Journal of Dynamical Systems & Geometric Theories
ISSN : 1726-037X (Print) 2169-0057 (Online)
Vol. 13(1) May 2015, pp. 51-70, DOI : 10.1080/1726037X.2015.1029332

ON THE PRACTICAL STABILIZATION OF DYNAMICAL SYSTEMS WITH APPLICATION

MOUNA ERREBII AND MOHAMED ALI HAMMAMI
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES OF SFAX, UNIVERSITY OF SFAX
ROAD OF SOUKRA, B.P. 1171, SFAX, 3000, TUNISHIA
E-MAIL: MOHAMEDALI.HAMMAMI@FSS.RNU.TN

(Received: 10 December 2014, Accepted:07 March 2015)

Abstract. In this paper, we investigate the problem of exponential stabilization of a class of nonlinear time varying systems. We use Lyapunov techniques to obtain exponential stability of the system in closed-loop with a controller in presence of nonlinear perturbation. The convergence of solutions are in the sense that they converge to a small neighborhood of the origin. An application to a D-C motor is given.

AMS Classification: 34K20, 39A11, 49K40, 37C28

Keywords: Time-varying systems; Practical uniform exponential stabilization; Lyapunov function; Perturbed differential equations

1. Introduction

It is well known that there are two major approaches to the Lyapunov stability analysis of nonlinear time-varying differential systems. The first one is to consider the linearization method and the second one the direct method. Stability of nonlinear systems can be investigated via the first linearization method, but the most powerful techniques is the second direct method [1]-[14]. For this method one usually assumes the existence of, so called Lyapunov function, a positive definite function with negative derivative along the trajectories of the system. The investigation of stability analysis of nonlinear systems using the second Lyapunov function method has produced a vast body of important results and been widely studied. This is due to the theoretical interests in powerful tools for system analysis and control design. It is recognized that the Lyapunov function method serves as a main technique to provide the asymptotic stability of nonlinear systems in presence of perturbations which has very useful applications to control theory. In particular, the problem of stabilization of nonlinear systems has attracted many researches in control theory in the last decade [9, 10, 11, 12, 15]. In [6] the authors studied the observer design problem for a class of nonlinear systems. Specifically, they designed an exponential observer for a separately excited DC motor and a stabilizing controller is designed for the system to ensure exponential stability of the solutions toward their desired values. The purpose of this paper is to establish sufficient conditions for the exponential stabilization of a class of nonlinear time-varying systems in the sense that the states approached the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner.

The paper is organized as follows. In the next section we present some definitions and preliminary results necessary to state our main theorems. In section 3, we present our main results: First, we propose new sufficient conditions using the extended Lyapunov functions for the practical exponential stabilization of nonlinear time-varying systems (1) and (2) with control. Second, we give an illustrative example to show the applicability of the main result. In section 4, we address the stabilization problem for a DC motor and finally we conclude in section 5.

We consider the following system:

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0 \tag{1}$$

where $t \geq t_0 \geq 0$, $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in (t, x) and locally Lipschitz in x. The second one, is a perturbed system:

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t)), \tag{2}$$

where $t \in \mathbb{R}_+$, $x(t) \in \mathbb{R}^n$, $f, g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous in (t, x) and locally Lipschitz in x.

Throughout this paper $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^n$, and B_r is the closed Ball in $\mathbb{R}^n$ centered at 0 with radius r.

2. Auxiliary Facts and Results

In this section, we introduce some basic definitions and preliminary facts which we shall need in the sequel.

Definition 2.1. A solution of system (1) is said to be globally uniformly bounded if for every $\alpha > 0$ there exists $c = c(\alpha)$ such that, for all $t_0 \geq 0$,

$$\|x_0\| \leq \alpha \Rightarrow \|x(t)\| \leq c(\alpha), \quad \forall\ t \geq t_0.$$

Definition 2.2. (i) B_r is uniformly stable if for all $\epsilon > r$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $t_0 \geq 0$,

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall\ t \geq t_0.$$

(ii) B_r is globally uniformly stable if it is uniformly stable and the solutions of system (1) are globally uniformly bounded.

