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A CONSTRUCTION OF HYPERBOLIC SPINORS ACCORDING TO FRENET FRAME IN MINKOWSKI SPACE

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Abstract. In this paper, spinors which consists of two hyperbolic components are introduced and hyperbolic spinor representation of the curve (spacelike or timelike) in Minkowski space is expressed. Then, the fundamental theorem of differential geometry has been restated by means of these hyperbolic spinors. Finally, some examples about this study are given.

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1. Introduction

Spinors for the first time was discovered by the French mathematician Elie Cartan, who also has a basic work about Lie groups, a modern theory. One of the main aims of Cartan in this study is to systematically develop theory of spinors giving the geometric definition of this mathematical statement and in addition to this, to make a contribution to differential geometry, group theory and mathematical physics, [5]. In another study, Vivarelli has provided a new approach to quaternions and one-index spinors derived by the vector formulation of Euler's theorem about displacement of a rigid body in three dimensional Euclidean space. Moreover, Vivarelli introduced a linear and one to one relation between quaternions and one-index spinors, [17]. Spinors in the field of physics have a lot of areas of use. Quantum mechanics is an example of these areas. In this area, spinors generate four wave functions which substantially are the components of spinors and the famous Dirac equations. Numerous studies have been published in this field. A perfect and essential example of these studies was conducted by Brauer and Weyl, [3]. But in almost all of these studies, the spinors have been introduced only in a formal way without intuitive geometric opinion. Unfortunately, this makes it rather difficult to understand existing literature on spinor. Many are assumed as modern algebra knowledge. But, in recent years, the clearer several studies have been made on the geometric sense about this subject. One of these is the study in [7]. In [7], Torres et al. have expressed a triplet consisted from the mutually orthogonal unit vectors in terms of a spinor which has two complex components. Moreover, in another study, Kişi and Tosun have given the spinor formulation of Darboux frame on oriented a surface and the relation in between the spinor representation of Frenet frame and Darboux frame, [9]. Similarly, in [16], the spinor Bishop equations of the curves in $\mathbb{E}^3$ have been expressed, [16].

In this study, the spinors with two hyperbolic components are investigated in their geometrical sense. Moreover, the hyperbolic spinor formulation of a curve in Minkowski space is expressed. In doing so, we consider that the cases of curve to be spacelike or timelike, separately. In addition, the fundamental theorem of differential geometry is expressed by these hyperbolic spinors. Then, we consider that the curve (spacelike or timelike) $\alpha \in \mathbb{R}^3_1$ is general helix and obtain that the hyperbolic spinor corresponding to the Frenet frame of this curve. With the aid of these hyperbolic spinors, we calculate the Frenet frame of this curve in terms of hyperbolic spinors. Finally, we give some examples about this representation.

2. Preliminaries

In three dimensional Minkowski space, the standard metric is defined as

$$g(\mathbf{u},\mathbf{v}) = u_1v_1 + u_2v_2 - u_3v_3 \tag{2.1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3_1$. Moreover, if we take $g(\mathbf{u},\mathbf{u}) > 0$ or $\mathbf{u} = 0$, $g(\mathbf{u},\mathbf{u}) = 0$ $(\mathbf{u} \neq 0)$ and $g(\mathbf{u},\mathbf{u}) < 0$, the vector $\mathbf{u} \in \mathbb{R}^3_1$ is spacelike, lightlike (null) and timelike vectors, respectively. In addition, if the first component of the timelike vector is positive and negative, the timelike vector is called future pointing and past pointing timelike vector, respectively. Also, with the aid of the equation (2.1), the norm of the vector $\mathbf{u} \in \mathbb{R}^3_1$ is

$$\|\mathbf{u}\|_L = \sqrt{|g(\mathbf{u},\mathbf{u})|}. \tag{2.2}$$

Moreover, the Lorentzian cross product of the vectors of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_1$ is

$$\mathbf{u} \wedge_L \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & -\mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \tag{2.3}$$

[11].

Let $\alpha : I \to \mathbb{R}^3_1$ be a curve in Minkowski space. For $\forall t \in I \subset \mathbb{R}$, if the velocity vector of the curve is spacelike, timelike and null, the curve α is spacelike, timelike and null, respectively. A triad consisted by mutually orthogonal unit vectors at each point on the differentiable curve in $\mathbb{R}^3_1$ can be constructed. In addition, the change of these vectors along the curve defines the curvature and torsion of the curve. The frame consisted of these vectors is called Serret-Frenet frame in $\mathbb{R}^3_1$, [13]. A lot of studies are made on the curve theory in $\mathbb{R}^3_1$. Some of these are [8, 12, 2, 6].

