

DYNAMICAL PROPERTIES OF RELATIVE SEMI-DYNAMICAL SYSTEMS

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ABSTRACT. In this paper, we introduce the product of two relative semi-dynamical systems and study the preservation of certain dynamical notions under the product. Also, a variant of conjugacy is introduced to get these properties shared by conjugate systems.

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1. INTRODUCTION

Mathematical models, whose success depends on getting the observations match the theory, found a strong basis with the introduction of fuzzy sets by Zadeh [9] in 1965. The fuzzy version of mathematical models got developed by the addition of the thought of an observer which is precisely a fuzzy set in the context of mathematics. We indicate dynamics, in fact any structure on a set from the viewpoint of the observer. Applying this idea for describing various geometrical structures on the set such as manifolds [8], vector fields [6] and metrics [7], has led to the

observer-relative perspective on the set. With the expansion of study to this extent, the notion of a relative semi-dynamical system [5] came into picture to explain the dynamics related to the observer's point of view.

While discussing a family of relative semi-dynamical systems, to examine their product is natural. In this paper, we introduce transitivity, exactness and conjugacy from the perspective of an observer. In Section 2, some necessary definitions and terminology have been given and the product of two relative semi-dynamical systems has been introduced. In the next section, μ_α -transitivity and μ_α -exactness have been introduced and it has been seen that the latter implies the former. In Section 4, by means of examples it has been shown that μ_α -transitivity and μ_α -minimality are not necessarily preserved under the product, while μ_α -exactness is preserved under the product. Finally, in Section 5, a variant of conjugacy has been introduced and it has been shown that conjugacy is preserved under product. Besides, we establish that μ_α -transitivity, μ_α -minimality and μ_α -exactness are shared in common by conjugate relative semi-dynamical systems.

2. PREREQUISITE

Throughout the paper, X denotes a nonempty set and the symbols $\mathbb{R}$, $\mathbb{N}$ and I , respectively, represent the set of real numbers, the set of integers and the unit closed interval $[0, 1]$ of the real line. By an observer, we mean a fuzzy subset on X , that is, a map from X to I . For a fuzzy subset μ of X , and $\alpha \in (0, 1]$, the set $\mu^{-1}(\alpha, 1]$, denoted by $(\mu)_\alpha$, is called the α -cut of μ . If $\{\lambda_i \mid i \in \Gamma\}$, where Γ is a nonempty arbitrary index set, is a family of fuzzy subsets of X , then the *join* and the *meet* of the family are defined for $x \in X$ by $\vee_{i \in \Gamma} \lambda_i(x) = \sup_{i \in \Gamma} \{\lambda_i(x)\}$ and $\wedge_{i \in \Gamma} \lambda_i(x) = \inf_{i \in \Gamma} \{\lambda_i(x)\}$, respectively.

For an observer μ , a family τ_μ of fuzzy subsets of X that are less than or equal to μ , containing μ and the constant map 0, which is closed under finite intersections (meets) and arbitrary unions (joins), is called a μ -relative topology on X , and the triple (X, μ, τ_μ) is called a μ -relative topological space. Also, it may be noted that for an $\alpha \in (0, 1]$, the collection $(\tau_\mu)_\alpha = \{\lambda_\alpha \mid \lambda \in \tau_\mu\}$ forms a topology on $(\mu)_\alpha$.

Let X and Y be nonempty sets and $f : X \rightarrow Y$ be a map. Then for a fuzzy subset λ of X , $f(\lambda)$ is a fuzzy subset of Y defined for $y \in Y$ by

$$f(\lambda)(y) = \begin{cases} \sup_{z \in f^{-1}(y)} \{\lambda(z)\}, & \text{if } y \in f(X), \\ 0, & \text{otherwise.} \end{cases}$$

The following lemma describes the relation between the α -cut of the image of a fuzzy subset λ of X and the image of the α -cut of λ under a selfmap f on X .