Definition 2.3. B_r is globally uniformly attractive if for all $\epsilon > r$ and $c > 0$, there exists $T(\epsilon, c) > 0$ such that for all $t_0 \geq 0$,

$$\|x(t)\| < \epsilon, \quad \forall\ t \geq t_0 + T(\epsilon, c), \quad \|x_0\| < c.$$

Definition 2.4. A function $\alpha : [0, a[\to [0, +\infty[$ is said to be of class $\mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, $a = +\infty$ and $\alpha(r) \to +\infty$ as $r \to +\infty$.

Definition 2.5. Let $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ a Lyapunov function:

$$\begin{cases} V(t, 0) = 0,\ \forall t \geq 0; \\ V(t, x) > 0,\ \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n \backslash \{0\}. \end{cases}$$

(i) $V(t, x)$ is positive definite, i.e., there exists a continuous, non-decreasing scalar function $\alpha(x)$ such that $\alpha(0) = 0$ and

$$0 < \alpha(\|x\|) < V(t, x), \quad \forall t \geq 0, \forall x \neq 0.$$

(ii) $\dot{V}(t, x)$ is negative definite, that is

$$\dot{V}(t, x) \leq -\gamma(\|x\|) < 0$$

where γ is a continuous non-decreasing scalar function such that $\gamma(0) = 0$.

(iii) $V(t, x) \leq \beta(\|x\|)$ where β is a continuous non-decreasing function and $\beta(0) = 0$, i.e. V is decrescent, i.e. the Lyapunov function is upper bounded.

(iv) V is radially unbounded, that is $\alpha(\|x\|) \to \infty$ as $\|x\| \to \infty$.

Let as recall the following result.

Theorem 2.1. (*see* [5]) *Consider system* (1) *with* $f(t, x)$ *is continuous in* (t, x) *and locally Lipschitz in* x. *Suppose that there exist a continuously differentiable real function* $V(\cdot, \cdot)$ *on* $\mathbb{R}_+ \times \mathbb{R}^n$, $\mathcal{K}_\infty$ *functions* $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, *a* $\mathcal{K}$ *function* $\alpha_3(\cdot)$ *and a small positive real number* ϱ *such that the following inequalities hold for all* $t \in \mathbb{R}_+$ *and* $x \in \mathbb{R}^n$,

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(\|x\|) + \varrho.$$

Then system (1) *is globally uniformly practically stable with* $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)$.

Note that, if the function f satisfies $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$, we shall study the asymptotic stability of system (1) at the origin viewed as an equilibrium point. In this case we have the standard definition of exponential stability.

Definition 2.6. The system (1) is globally uniformly exponentially stable if there exist $\gamma > 0$ and $k > 0$ such that for all $t \geq t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$,

$$\|x(t)\| \leq k\|x_0\| \exp(-\gamma(t - t_0)).$$

Therefore, if the function f satisfies $f(t, 0) \neq 0$ for certain $t \in \mathbb{R}_+$, we shall study the asymptotic stability of system (1) at a neighborhood of the origin viewed as a small ball centered at the origin. The state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner. Here, we study the asymptotic behavior of a small ball centered at the origin for $0 \leq \|x(t)\| - r$. The next definition concerns the practical global uniform exponential stability of a small ball centered at the origin.

Definition 2.7. B_r is globally uniformly exponentially stable if there exist $\gamma > 0$ and $k > 0$ such that for all $t \geq t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$,

$$\|x(t)\| \leq k\|x_0\| \exp(-\gamma(t - t_0)) + r.$$

System (1) is said to be globally practically uniformly exponentially stable if there exists $r > 0$ such that B_r is globally uniformly exponentially stable.

Note that, that if $r = 0$ in the above definition we find the classical definition of the uniform exponential stability of the origin viewed as an equilibrium point.
The following result gives sufficient conditions for practical global exponential stability, (see [1]).

Theorem 2.2. *Consider system* (1). *Let* $V : [0, +\infty[\times\mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable Lyapunov function such that*

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\ f(t,x) \leq -c_3V(t,x) + \varrho$$

for all $t \geq 0$ *and* $x \in \mathbb{R}^n$, *where* $c_1,\ c_2$ *and* c_3 *are positive constants. Then* B_r *is globally uniformly exponentially stable, with* $r = \sqrt{\varrho/c_1c_2}$.