Let $\alpha : I \to \mathbb{R}^3_1$ be a non-null arbitrary regular curve with arc-length parameterized. Throughout this work, the curves will be considered as arc-length parameterized, non-null and regular. The Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ of the curve α in Minkowski space is calculated that

$$\mathbf{T}(s) = \alpha'(s), \qquad \mathbf{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \qquad B(s) = \varepsilon_{\mathbf{B}}\left(\mathbf{T}(s) \wedge \mathbf{N}(s)\right) \tag{2.4}$$

where $s \in I$ is the arc-length parameter of the curve α. Also, the Frenet derivative formulas are

$$\begin{aligned} \mathbf{T}' &= \kappa \mathbf{N} \\ \mathbf{N}' &= \varepsilon_B \kappa \mathbf{T} + \tau \mathbf{B} \\ \mathbf{B}' &= \varepsilon_T \tau \mathbf{N} \end{aligned} \tag{2.5}$$

where $\varepsilon_B = g(\mathbf{B}, \mathbf{B})$, $\varepsilon_{\mathbf{T}} = g(\mathbf{T}, \mathbf{T})$ and κ, τ are the curvature and torsion functions, [8].

Theorem 2.1. *An arbitrary curve α is a general helix necessary and sufficient condition the harmonic curvature H (the rate of torsion to curvature of curve) is constant,* [10, 15].

We consider that numbers of the form $x + jy$ where x and y real numbers and j is a commuting element satisfying the relation $j^2 = 1$. This number system is known as the "hyperbolic number system" symbolized $\mathbb{H}$, [18]. Let us suppose A be an $n \times n$ matrix defined on $\mathbb{H}$. So, $A^\dagger$ is $n \times n$ matrix obtained by transposing and conjugating of A, $(A^\dagger = \overline{A^t})$]. If A is Hermitian (or anti-Hermitian) with respect to $\mathbb{H}$, the statement $A^\dagger = A$ ($or\ A^\dagger = -A$) is valid. So, let A be an $n \times n$ Hermitian matrix. Then for $U = e^{jA}$, the following statement is hold;

$$U^\dagger.U = U.U^\dagger = 1.$$

The set of all $n \times n$ matrices on $\mathbb{H}$ satisfying the last equation forms a group. The group is denoted as $U(n, \mathbb{H})$ and called hyperbolic unitary group. Especially, if all $n \times n$ matrices in this group satisfy the statement $\det U = 1$, this special group is denoted by $SU(n, \mathbb{H})$, [1].

The Lorentz group is defined as the group of all Lorentz transformations of Minkowski space in physics and mathematics,. Also, this group is a subgroup of the Poincaré group because Poincaré group is the group of all isometries of Minkowski space. Lorentz transformations which preserve the direction of time are called orthochronous. The subgroup of orthochronous transformations is often denoted $O^+(1,3)$. They have determinant $+1$ and they preserve the orientation. This subgroup is denoted by $SO(1,3)$, [4].

3. Spinors with two hyperbolic components

There is a homomorphism between the group of rotation about the origin; $SO(1,3)$ and the group of unitary matrix in the type of 2×2; $SU(2, \mathbb{H})$. While the elements of $SO(1,3)$ activate the vectors with three real component, the elements of $SU(2, \mathbb{H})$ activate the spinors, [14].

Suppose that the three dimensional space $\mathbb{R}^3_1$ is referred to a system of orthogonal coordinates; let (x_1, x_2, x_3) be an isotropic vector, i.e., have zero length. So, the components of this vector satisfy $x_1^2 + x_2^2 - x_3^2 = 0$. We can associate with this vector the hyperbolic spinor ψ. So, we have

$$\mathbf{x} \; = (x_1, x_2, x_3) = ({\psi_1}^2 - {\psi_2}^2, j({\psi_1}^2 + {\psi_2}^2), -2\psi_1\psi_2)$$

The equations for the isotropic direction of the vector associated with a hyperbolic spinor ψ can be written as

$$\begin{aligned} &\psi_1 x_3 + \psi_2(x_1 - jx_2) = 0, \\ &\psi_1\,(x_1 + jx_2) - \psi_2 x_3 = 0. \end{aligned}$$

