\Lambda A = A\Lambda$. Therefore all conditions for GLAS are satisfied. Here the driven Rossler system is given by

$$(11) \quad \begin{aligned} \dot{y}_1 &= -2y_1 - y_2 - 2\lambda x_1(t-\tau) + \lambda x_3(t-\tau), \\ \dot{y}_2 &= -y_2 - \lambda x_1(t-\tau) - \lambda(a+1)x_3(t-\tau), \\ \dot{y}_3 &= -cy_3 - b\lambda - \lambda x_1(t-\tau)x_3(t-\tau). \end{aligned}$$

The results are shown in Fig.1-Fig.4. We select $\lambda = 1.0$. Therefore the driving system and the driven system are in linear transformations

$$(12) \quad \begin{aligned} y_1(t - \tau) &= -x_1(t), \\ y_2(t - \tau) &= -x_2(t), \\ y_3(t - \tau) &= -x_3(t). \end{aligned}$$

Simulation II

In this case, we choose

$$(13) \quad \Lambda = A = \begin{pmatrix} -2 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -c \end{pmatrix}.$$

Clearly Λ commutes with A . Therefore all conditions for GLAS are satisfied. In this case the driven Rossler system is given by

$$(14) \quad \begin{aligned} \dot{y}_1 &= -2y_1 - y_2 + 5x_1(t - \tau) + (a + 1)x_2(t - \tau) - 2x_3(t - \tau), \\ \dot{y}_2 &= -y_2 + x_1(t - \tau) + (a + 1)x_2(t - \tau), \\ \dot{y}_3 &= -cy_3 + bc + cx_1(t - \tau)x_3(t - \tau). \end{aligned}$$

If the GLAS between the driven and driving systems are reached, the following relations should be satisfied

$$(15) \quad \begin{aligned} y_1(t - \tau) &= -2x_1(t) - x_2(t), \\ y_2(t - \tau) &= -x_2(t), \\ y_3(t - \tau) &= -cx_3(t). \end{aligned}$$

The simulation result is shown in Fig.5.

Simulation III

In this case, we choose

$$(16) \quad \Lambda = A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{c} \end{pmatrix}.$$

Clearly A^{-1} commutes with A . Therefore all conditions for GLAS are satisfied. In this case the driven system is given by

$$\begin{aligned}
 \dot{y}_1 &= -2y_1 - y_2 + \frac{1}{2}x_1(t - \tau) - \frac{a+1}{2}x_2(t - \tau) - \frac{1}{2}x_3(t - \tau), \\
 \dot{y}_2 &= -y_2 + x_1(t - \tau) + (a+1)x_2(t - \tau), \\
 (17) \quad \dot{y}_3 &= -cy_3 + \frac{b}{c} + \frac{1}{c}x_1(t - \tau)x_3(t - \tau).
 \end{aligned}$$

The GLAS between the driven and driving systems are achieved if the following connections between driven and driving systems are holds

$$\begin{aligned}
 y_1(t - \tau) &= -\frac{1}{2}x_1(t) + \frac{1}{2}x_2(t), \\
 y_2(t - \tau) &= -x_2(t), \\
 (18) \quad y_3(t - \tau) &= -\frac{1}{c}x_3(t).
 \end{aligned}$$

The simulation result is shown in Fig.6.

A generalized approach for constructing chaotically generalized lag anti-synchronized systems via linear transformation with constant time delay is proposed. The Rossler system is treated as an example. The proposed method is simpler than the conventional anti-synchronization approach. This method is very useful for the anti-synchronization of Rossler systems because here we need not required to calculate the Lyapunov exponents. In the proposed method the functional relationship between the states of drive-driven systems can be determined. Therefore knowing driving system the behaviour of the driven system can be predicted in advance and here the prediction of the behaviour of the driving system is also possible since the matrix associated with the linear transformation is invertible.