Lemma 2.1. *Let f be a selfmap on a set X . Then for a fuzzy subset λ of X , $(f(\lambda))_\alpha = f((\lambda)_\alpha)$.*

Proof. Let $x \in f((\lambda)_\alpha)$. Then $x = f(y)$ for some $y \in (\lambda)_\alpha$. Hence $f(\lambda)(x) = \sup_{z \in f^{-1}(x)} \{\lambda(z)\} > \alpha$, which shows that $x \in (f(\lambda))_\alpha$. Next, consider $y \in (f(\lambda))_\alpha$. Then, $f(\lambda)(y) > \alpha$. This provides an $x \in f^{-1}(y)$ such that $\lambda(x) > \alpha$. Thus $y = f(x) \in f((\lambda)_\alpha)$. $\square$

Let (X, μ, τ_μ) and (Y, ν, τ_ν) be μ -relative and ν -relative topological spaces, respectively. Then a map $f : (X, \mu, \tau_\mu) \rightarrow (Y, \nu, \tau_\nu)$ is called (μ, ν) -continuous [1] if for every $\lambda \in \tau_\nu$, $f^{-1}(\lambda) = (\lambda \circ f) \wedge \mu \in \tau_\mu$.

By a *relative semi-dynamical system*, written RSD-system for short, we mean a triple (f, X, τ_μ) , where $f : X \rightarrow X$ is a (μ, μ) -continuous map. For an element x of an RSD-system (f, X, τ_μ) , we denote the set $\{x\} \cup \{f^n(x) \mid n \in \mathbb{N}\}$ by $O_f(x)$, and call it to be the *orbit* of x .

Definition 2.1 ([1]). An RSD-system (f, X, τ_μ) is called μ_α -minimal if the set $(\mu)_\alpha$ is invariant under f , and for all $x \in (\mu)_\alpha$, the orbit of x is dense in $(\mu)_\alpha$.

If μ and ν are fuzzy subsets of X and Y , respectively, then the product $(\mu \times \nu)$ of μ and ν is a fuzzy subset of $X \times Y$, defined by

$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}, \quad x \in X, y \in Y.$$

Let (X, μ, τ_μ) and (Y, ν, τ_ν) be μ -relative and ν -relative topological spaces, respectively. Then the *product $(\mu \times \nu)$ -topology*, $\tau_{(\mu \times \nu)}$, on $X \times Y$ is defined to be the $(\mu \times \nu)$ -topology generated by the collection $\{\lambda \times \gamma \mid \lambda \in \tau_\mu, \gamma \in \tau_\nu\}$.

If $f : X \rightarrow X$ and $g : Y \rightarrow Y$ are (μ, μ) -continuous and (ν, ν) -continuous maps, respectively, then it is straightforward to verify that $f \times g : X \times Y \rightarrow X \times Y$ defined

by

$$(f \times g)(x, y) = (f(x), g(y)), \quad x \in X, y \in Y,$$

is a $(\mu \times \nu, \mu \times \nu)$ -continuous map. Thus we have the following:

Definition 2.2. Let (f, X, τ_μ) and (g, Y, τ_ν) be two RSD-systems. Then their *product relative semi-dynamical system* is defined to be the triple $(f \times g, X \times Y, \tau_{(\mu \times \nu)})$ as described above.

3. μ_α -TRANSITIVITY AND μ_α -EXACTNESS

In this section, we introduce the notions of μ_α -transitivity and μ_α -exactness for a relative semi-dynamical system and observe that a μ_α -exact RSD-system is μ_α -transitive.

Definition 3.1. For an $\alpha \in (0, 1]$, a relative semi-dynamical system (f, X, τ_μ) is called μ_α -transitive if for every $\lambda, \nu \in \tau_\mu$ with $(\lambda)_\alpha$ and $(\nu)_\alpha$ nonempty, there is an $n \in \mathbb{N}$ such that $(f^n(\lambda) \wedge \nu)_\alpha \neq \phi$, or equivalently, $(f^n(\lambda))_\alpha \cap (\nu)_\alpha \neq \phi$.