This suggest that the matrix

$$X = \begin{pmatrix} x_3 & x_1 - jx_2 \\ x_1 + jx_2 & -x_3 \end{pmatrix}.$$

If we take $K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the equation $KX = -K^T X$ is hold. Moreover, with the aid of the matrix X, the matrices corresponding to the basis vectors of $\mathbb{R}^3_1$ are called Pauli matrices $P = (p_1, p_2, p_3)$ as

$$p_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \tag{3.1}$$

[1]. Here $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector consisting of hyperbolic symmetric matrices and the components of this vector are defined as

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{3.2}$$

A spinor consists of two hyperbolic components is introduced as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{3.3}$$

For three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3_1$, we can write

$$\mathbf{a} + j\mathbf{b} = \psi^t \sigma \psi, \qquad \mathbf{c} = -\widehat{\psi}^t \sigma \psi \tag{3.4}$$

where $j^2 = 1$ and t is transpose. Moreover, the conjugate of ψ is $\overline{\psi}$. Also, the following equations are hold;

$$\widehat{\psi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\psi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\psi_1} \\ \overline{\psi_2} \end{pmatrix} = \begin{pmatrix} -\overline{\psi_2} \\ \overline{\psi_1} \end{pmatrix} \tag{3.5}$$

where the mate of ψ is $\widehat{\psi}$.

So, with the aid of the equations (3.2), (3.3), (3.4) and (3.5), the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}_1^3$ are obtained that

$$\begin{aligned} a + jb &= ({\psi_1}^2 - {\psi_2}^2, j({\psi_1}^2 + {\psi_2}^2), -2\psi_1\psi_2) \\ c &= \left(\psi_1\overline{\psi_2} + \overline{\psi_1}\psi_2, j(\psi_1\overline{\psi_2} - \overline{\psi_1}\psi_2), |\psi_1|^2 - |\psi_2|^2\right) \end{aligned} \tag{3.6}$$

where the mutually orthogonal vectors and $\|\mathbf{a}\|_L = \|\mathbf{b}\|_L = \|\mathbf{c}\|_L = \overline{\psi}^t\psi$. Also, the vector $\mathbf{c} \in \mathbb{R}_1^3$ can be obtained as $\mathbf{c} = \overline{\psi}^t P\psi$. On the other hand, there is a hyperbolic spinor providing the equation (3.4) for mutually orthogonal vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}_1^3$ which have same magnitude.

For an arbitrary matrix $U \in SU(2, \mathbb{H})$, the equation $\overline{\psi'}^t\psi' = \overline{\psi}^t\psi$ is hold where $\psi' = U\psi$. So, the magnitudes of the spacelike (or timelike) vectors $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ corresponding to the hyperbolic spinor ψ' are equal to the magnitudes of the spacelike (or timelike) vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ corresponding to the hyperbolic spinor ψ. Hence, each element of $SU(2, \mathbb{H})$ creates a transformation that converts the orthogonal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ of $\mathbb{R}_1^3$ to the orthogonal basis $\{\mathbf{a}', \mathbf{b}', \mathbf{c}'\}$. This conversion is a two to one. Namely, two elements of $SU(2, \mathbb{H})$; U and $-U$ generate the same ordered triad of $\mathbb{R}_1^3$. While the triad $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ corresponds to the hyperbolic spinor ψ, the ordered triads $\{\mathbf{b}, \mathbf{c}, \mathbf{a}\}$ and $\{\mathbf{c}, \mathbf{a}, \mathbf{b}\}$ correspond to different hyperbolic spinors. In addition, the spinors ψ and $-\psi$ correspond to the same triad. So, the following proposition can be given with the aid of the equations (3.2), (3.3) and (3.5).

Proposition 3.1. *For two arbitrary hyperbolic spinors φ and ψ, the following statements are hold;*

$$\begin{aligned} &\textbf{\textit{i)}} \quad \overline{\varphi^t\sigma\psi} = -\hat{\varphi}^t\sigma\hat{\psi} \\ &\textbf{\textit{ii)}} \quad \left(\widehat{a\varphi + b\psi}\right) = \bar{a}\hat{\varphi} + \bar{b}\hat{\psi} \\ &\textbf{\textit{iii)}} \ \hat{\hat{\psi}} = -\psi \end{aligned} \tag{3.7}$$

where a, b are two hyperbolic numbers.