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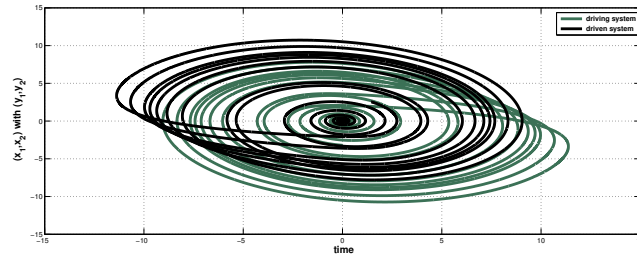


FIGURE 1. Trajectories of (x_1, x_2) with (y_1, y_2) with lag $\tau = 0.5$ for Simulation I.

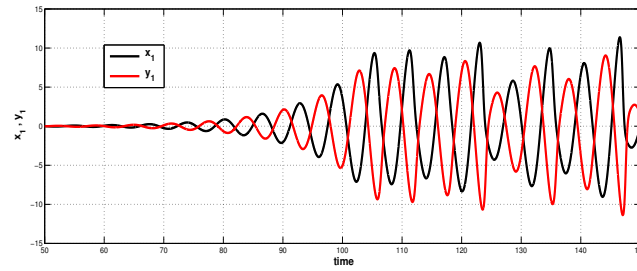


FIGURE 2. Time response of states for driving system and driven system (signal x_1 and y_1) with lag $\tau = 0.5$ for Simulation I.

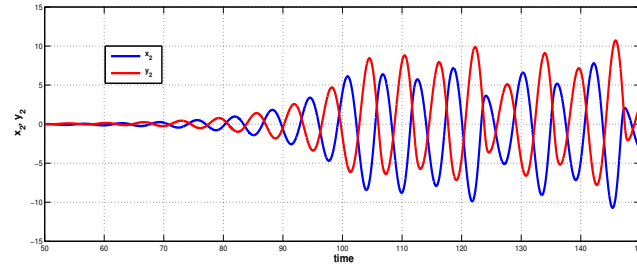


FIGURE 3. Time response of states for driving system and driven system (signal x_2 and y_2) with lag $\tau = 0.5$ for Simulation I.

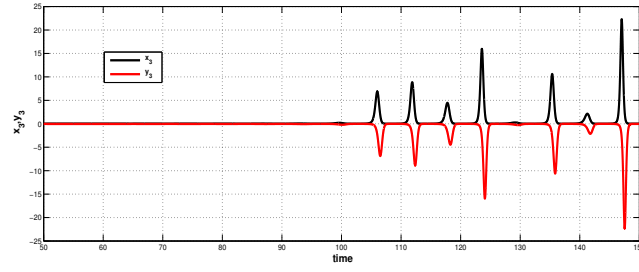


FIGURE 4. Time response of states for driving system and driven system (signal x_3 and y_3) with lag $\tau = 0.5$ for Simulation I.

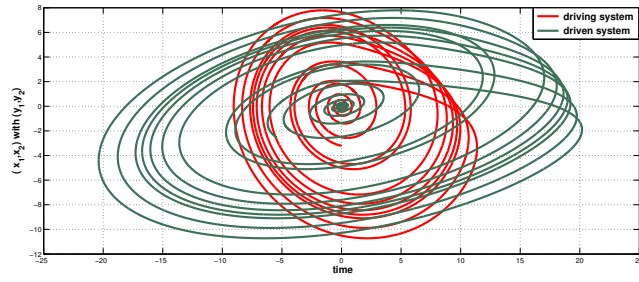


FIGURE 5. Trajectories of (x_1, x_2) with (y_1, y_2) with lag $\tau = 0.5$ for Simulation II.

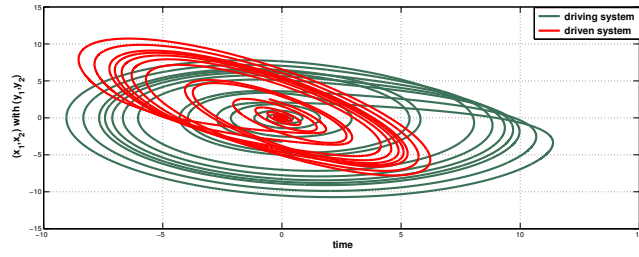


FIGURE 6. Trajectories of (x_1, x_2) with (y_1, y_2) with lag $\tau = 0.5$ for Simulation III.