Example 3.1. Let $X = [0, 1]$ and $\mu : X \rightarrow I$ be a fuzzy subset of X defined by $\mu(x) = 2x \bmod 1$, $x \in X$. Then the family $\tau_\mu = \{0\} \cup \{\frac{\mu}{2^n} \mid n \in \mathbb{N} \cup \{0\}\}$ is a μ -topology on X . Clearly, the identity map I_X on X is (μ, μ) -continuous, and hence (I_X, X, τ_μ) is an RSD-system. Since $1/2$ lies in each open set of $(\mu)_\alpha$, the system (I_X, X, τ_μ) is μ_α -transitive.

Example 3.2. Let X denote the polynomial ring $\mathbb{R}[x]$ in x with coefficients in $\mathbb{R}$. Define $\mu : X \rightarrow [0, 1]$ by

$$\mu(f) = \begin{cases} \frac{1}{\deg f}, & \text{if } \deg f \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $f \in X$ and $\deg f$ denotes the degree of f , and for each $i \in \mathbb{N}$, $\lambda^i : X \rightarrow [0, 1]$ by

$$\lambda^i(f) = \begin{cases} \frac{1}{i}, & \text{if } \deg f = i, \\ 0, & \text{otherwise.} \end{cases}$$

Denote by τ_μ , the μ -topology generated by $\{\lambda^i \mid i \in \mathbb{N}\}$, so that (X, μ, τ_μ) is a μ -relative topological space. Consider the map $F : X \rightarrow X$ which sends $f \in X$ to its derivative f' . It is (μ, μ) -continuous [1]. On account of the fact that for $i, j \in \mathbb{N}$,

with $j > i$, $F^n((\lambda^i)_\alpha) \cap (\lambda^j)_\alpha = \emptyset$, for every $n \in \mathbb{N}$, the RSD-system (F, X, τ_μ) is not μ_α -transitive.

Remark 3.1. If $x \in (\lambda_1)_\alpha$, where $(\lambda_1)_\alpha \in \tau_\mu$. Then $O_f(x)$ is dense in $(\mu)_\alpha$ implies that for every nonempty open set $(\lambda_2)_\alpha$ of $(\mu)_\alpha$, $f^n((\lambda_1)_\alpha) \cap (\lambda_2)_\alpha$ is nonempty for some $n \in \mathbb{N}$. Hence, a μ_α -minimal RSD-system is μ_α -transitive.

Definition 3.2. For an $\alpha \in (0, 1]$, a relative semi-dynamical system (f, X, τ_μ) is called μ_α -exact if

- (i) $f(\mu_\alpha) \supseteq \mu_\alpha$, and
- (ii) for $\lambda \in \tau_\mu$ with $(\lambda)_\alpha$ nonempty, there is an $n \in \mathbb{N}$ such that $f^n(\lambda) \supseteq \mu$ on $(\mu)_\alpha$, or equivalently, $f^n((\lambda)_\alpha) \supset (\mu)_\alpha$.

Remark 3.2. That a μ_α -exact RSD-system is μ_α -transitive follows from the definition.

Example 3.3. Let $X = \{\omega : \mathbb{N} \rightarrow \mathbb{N} \mid \omega \text{ is nondecreasing}\}$. Define $\mu : X \rightarrow [0, 1]$ by $\mu(\omega) = \frac{1}{\omega(1)}$, $\omega \in X$, and for $i \in \mathbb{N}$, $\lambda^i : X \rightarrow [0, 1]$ by $\lambda^i(\omega) = \frac{1}{\omega(i)}$. Then the family $\tau_\mu = \{0\} \cup \{\lambda^i \mid i \in \mathbb{N}\}$ is a μ -topology on X . Consider the map $F : X \rightarrow X$ given by $F(\omega)(n) = \omega(n+1)$, where $\omega \in X$, $n \in \mathbb{N}$. That F is (μ, μ) -continuous follows by noting that $\lambda^i \circ F = \lambda^{i+1}$. Let $x \in (\mu)_\alpha$. Then the map $y : \mathbb{N} \rightarrow \mathbb{N}$ defined for $n \in \mathbb{N}$ by

$$y(n) = \begin{cases} 1, & \text{if } n \leq i, \\ x(n-i), & \text{otherwise,} \end{cases}$$

is such that $y \in (\lambda^i)_\alpha$ and $F^i(y) = x$. Thus $x \in F^i((\lambda^i)_\alpha)$, and hence the RSD-system (F, X, τ_μ) is μ_α -exact.