Using the features of this proposition, it can be easily seen that the vector $\mathbf{c}$ in the equation (3.4) occurs from real components. Moreover, the equation $\varphi^t\sigma\psi = \psi^t\sigma\varphi$ can be written by the symmetric matrices in the equation (3.2).

Let the hyperbolic spinor ψ be non-zero, then we consider that

$$\det(\psi,\widehat{\psi}) = \begin{vmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{vmatrix} = |\psi_1|^2 + |\psi_2|^2.$$

This sum solely is zero when both ψ_1 and ψ_2 components are zero. So, $\left\{\psi,\widehat{\psi}\right\}$ is linear independent.

Especially, if we choose $\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\hat{\psi}$ is equal to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. From the equation (3.6) we can know that the triad $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ create the canonical basis of $\mathbb{R}_1^3$. So, we can see that $\left\{\psi,\widehat{\psi}\right\}$ create a basis for the spinors with two hyperbolic components.

4. The Hyperbolic Spinors Corresponding to Minkowski Space

4.1. **The Hyperbolic Spinor Representation of Spacelike Curves.**

In this section, the hyperbolic spinor formulation of a spacelike curve in Minkowski space $\mathbb{R}_1^3$ is given. Moreover, the basic theorem of differential geometry is proved with the aid of hyperbolic spinors with two hyperbolic components. In addition to this, the case of the curve to be a general helix is introduced and the Frenet frame of this curve is given in terms of these hyperbolic spinors.

Let $\alpha : I \to \mathbb{R}_1^3$ be an arbitrary spacelike curve. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be Frenet vector fields in $\mathbb{R}_1^3$. So the Frenet derivative equations for spacelike curves are

$$\begin{aligned} \mathbf{T}' &= \kappa\mathbf{N} \\ \mathbf{N}' &= \varepsilon_B\kappa\mathbf{T} + \tau\mathbf{B} \\ \mathbf{B}' &= \tau\mathbf{N} \end{aligned} \tag{4.1}$$

where $\varepsilon_B = g(\mathbf{B},\mathbf{B})$ and κ, τ are curvature and torsion functions, [8]. According to the results in the previous section, there is a hyperbolic spinor for each ordered triad in Minkowski space. So, for the hyperbolic spinor ψ and the Frenet frame $\{\mathbf{N},\mathbf{B},\mathbf{T}\}$, the equalities

$$\mathbf{N} + j\mathbf{B} = \psi^t\sigma\psi \quad \textit{and} \quad \mathbf{T} = -\hat{\psi}^t\sigma\psi \tag{4.2}$$

are valid. Here $\bar{\psi}^t\psi = 1$. So, the hyperbolic spinor ψ represent the triad $\{\mathbf{N},\mathbf{B},\mathbf{T}\}$ and $\frac{d\psi}{ds}$ represent the change of this triad during the curve. Thus, we calculate that $\frac{d\psi}{ds}$.

Differentiating the first equation in (4.2) and considering the equation (4.1) we find easily that

$$\varepsilon_B \kappa \mathbf{T} + \tau \mathbf{B} + j\tau \mathbf{N} = \left(\frac{d\psi}{ds}\right)^t \sigma\psi + \psi^t \sigma \left(\frac{d\psi}{ds}\right). \tag{4.3}$$

Since the binary $\left\{\psi, \widehat{\psi}\right\}$ create a basis for the hyperbolic spinors, the statement

$$\frac{d\psi}{ds} = f\psi + g\widehat{\psi} \tag{4.4}$$

is hold where f and g are two hyperbolic functions. Thus, considering the equations (4.2), (4.3) and (4.4) we find that

$$f = \frac{j\tau}{2}, \quad g = -\varepsilon_B \frac{\kappa}{2}. \tag{4.5}$$

The second statement in the equation (4.2) does not give additional correlations. So, the following theorem can be given with the aid of the equations (4.4) and (4.5).