3. Stabilizability analysis

Now we state the stabilizability problem associated with system (1) where we include a control function. The problem is to find a suitable controller which stabilizes the control system. We consider a nonlinear time-varying control system:

$$\dot{x}(t) = f(t,x,u), \quad t \geq 0, \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control and $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a C^1-function.

Definition 3.1. Control system (3) is globally uniformly asymptotically (resp. exponentially) stabilizable by the feedback control $u = u(t,x)$, where $u : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^m$, if the closed-loop system

$$\dot{x}(t) = f(t,x(t),u(t,x(t))) \tag{4}$$

is globally uniformly asymptotically (resp. exponentially) stable.

Definition 3.2. B_r is globally uniformly asymptotically (resp. exponentially) stabilizable by the feedback control $u = u(t,x)$ if the system (4) is globally practically uniformly asymptotically (resp. exponentially) stable.

From Theorem 2.1, one has the following result which concerns the asymptotic stabilizability problem of system (3).

Theorem 3.1. *Suppose that there exist a feedback controller $u = u(t,x)$ for control system* (3) *and a continuously differentiable function $V(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, a $\mathcal{K}$ function $\alpha_3(\cdot)$ and a small positive real number ϱ such that the following inequalities hold for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$,*

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\ f(t,x,u(t,x)) \leq -\alpha_3(\|x\|) + \varrho.$$

Then system (3) *in closed-loop with the feedback controller $u = u(t,x)$ is globally uniformly practically asymptotically stable with $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)$.*

Definition 3.3. Control system (3) is globally uniformly asymptotically stabilizable (resp. globally uniformly practically asymptotically stabilizable) by the feedback control $u = u(t,x)$, if the closed-loop system (4) is globally uniformly asymptotically stable (resp. globally uniformly practically asymptotically stable).

Definition 3.4. B_r is globally uniformly exponentially stabilizable by the feedback control $u = u(t,x)$ if there exist $\gamma > 0$ and $k > 0$ such that for all $t \geq t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$, the solution $x(t)$ of the closed-loop system (4) satisfies:

$$\|x(t)\| \leq k\|x_0\| exp(-\gamma(t - t_0)) + r.$$

In this case, system (3) is globally practically uniformly exponentially stabilizable by the feedback control $u = u(t,x)$.

From Theorem 2.2, one has the following result which concerns the exponential stabilizability problem of system (3).

Theorem 3.2. *Consider system* (3)*. Let $u = u(t,x)$ be an exponential feedback law and $V : [0,+\infty[\times\mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable Lyapunov function such that*

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\ f(t,x,u(t,x)) \leq -c_3 V(t,x) + \varrho$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$, where c_1, c_2 and c_3 are positive constants. Then B_r is globally uniformly exponentially stable, with $r = \sqrt{\varrho/c_1 c_2}$, with respect the closed-loop system (4)*.*

Next, we will study the practical exponential stability problem of a class of nonlinear systems of form (4). It is worth to notice that the origin is not required to be an equilibrium point for the system (4). This may be in many situations meaningful from a practical point of view specially, when stability for control systems is investigated.

Consider the class of systems that can be modeled by (3). We suppose that there exists a feedback control $u = u(t, x)$, such that the closed-loop system (4) is globally practically exponentially stable. The practical uniform exponential stability can be established by requiring the existence of a Lyapunov function that satisfies certain conditions. The next theorem spells out conditions. Under these conditions, the system (4) is practically globally uniformly exponentially stable.

Theorem 3.3. *Consider system* (4). *Let* $V : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a continuous differentiable Lyapunov function such that*

$$c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p + a(t) \tag{5}$$

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u(t,x)) \leq -c_3\|x\|^p + K(t)\ \exp(-\delta t) \tag{6}$$

for all $t \geq 0$ *and* $x \in \mathbb{R}^n$, *where* $c_1,\ c_2,\ c_3,\ p,$ *et* δ *are positive constants with* $p > 1$. *The functions* $a(\cdot)$ *and* $K(\cdot)$ *are continuous and supposed bounded for all* $t \geq 0$ *respectively by* $a > 0$ *and* $K > 0$. *Then* (4) *is practically globally uniformly exponentially stable.*