4. PRESERVATION OF DYNAMICAL PROPERTIES OF RELATIVE SEMI-DYNAMICAL SYSTEMS UNDER PRODUCT

In this section, by means of examples we show that the product of a μ_α -transitive (μ_α -minimal) RSD-system and a ν_α -transitive (ν_α -minimal) RSD-system need not be a $(\mu \times \nu)_\alpha$ -transitive $((\mu \times \nu)_\alpha$ -minimal) RSD-system. However, we show that the product of a μ_α -exact RSD-system and a ν_α -exact RSD-system is a $(\mu \times \nu)_\alpha$ -exact RSD-system.

Example 4.1. Let (I_X, X, τ_μ) be the RSD-system as described in Example 3.1. Then $(\frac{1}{2}, \frac{1}{2})$ lies in each nonempty open set of $(\mu \times \mu)_\alpha$. Therefore, $(I_X \times I_X, X \times X, \tau_{(\mu \times \mu)})$ is $(\mu \times \mu)_\alpha$ -transitive.

Example 4.2. Let X be the set $\mathbb{Z}_n$ of residue classes modulo n , where $n \geq 2$, and $\mu : X \rightarrow I$ be a fuzzy subset of X defined by $\mu(x) = 1$. Take τ_μ to be the collection of all fuzzy subsets of X . Then (X, μ, τ_μ) is a μ -relative topological space. Define $f : X \rightarrow X$ by $f(x) = (x + 1) \bmod n$. Then f is (μ, μ) -continuous. Also, for $\alpha \in (0, 1]$, the RSD-system (f, X, τ_μ) is μ_α -transitive, being μ_α -minimal. For open sets $\{(1, 1)\}$ and $\{(1, 2)\}$ of $(\mu \times \mu)_\alpha$, $f^k(\{(1, 1)\}) \cap (\{(1, 2)\}) = \emptyset$, for every $k \in \mathbb{N}$. Hence $(f \times f, X \times X, \tau_{(\mu \times \mu)})$ is not $(\mu \times \mu)_\alpha$ -transitive.

Example 4.3. Let $X = \{0, 1\}^\mathbb{N}$ and for $x \in X$, n_x be the smallest non-negative integer such that $x(n_x + 1) = 0$. Define $\mu : X \rightarrow I$ by

$$\mu(x) = \frac{n_x}{n_x + 1},$$

and for $i \in \mathbb{N}$, $\lambda^i : X \rightarrow I$ by

$$\lambda^i(x) = \begin{cases} \frac{1}{2}, & \text{if } x(j) = 1 \text{ for every } j \in 1, \dots, i, \\ 0, & \text{otherwise,} \end{cases}$$

where $x \in X$. Then $\tau_\mu = \{0, \mu\} \cup \{\lambda^i \mid i \in \mathbb{N}\}$ is a μ -topology on X . Define $f : X \rightarrow X$ by

$$f(x)(n) = \begin{cases} 1, & \text{if } n = 1, \\ x(n - 1), & \text{if } n \neq 1, \end{cases}$$