Theorem 4.1. *If the hyperbolic spinor ψ represents the oriented triad $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ of the spacelike curve α in Minkowski space $\mathbb{R}^3_1$, the Frenet derivative equations are equivalent a single hyperbolic spinor equation as*

$$\frac{d\psi}{ds} = \frac{1}{2}\left(j\tau\psi - \varepsilon_B \kappa \hat{\psi}\right). \tag{4.6}$$

The fundamental theorem of differential geometry (two spacelike (or timelike) curves which has the same curvature and torsion functions coincide at all points after one of them rotate and translate) can be regained by means of the hyperbolic spinors. To demonstrate this, let φ and ψ be two hyperbolic spinors which have the future $\bar{\varphi}^t \varphi = 1 = \bar{\psi}^t \psi$. So the statement

$$\left(\overline{\psi \pm \varphi}\right)^t (\psi \pm \varphi) = 2 \pm \bar{\psi}^t \varphi \pm \psi^t \bar{\varphi} = 2 \pm 2\mathrm{Re}\left(\bar{\psi}^t \varphi\right) \tag{4.7}$$

is hold. On the other hand, the hyperbolic spinors φ and ψ belong to two different spacelike (or timelike) curves which have the same curvature and torsion functions in Minkowski space. So, considering the equation (4.6) we can obtain that

$$\frac{d\left(\bar{\psi}^t \varphi\right)}{ds} = -\varepsilon_B \frac{\kappa}{2}\left[\bar{\hat{\psi}}^t \varphi + \bar{\psi}^t \hat{\varphi}\right].$$

Thus, the statement $\frac{d\left(\bar{\psi}^t \varphi\right)}{ds}$ is only imaginer and the equation $\left(\overline{\psi \pm \varphi}\right)^t (\psi \pm \varphi)$ in the equation (4.7) is constant. In this way, if the triad $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$ of these curves coincide for some values of s, on the points s, the hyperbolic spinor φ coincides with

ψ or $-\psi$. This emphasizes that the tangents of these curves coincide for all points s and two curves are far apart just a constant vector. So, these curves coincide for all values s.

Let us consider the spacelike curve $\alpha : I \to \mathbb{R}^3_1$. According to Theorem 2.1, if the curve α is general helix, the harmonic curvature $H = {}^{\tau}/_{\kappa}$ is constant. The matrix representation of the harmonic curvature is $H(s) = \frac{j}{2}\begin{pmatrix} \tau(s) & j\varepsilon_B\kappa(s) \\ -j\varepsilon_B\kappa(s) & -\tau(s) \end{pmatrix}$.
So, we can take ${}^{\tau}/_{\kappa} = \coth\theta$ where the angle θ is constant. So, using the equations (4.6) and ${}^{\tau}/_{\kappa} = \coth\theta$ we obtain that

$$\frac{d\psi}{ds} = \frac{j\kappa}{2\sinh\theta}\left(\cosh\theta\psi - j\varepsilon_B sinh\theta\widehat{\psi}\right) \tag{4.8}$$

and

$$\frac{d\widehat{\psi}}{ds} = \frac{-j\kappa}{2\sinh\theta}\left(\cosh\theta\widehat{\psi} - j\varepsilon_B\sinh\theta\psi\right). \tag{4.9}$$

In addition, from the equations (4.8) and (4.9) we get

$$\frac{d\left(\cosh(\theta/2)\psi - j\varepsilon_B\sinh(\theta/2)\widehat{\psi}\right)}{ds} = \frac{j\kappa}{2\sinh\theta}\left(\cosh(\theta/2)\psi - j\varepsilon_B\sinh(\theta/2)\widehat{\psi}\right). \tag{4.10}$$

Here for ϕ which is not dependent on s, we obtain that

$$\cosh(\theta/2)\psi + j\sinh(\theta/2)\hat{\psi} = e^{\frac{j}{2\sinh\theta}\int_0^s \kappa(s')ds'}\phi \tag{4.11}$$

where $\bar{\phi}^t\phi = 1$. Similarly, if we get

$$\cosh(\theta/2)\widehat{\psi} - j\varepsilon_B\sinh(\theta/2)\psi = e^{\frac{-j}{2\sinh\theta}\int_0^s \kappa(s')ds'}\widehat{\phi}. \tag{4.12}$$

So, from the equations (4.11) and (4.12) we find the hyperbolic spinors ψ and $\widehat{\psi}$ that

$$\psi = e^{\frac{j}{2\sinh\theta}\int_0^s \kappa(s')ds'}\cosh(\theta/2)\phi + j\varepsilon_B e^{\frac{-j}{2\sinh\theta}\int_0^s \kappa(s')ds'}\sinh(\theta/2)\widehat{\phi} \tag{4.13}$$

and

$$\widehat{\psi} = e^{\frac{-j}{2\sinh\theta}\int_0^s \kappa(s')ds'}\cosh(\theta/2)\widehat{\phi} + j\varepsilon_B e^{\frac{j}{2\sinh\theta}\int_0^s \kappa(s')ds'}\sinh(\theta/2)\phi. \tag{4.14}$$