Proof. The derivative of V along the trajectories of system (4) is given by

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\ f(t,x,u(t,x)).$$

Keeping in the mind that $a(t)$ and $K(t)$ are bounded by a and K respectively, and using condition (5) we infer that

$$-\|x\|^p \leq -\left[\frac{V(t,x) - a}{c_2}\right]. \tag{7}$$

Therefore, taking into account (6), for all $t \geq t_0$ and $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\dot{V}(t,x) &\leq -c_3\Big(\frac{V(t,x)-a}{c_2}\Big) + K\ \exp(-\delta t)\\ &\leq -\frac{c_3}{c_2}\ V(t,x) + \frac{a\ c_3}{c_2} + K\ \exp(-\delta t)\\ &\leq -\frac{c_3}{c_2}\ V(t,x) + \frac{a\ c_3}{c_2} + K.\end{aligned}$$

Thus,

$$V(t,x) \leq V(t_0,x_0)\ \exp\left(-\frac{c_3}{c_2}(t-t_0)\right) + \left(a + \frac{c_2\ K}{c_3}\right), \quad \forall\ t \geq t_0 \geq 0.$$

From condition (5), it follows that

$$V(t_0,x_0) \leq c_2\|x_0\|^p + a$$

$$\|x(t)\| \leq \left[\frac{V(t,x(t))}{c_1}\right]^{\frac{1}{p}}.$$

Form this, we deduce that

$$\begin{aligned}\|x(t)\| &\leq \left(\left(\frac{c_2}{c_1}\|x_0\|^p + \frac{a}{c_1}\right)\exp\left(-\frac{c_3}{c_2}\ (t-t_0)\right) + \frac{a}{c_1} + \frac{c_2\ K}{c_1c_3}\right)^{\frac{1}{p}}\\ &\leq \left(\frac{c_2}{c_1}\right)^{\frac{1}{p}}\|x_0\|\ \exp\left(-\frac{c_3}{c_2\ p}(t-t_0)\right) + \left(\frac{a}{c_1}\right)^{\frac{1}{p}}\exp\left(-\frac{c_3}{c_2\ p}(t-t_0)\right)\\ &+ \left(\frac{a}{c_1} + \frac{c_2\ K}{c_1c_3}\right)^{\frac{1}{p}}\\ &\leq \left(\frac{c_2}{c_1}\right)^{\frac{1}{p}}\|x_0\|\ \exp\left(-\frac{c_3}{c_2\ p}(t-t_0)\right) + (1+\varepsilon)\Big(\ \frac{a}{c_1} + \frac{c_2\ K}{c_1c_3}\Big)^{\frac{1}{p}}, \quad \forall\ t \geq T,\end{aligned}$$

where $T > 0$ is large enough such that,

$$\left(\frac{a}{c_1}\right)^{\frac{1}{p}}\exp(-\frac{c_3}{c_2\ p}(t-t_0)) \leq \varepsilon\Big(\ \frac{a}{c_1} + \frac{c_2\ K}{c_1c_3}\Big)^{\frac{1}{p}},\ \varepsilon \in [0,1],$$

this is because the function $\exp(-\frac{c_3}{c_2\ p}(t-t_0))$ tends to zero when t goes to infinity.

The last inequality on the trajectories shows that (4) is practically globally uniformly exponentially stable.

Hence, B_r is globally uniformly exponentially stable with respect to the closed-loop system (4) where

$$r = (1+\varepsilon)\left(\frac{a}{c_1} + \frac{c_2\ K}{c_1 c_3}\right)^{\frac{1}{p}}.$$

□

Note that, the above radius r is not the smallest one, since the function

$$t \mapsto \exp(-\frac{c_3}{c_2\ p}(t-t_0))$$

tends to zero when t goes to infinity, we can find $\tilde{r} < r$ such that $B_{\tilde{r}}$ is globally uniformly exponentially stable with respect to the closed-loop system (4). Also, it suffices to take ε small enough.

Next, we give an example of system satisfying the assumptions of Theorem 3.3.