where $x \in X, n \in \mathbb{N}$. That f is (μ, μ) -continuous follows by noting that $(\lambda^1 \circ f) \wedge \mu = \lambda^1$, for $i \neq 1$, $\lambda^i \circ f = \lambda^{i-1}$ and $(\mu \circ f) \wedge \mu = \mu$. Also, for $x \in (\mu)_\alpha$ and $\lambda^i \in \tau_\mu$ with $(\lambda^i)_\alpha \neq \emptyset$, we have $f^i(x) \in (\lambda^i)_\alpha$. Therefore, $O_f(x)$ is dense in $(\mu)_\alpha$. Since $f((\mu)_\alpha) \subseteq (\mu)_\alpha$, we obtain (f, X, τ_μ) to be μ_α -minimal. Now, consider $(f \times f, X \times X, \tau_{(\mu \times \mu)})$. Clearly, $(f \times f)((\mu \times \mu)_\alpha)$ is contained in $(\mu \times \mu)_\alpha$. Let $(x, y) \in X \times X$ and $(\lambda^i \times \lambda^j)_\alpha$ be a nonempty basic open set of $(\mu \times \mu)_\alpha$. Then $(f \times f)^k(x, y) \in (\lambda^i \times \lambda^j)_\alpha$, where $k = \max\{i, j\}$. Hence, $O_{f \times f}(x, y)$ is dense in $(\mu \times \mu)_\alpha$, showing that $(f \times f, X \times X, \tau_{(\mu \times \mu)})$ is $(\mu \times \mu)_\alpha$ -minimal.

Example 4.4. The RSD-system (f, X, τ_μ) as described in Example 4.2 is μ_α -minimal, but the product RSD-system $(f \times f, X \times X, \tau_{(\mu \times \mu)})$ is not $(\mu \times \mu)_\alpha$ -minimal.

Theorem 4.1. *Let (f, X, τ_μ) be μ_α -exact and (g, Y, τ_ν) be ν_α -exact. Then the product RSD-system $(f \times g, X \times Y, \tau_{(\mu \times \nu)})$ is $(\mu \times \nu)_\alpha$ -exact.*

Proof. Let $(\lambda \times \gamma)_\alpha$ be a nonempty basic open set of $(\mu \times \nu)_\alpha$. Then the μ_α -exactness and ν_α -exactness of (f, X, τ_μ) and (g, Y, τ_ν) , respectively, provide $n_1, n_2 \in \mathbb{N}$ such that $f^{n_1}((\lambda)_\alpha) \supseteq (\mu)_\alpha$, and $g^{n_2}((\gamma)_\alpha) \supseteq (\nu)_\alpha$. Thus $(f \times g)^k((\lambda \times \gamma)_\alpha) = (f^k((\lambda)_\alpha) \times g^k((\gamma)_\alpha)) \supseteq (\mu \times \nu)_\alpha$, where $k = \max\{n_1, n_2\}$, and hence the result. $\square$

Theorem 4.2. *Let (f, X, τ_μ) and (g, Y, τ_ν) be RSD-systems. Then the following are true:*

- (i) *if the product RSD-system $(f \times g, X \times Y, \tau_{(\mu \times \nu)})$ is $(\mu \times \nu)_\alpha$ -transitive, then (f, X, τ_μ) is μ_α -transitive and (g, Y, τ_ν) is ν_α -transitive;*
- (ii) *if the product RSD-system $(f \times g, X \times Y, \tau_{(\mu \times \nu)})$ is $(\mu \times \nu)_\alpha$ -minimal, then (f, X, τ_μ) is μ_α -minimal and (g, Y, τ_ν) is ν_α -minimal;*
- (iii) *if the product RSD-system $(f \times g, X \times Y, \tau_{(\mu \times \nu)})$ is $(\mu \times \nu)_\alpha$ -exact, then (f, X, τ_μ) is μ_α -exact and (g, Y, τ_ν) is ν_α -exact.*