With the aid of the equations (3.5), (4.2), (4.4) and (4.13) we obtain that

$$\mathbf{T} = \cosh\theta\mathbf{c} - \varepsilon_B\sinh\theta\left[\sinh\left(\frac{1}{\sinh\theta}\int_0^s \kappa(s')\,ds'\right)\mathbf{a} + \cosh\left(\frac{1}{\sinh\theta}\int_0^s \kappa(s')\,ds'\right)\mathbf{b}\right]. \tag{4.15}$$

Considering the similar operations we can take the normal and binormal vectors of α as

$$\mathbf{N} = \cosh\left(\frac{1}{\sinh\theta}\int_0^s \kappa\left(s'\right)ds'\right)\mathbf{a} + \sinh\left(\frac{1}{\sinh\theta}\int_0^s \kappa\left(s'\right)ds'\right)\mathbf{b} \tag{4.16}$$

and

$$\mathbf{B} = -\varepsilon_B \sinh\theta\mathbf{c} + \cosh\theta\left[\sinh\left(\frac{1}{\sinh\theta}\int_0^s \kappa\left(s'\right)ds'\right)\mathbf{a} + \cosh\left(\frac{1}{\sinh\theta}\int_0^s \kappa\left(s'\right)ds'\right)\mathbf{b}\right] \tag{4.17}$$

where $\bar{\phi}^t\phi = 1$ and for the hyperbolic spinor ϕ corresponding to the unit constant triad of vector $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, $\mathbf{a} + j\mathbf{b} = \phi^t\sigma\phi$ and $\mathbf{c} = -\widehat{\phi}^t\sigma\phi$ are hold.

As emphasised in the previous section, there is a two to one homomorphism of the group $SU(2,\mathbb{H})$ on the rotation groups $SO(1,3)$ in 3-dimensional Minkowski space. Let us consider $\bar{\psi}^t\psi = 1$. So, the matrix of type 2×2

$$Q = \begin{pmatrix} \psi_1 & \widehat{\psi}_1 \\ \psi_2 & \widehat{\psi}_2 \end{pmatrix} \tag{4.18}$$

is element of $SU(2,\mathbb{H})$ and this matrix corresponds to the rotation which transport the canonical basis to the orthonormal basis $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$. So, if we consider the equations (3.7), (4.6) and (4.18), we obtain that

$$\frac{dQ}{ds} = QH\left(s\right). \tag{4.19}$$

Here there are two cases in which the integration of (4.19) is relatively simple. First of them, the function $H(s)$ can be independent from s. This case corresponds to the trivial case. If so, $H(s)$ is constant and from the equation (4.19) the statement $Q\left(s\right) = Q\left(0\right)e^{H(s)}$ is hold. In other case, let us consider that $H(s)$ commutes with $H(s')$ for all s'. The solution of the equation (4.19) is $Q\left(s\right) = Q\left(0\right)e^{\int_0^s H(s')ds'}$.

4.2. **The Hyperbolic Spinor Formulation of Timelike Curve.**

In this section, the hyperbolic spinor formulation of a timelike curve is obtained in Minkowski space $\mathbb{R}^3_1$. Moreover, with the aid of these spinors the Frenet frame of timelike curve is calculated. Finally, some examples about the hyperbolic spinor representations of the spacelike and timelike curves are given.

Let $\alpha : I \to \mathbb{R}_1^3$ be an arbitrary timelike curve. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be Frenet vector fields in $\mathbb{R}_1^3$. So the Frenet derivative equations of timelike curve are

$$\begin{aligned} \mathbf{T}' &= \kappa \mathbf{N} \\ \mathbf{N}' &= \kappa \mathbf{T} + \tau \mathbf{B} \\ \mathbf{B}' &= -\tau \mathbf{N} \end{aligned} \tag{4.20}$$

where κ, τ are curvature and torsion functions, [8]. According to the results in the section 4.1, for the hyperbolic spinor ξ the equalities, we obtain that

$$\mathbf{N} \ + j\mathbf{T} = \xi^t \sigma \xi, \qquad \mathbf{B} = -\widehat{\xi}^t \sigma \xi \tag{4.21}$$

where $\overline{\xi}^t \xi = 1$. So, the hyperbolic spinor ξ represent the triad $\{\mathbf{N}, \mathbf{T}, \mathbf{B}\}$ and $\frac{d\xi}{ds}$ represent the change of this triad during the curve. So, considering the similar operations to previous section we obtain that

$$\frac{d\xi}{ds} = f\xi + g\widehat{\xi} \tag{4.22}$$

where two hyperbolic functions f and g are

$$f = \frac{j\kappa}{2}, \quad g = -\frac{\tau}{2}. \tag{4.23}$$

So, the following theorem can be presented.