Example 1. *Let consider the following non linear control system:*

$$\dot{x} = u + \frac{x^{\frac{5}{3}}}{1+x^2}\ e^{-t}, \quad t \geq 0. \tag{8}$$

Consider a feedback controller as $u(t,x) = -x$ *and a Lyapunov function as*

$$V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+,$$
$$V(t,x) = x^{\frac{4}{3}} + e^{-t}.$$

Note that

$$|x|^{\frac{4}{3}} \leq V(t,x) \leq |x|^{\frac{4}{3}} + 1, \quad \forall\ x \in \mathbb{R}.$$

Then condition (5) *holds with*

$$c_1 = c_2 = 1,\ p = \frac{4}{3} \text{ and } a = 1.$$

On the other hand, we have

$$\dot{V}(t,x) = -\ e^{-t} + \frac{4}{3}\ x^{\frac{1}{3}}\ \dot{x} = -\ e^{-t} - \frac{4}{3}\ x^{\frac{4}{3}} + \frac{4}{3}\ \frac{x^2}{1+x^2}\ e^{-t}.$$

Therefore,

$$\begin{aligned} \dot{V}(t,x) &\leq -\ e^{-t} - \frac{4}{3}\ x^{\frac{4}{3}} + \frac{4}{3}\ e^{-t} \\ &\leq -\frac{4}{3}\ |x|^{\frac{4}{3}} + \frac{1}{3}\ e^{-t}, \quad \forall\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \end{aligned}$$

The condition (6) *of Theorem* 3.3 *is also satisfied with*

$$c_3 = \frac{4}{3},\ K = \frac{1}{3},\ \delta = 1\ et\ p = \frac{4}{3}.$$

So, (8) *is globally practically uniformly exponentially stabilizable by the controller*

$$u(t,x) = -x.$$

Hence, B_r*,* $r = 2.365$*, is globally uniformly exponentially stable with respect to the system* (8) *in closed-loop with the above controller. In this case, all state trajectories are bounded and approach a sufficiently small neighborhood of the origin. One also desires that the state approaches the origin. Therefore, since the function* e^{-t} *tends to zero when* t *goes to infinity, system* (8) *in closed-loop with the above control converges to a small ball* $B(0, r(t))$ *where* $r(t)$ *is small enough when* t *goes to* $+\infty$*.*

Example 2. *We present now an example in dimensional two that implements the previous theorem. Consider the following planar system:*

$$\dot{x} \;=\; \begin{pmatrix} -x_1 + u \\ -x_2 + \varepsilon \frac{x_2}{1+x_2^2} \exp(-t^2) \end{pmatrix} + \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} a(t) \tag{9}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$*,* $\varepsilon > 0$ *and* $a(t)$ *is a continuous bounded function. Let*

$$u(t,x) = \varepsilon \frac{x_1}{1+x_1^2} \exp(-x_1^2),$$

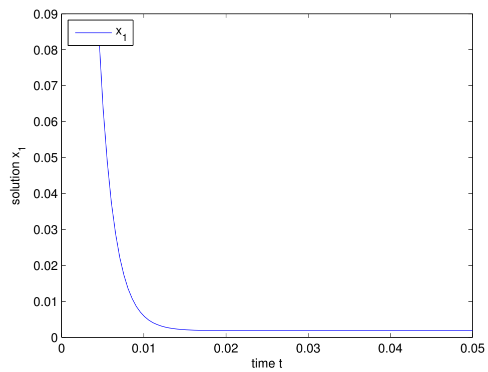
and choose the quadratic Lyapunov function

$$V(t,x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2.$$

The time derivative of V *along the trajectories of the system in closed-loop with* $u = u(t,x)$ *is given by*

$$\dot{V}(t,x) \quad \leq \quad -2V(t,x) + 2\varepsilon.$$

Hence, by application of Theorem 3.3*, this planar system is globally practical exponential stabilizable by the feedback controller* $u = u(t,x)$*. This example shows the convergence of solutions when* t *goes to* $+\infty$ *for* ε *small enough. In Figure1, we see the convergence of the components* $x_1(t)$ *and* $x_2(t)$ *of the system* (9) *in closed-loop with the feedback controller* $u(t,x)$*.*