Proof. We prove the result for one of the factors, say (f, X, τ_μ) . Let $\lambda \in \tau_\mu$ be such that $(\lambda)_\alpha$ is nonempty. Then $(\lambda \times \nu) \in \tau_{(\mu \times \nu)}$ and $(\lambda \times \nu)_\alpha$ is nonempty. For (i), let λ_1 and λ_2 be in τ_μ with $(\lambda_1)_\alpha$ and $(\lambda_2)_\alpha$ nonempty. Then the $(\mu \times \nu)_\alpha$ -transitivity of $f \times g$ provides an $n \in \mathbb{N}$ such that $((f \times g)^n(\lambda_1 \times \nu) \wedge (\lambda_2 \times \nu))_\alpha \neq \phi$. In other words, $((f^n(\lambda_1))_\alpha \times (g^n(\nu))_\alpha) \cap ((\lambda_2)_\alpha \times (\nu)_\alpha) \neq \phi$, which shows that $f^n((\lambda_1)_\alpha) \cap (\lambda_2)_\alpha \neq \phi$. For (ii), let $x \in X$ and $\lambda \in \tau_\mu$ with $(\lambda)_\alpha$ nonempty. Since for $y \in Y$, $O_{(f \times g)}(x, y)$ is dense in $(\mu \times \nu)_\alpha$, there is an $n \in \mathbb{N}$ such that $(f \times g)^n(x, y) \in (\lambda \times \nu)_\alpha$. Hence, $f^n(x) \in (\lambda)_\alpha$. Finally, (iii) follows by noting that $(f \times g)^n(\lambda \times \nu) \supseteq (\mu \times \nu)_\alpha$ implies that $f^n((\lambda)_\alpha) \supseteq (\mu)_\alpha$. $\square$

5. CONJUGACY IN RELATIVE SEMI-DYNAMICAL SYSTEMS

In this section, the definition of conjugacy for crisp dynamical systems has been elevated to relative semi-dynamical systems and it is found that the dynamical properties viz. μ_α -minimality, μ_α -transitivity and μ_α -exactness of a relative semi-dynamical system are carried over to its conjugate system.

Definition 5.1. Let (X, μ, τ_μ) and (Y, ν, τ_ν) be μ -relative and ν -relative topological spaces, respectively. Then a bijective map $h : X \rightarrow Y$ is called a *relative homeomorphism* if the following hold:

- (1) h is (μ, ν) -continuous,
- (2) h^{-1} is (ν, μ) -continuous, and
- (3) $\nu \circ h = \mu$.

Lemma 5.1. *Let (X, μ, τ_μ) and (Y, ν, τ_ν) be μ -relative and ν -relative topological spaces, respectively, and $h : X \rightarrow Y$ be a bijective map. Then for an $\alpha \in (0, 1]$ and $\lambda \in \tau_\nu$, $(\lambda)_\alpha = h((\lambda \circ h)_\alpha)$.*

Proof. Let $y \in Y$. Then $\lambda(y) > \alpha$ if and only if $\lambda \circ h(h^{-1}(y)) > \alpha$. Thus $y \in (\lambda)_\alpha$ if and only if $h^{-1}(y) \in (\lambda \circ h)_\alpha$, or if and only if $y \in h((\lambda \circ h)_\alpha)$. $\square$

Lemma 5.2. *Let (X, μ, τ_μ) and (Y, ν, τ_ν) be μ -relative and ν -relative topological spaces, respectively, and $h : X \rightarrow Y$ be a relative homeomorphism. Then for $\lambda \in \tau_\nu$, $\lambda \circ h \in \tau_\mu$.*

Proof. Let $\lambda \in \tau_\nu$ and $x \in X$. Then $\lambda(h(x)) \leq \nu(h(x)) = \mu(x)$, showing that $\lambda \circ h \leq \mu$. Since h is (μ, ν) -continuous, the result follows. $\square$

Definition 5.2. Two relative semi-dynamical systems (f, X, τ_μ) and (g, Y, τ_ν) are said to be *conjugate* if there is a relative homeomorphism $h : X \rightarrow Y$ satisfying $h \circ f = g \circ h$.

It is straightforward to check that the relation of conjugacy is an equivalence relation in the set of all relative semi-dynamical systems.