Theorem 4.2. *Let the hyperbolic spinor ξ be represents the oriented triad $\{\boldsymbol{N}, \boldsymbol{T}, \boldsymbol{B}\}$ of the timelike curve α. So, the Frenet derivative equations are equivalent a single hyperbolic spinor equation as*

$$\frac{d\xi}{ds} = \frac{1}{2}\left(j\kappa\xi - \tau\widehat{\xi}\right). \tag{4.24}$$

Similar to the previous section, let us look at the case of the harmonic curvature $H = {}^{\tau}\!/_{\kappa}$ is constant where $H(s) = \frac{j}{2}\begin{pmatrix} \kappa(s) & j\tau(s) \\ -j\tau(s) & -\kappa(s) \end{pmatrix}$. So the timelike curve α is a general helix. The hyperbolic spinors ξ and $\widehat{\xi}$ can be obtained that

$$\xi = \sinh(\theta/2)\, e^{\frac{-j}{2\sinh\theta}\int_0^s \kappa(s')ds'}\,\widehat{\eta} - j\cosh(\theta/2)\, e^{\frac{j}{2\sinh\theta}\int_0^s \kappa(s')ds'}\,\eta$$

and

$$\widehat{\xi} = -\sinh(\theta/2)\, e^{\frac{j}{2\sinh\theta}\int_0^s \kappa(s')ds'}\,\eta + j\cosh(\theta/2)\, e^{\frac{-j}{2\sinh\theta}\int_0^s \kappa(s')ds'}\,\widehat{\eta}$$

where $\overline{\eta}^t\eta = 1$ and the hyperbolic spinor η is independent from s. So, the Frenet frame of the timelike curve α can be written as

$$\mathbf{T} = \sinh\theta\mathbf{z} + \cosh\theta\left[\sinh\left(\frac{1}{\sinh\theta}\int_0^s \kappa\left(s'\right)ds'\right)\mathbf{x} + \cosh\left(\frac{1}{\sinh\theta}\int_0^s \kappa\left(s'\right)ds'\right)\mathbf{y}\right],$$

$$\mathbf{N} = \cosh\left(\frac{1}{\sinh\theta}\int_0^s \kappa\left(s'\right)ds'\right)\mathbf{x} + \sinh\left(\frac{1}{\sinh\theta}\int_0^s \kappa\left(s'\right)ds'\right)\mathbf{y},$$

$$\mathbf{B} = -\cosh\theta\ \mathbf{z} + \sinh\theta\left[-\sinh\left(\frac{1}{\sinh\theta}\int_0^s \kappa\left(s'\right)ds'\right)\ \mathbf{x} - \cosh\left(\frac{1}{\sinh\theta}\int_0^s \kappa\left(s'\right)ds'\right)\ \mathbf{y}\right].$$

Here, for the hyperbolic spinor η corresponding to the unit constant triad of vector $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$, $\mathbf{x}+j\mathbf{y} = \eta^t\sigma\eta$ and $\mathbf{z} = -\widehat{\eta}^t\sigma\eta$ are hold. Considering the similar equations in the previous section, if we take $\overline{\xi}^t\xi = 1$, then the matrix of type 2×2

$$Q = \begin{pmatrix} \xi_1 & \widehat{\xi}_1 \\ \xi_2 & \widehat{\xi}_2 \end{pmatrix}$$

is element of $SU(2,\mathbb{H})$ and this matrix corresponds to the rotation which transport the canonical basis to the orthonormal basis $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$. Here if we consider the equations (3.8), (4.6) and (4.18), we obtain that

$$\frac{dQ}{ds} = QH\left(s\right).$$

Example 1: Let $\alpha(s) = (\cos(s), \sin(s), 1) \in \mathbb{R}^3_1$ be a unit speed spacelike curve with timelike binormal. So, with the aid of the equation (2.4) we obtain the Frenet vector fields of this curve that