Theorem 5.1. *Let (f_1, X_1, τ_{μ_1}) be conjugate to (g_1, Y_1, τ_{ν_1}) , and (f_2, X_2, τ_{μ_2}) be conjugate to (g_2, Y_2, τ_{ν_2}) . Then $(f_1 \times f_2, X_1 \times X_2, \tau_{(\mu_1, \mu_2)})$ is conjugate to $(g_1 \times g_2, Y_1 \times Y_2, \tau_{(\nu_1, \nu_2)})$.*

Proof. Let h_1 and h_2 be the relative homeomorphisms from X_1 to Y_1 and X_2 to Y_2 , respectively. Then $h_1 \times h_2$ is the required relative homeomorphism between $X_1 \times X_2$ and $Y_1 \times Y_2$. $\square$

Theorem 5.2. *Let (f, X, τ_μ) and (g, Y, τ_ν) be conjugate RSD-systems. Then (f, X, τ_μ) is μ_α -transitive if and only if (g, Y, τ_ν) is ν_α -transitive.*

Proof. Let (f, X, τ_μ) be μ_α -transitive and $\lambda_1, \lambda_2 \in \tau_\nu$ be such that $(\lambda_1)_\alpha$ and $(\lambda_2)_\alpha$ are nonempty. Then, by Lemmas 5.2 and 5.1, $\lambda_1 \circ h$ and $\lambda_2 \circ h$ are in τ_μ with $(\lambda_1 \circ h)_\alpha$ and $(\lambda_2 \circ h)_\alpha$ nonempty. Thus, there is an $n \in \mathbb{N}$ such that $f^n((\lambda_1 \circ h)_\alpha) \cap (\lambda_2 \circ h)_\alpha \neq \emptyset$.

ϕ . Applying h on $f^n((\lambda_1 \circ h)_\alpha) \cap (\lambda_2 \circ h)_\alpha$, and using Lemma 5.1, (g, Y, τ_ν) is obtained to be ν_α -transitive. The converse follows because the relation of conjugacy is a symmetric relation. $\square$

Theorem 5.3. *Let (f, X, τ_μ) and (g, Y, τ_ν) be conjugate RSD-systems. Then (f, X, τ_μ) is μ_α -minimal if and only if (g, Y, τ_ν) is ν_α -minimal.*

Proof. Let (f, X, τ_μ) be μ_α -minimal. Then the invariance of $(\mu)_\alpha$ under f provides that $(h \circ f)((\mu)_\alpha) \supseteq h((\mu)_\alpha) = h((\nu \circ h)_\alpha) = (\nu)_\alpha$. Hence $g \circ h((\mu)_\alpha) \supseteq (\nu)_\alpha$. Using the fact that $h((\mu)_\alpha) = (\nu)_\alpha$, we have $g((\nu)_\alpha) \supseteq (\nu)_\alpha$. To see that the orbit of each element of $(\nu)_\alpha$ is dense in $(\nu)_\alpha$, notice that the orbit of $x \in (\mu)_\alpha$ is dense in $(\mu)_\alpha$ implies that the orbit of $h(x)$ is dense in $(\nu)_\alpha$. The converse follows from the symmetry of the relation of conjugacy. $\square$

Theorem 5.4. *Let (f, X, τ_μ) and (g, Y, τ_ν) be conjugate RSD-systems. Then (f, X, τ_μ) is μ_α -exact if and only if (g, Y, τ_ν) is ν_α -exact.*

Proof. Let (f, X, τ_μ) be μ_α -exact and $(\lambda)_\alpha$ be a nonempty open set of $(\nu)_\alpha$. Then by Lemmas 5.1 and 5.2, $h^{-1}((\lambda)_\alpha) = (\lambda \circ h)_\alpha$ is a nonempty open set of $(\mu)_\alpha$. The μ_α -exactness of f gives an $n \in \mathbb{N}$ such that $(\mu)_\alpha \subseteq f^n((\lambda \circ h)_\alpha)$. Applying the map h , we get $(\nu)_\alpha \subseteq (g^n \circ h)((\lambda \circ h)_\alpha)$. Thus $(\nu)_\alpha \subseteq g^n((\lambda)_\alpha)$, which shows that (g, Y, τ_ν) is ν_α -exact. For the converse, note that the relation of conjugacy is a symmetric relation. $\square$

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