$$\begin{aligned} \mathbf{T} &= (-\sin s, \mathrm{coss}, 0) \\ \mathbf{N} &= (-\cos s, -\sin s, 0) \\ \mathbf{B} &= (0, 0, 1) \end{aligned}$$

where $g\left(\mathbf{T},\mathbf{T}\right) = 1$, $g\left(\mathbf{N},\mathbf{N}\right) = 1$ and $g\left(\mathbf{B},\mathbf{B}\right) = -1$. Let the hyperbolic spinor with two hyperbolic components ψ be corresponding to the Frenet frame of this curve. Moreover, from the last equation and the equation (4.2) we obtain that

$$\frac{d\psi}{ds} = \frac{\widehat{\psi}}{2} \quad \text{and} \quad \frac{d\widehat{\psi}}{ds} = \frac{\widehat{\widehat{\psi}}}{2} = -\frac{\psi}{2}.$$

If we solve that two differential equations, we get

$$\psi = c_1\cos\left(\frac{s}{2}\right) + c_2\sin\left(\frac{s}{2}\right) \quad \text{and} \quad \widehat{\psi} = -c_1\sin\left(\frac{s}{2}\right) + c_2\cos\left(\frac{s}{2}\right)$$

where $c_1, c_2 \in \mathbb{H}$.

Example 2: We consider that $\alpha(s) = (\sqrt{2}\sinh(s), s, -\sqrt{2}\cosh(s))$ is timelike curve in Minkowski space $\mathbb{R}^3_1$. Thus, the Frenet vector fields of timelike curve are as follows:

$$\begin{aligned}\mathbf{T} &= (\sqrt{2}\cosh s, 1, -\sqrt{2}\sinh s),\\ \mathbf{N} &= (\sinh s, 0, -\cosh s),\\ \mathbf{B} &= (\cosh s, \sqrt{2}, -\sinh s).\end{aligned}$$

Here $g(\mathbf{T},\mathbf{T}) = -1$, $g(\mathbf{N},\mathbf{N}) = 1$ and $g(\mathbf{B},\mathbf{B}) = 1$. In addition, if we consider that the hyperbolic spinor ξ corresponds to the orthonormal frame $\{\mathbf{N},\mathbf{T},\mathbf{B}\}$, from the equation (4.2) and the last equation we obtain:

$$\frac{d\xi}{ds} = j\frac{\sqrt{2}}{2}\xi + \frac{1}{2}\widehat{\xi} \quad \text{and} \quad \frac{d\widehat{\xi}}{ds} = -j\frac{\sqrt{2}}{2}\widehat{\xi} - \frac{1}{2}\xi.$$

From the last differential equations we have

$$\xi = c_1 e^{\frac{1}{2}s} + c_2 e^{-\frac{1}{2}s} \quad \text{and} \quad \hat{\xi} = \frac{-\frac{1}{2}c_1\frac{2}{\sqrt{2}j+1}e^{\left(\frac{\sqrt{2}j+1}{2}\right)s} - \frac{1}{2}c_2\frac{2}{\sqrt{2}j-1}e^{\left(\frac{\sqrt{2}j-1}{2}\right)s} + c}{e^{j\frac{1}{\sqrt{2}}s}}.$$

5. Conclusion

The study aims to attain a representation of Serret-Frenet formulas defined on a (spacelike or timelike) curve in Minkowski space. This was done through spinors with hyperbolic components which have a vast field of study in physics. Because of its structure, spinors allow us to present a lot of statements in physics and mathematics in a simpler and shorter way, they provide much convenience to scientists in particularly complex areas such as quantum mechanics. In addition, the study areas of hyperbolic spinors are quite large especially in areas such as relativistic and non-relativistic physics. If we consider that there is a homomorphism between the group of rotation about the origin $SO(1,3)$ and the group of unitary matrix in the type of 2×2; $SU(2,\mathbb{H})$, then it is inevitable that some characteristic statements in Minkowski space can be expressed with these spinors with a simpler way taking advantage of this brevity. In this study, we have showed that the Serret-Frenet formulas of a curve (spacelike or timelike) in Minkowski space can be expressed as a single equation by means of the hyperbolic spinors by utilizing these statements. In addition, we re-proved the fundamental theorem of differential geometry by means of these spinors. Then we give some examples about these hyperbolic spinors considering the special cases of the curve (as helix). So, we think that this study would be important to the sciences of mathematics, engineering and astronomy. There is no doubt that this study will add innovation to science.

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