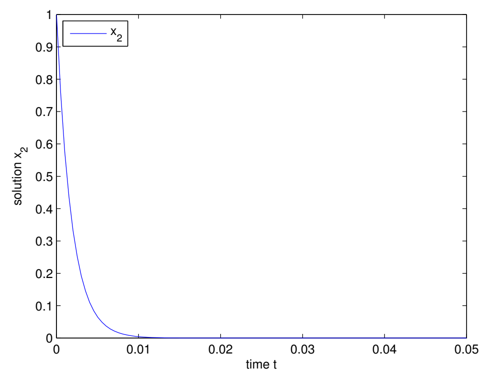


FIGURE 1. Time evolution of the state $(x_1(t), x_2(t))$

3.1. **Perturbed systems.**

Now, we consider the perturbed systems

$$\dot{x} = f(t, x, u) + g(t, x) \tag{10}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a C^1-function and $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous in (t, x) and locally Lipschitz in x on $\mathbb{R}_+ \times \mathbb{R}^n$. In the sequel, we can prove the practical global uniform exponential stability of system (10), under some conditions on the perturbation, by using Lyapunov technique, and using the same assumptions as in Theorem 3.3. We assume that (10) satisfies the following assumptions:

(H_1) There exists a feedback controller $u = u(t,x)$ and a function $V : [0,+\infty[\times\mathbb{R}^n \to \mathbb{R}$ that satisfies the inequalities:

$$c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p + a(t) \tag{11}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u(t,x)) \leq -c_3\|x\|^p + K(t)\ \exp(-\delta t) \tag{12}$$

$$\Big\|\frac{\partial V}{\partial x}\Big\| \leq c_4\|x\|^{p-1}, \tag{13}$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$, where $c_1,\ c_2,\ c_3, c_4,\ p,\ et\ \delta$ are positive constants with $p > 1$. $a(\cdot)$ and $K(\cdot)$ are continuous positive functions and supposed bounded for all $t \geq 0$ respectively by $a > 0$ and $K > 0$.

(H_2) The perturbation term $g(t,x)$ satisfies the following assumption:

$$\|g(t,x)\| \leq \gamma(t)\|x\|, \quad \forall t \geq 0 \tag{14}$$

where $\gamma(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ a positive function satisfies,

$$\gamma(t) \leq \frac{c_3}{c_4}, \tag{15}$$

for all $t \geq t_0 \geq 0$, with

$$\int_{t_0}^{t} \gamma(s)\, ds \leq M_\gamma < +\infty. \tag{16}$$

Theorem 3.4. *Under assumptions (H_1) and (H_2) the system* (10) *in closed-loop with the controller $u = u(t,x)$ is globally practically uniformly exponentially stable.*

Proof The derivative of V along the trajectories of the system (10) in closed-loop with the feedback $u = u(t,x)$ is given by

$$\begin{aligned} \dot{V}(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u(t,x)) + \frac{\partial V}{\partial x} g(t,x) \\ &\leq \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u(t,x)) + \|\frac{\partial V}{\partial x}\|\|g(t,x)\|. \end{aligned}$$

Using the assumptions (H_1) and (H_2), it follows that along any solution of system (10),

$$\dot{V}(t,x) \leq -c_3\|x\|^p + K(t)\ \exp(-\delta t) + c_4\|x\|^{p-1}\gamma(t)\|x\|.$$

Thus,

$$\begin{aligned}\dot V(t,x) &\leq -\Big(\tfrac{c_3-c_4\gamma(t)}{c_2}\Big)(V(t,x)-a)+K\ \exp(-\delta t)\\ &\leq -\Big(\tfrac{c_3-c_4\gamma(t)}{c_2}\Big)V(t,x)+\Big(\tfrac{c_3-c_4\gamma(t)}{c_2}\Big)a+K,\end{aligned}$$

for all $t\geq t_0$ and $x\in\mathbb{R}^n$.

To simplify the presentation, let

$$\alpha(t)=\frac{c_3-c_4\gamma(t)}{c_2},\quad \text{since } \gamma(t)<\frac{c_3}{c_4}, \text{we deduce that } \alpha(t)>0,$$

and

$$\beta(t)=\Big(\frac{c_3-c_4\gamma(t)}{c_2}\Big)a+K.$$

By the above expression, we give also the following estimate

$$\dot V(t,x)\leq -\alpha(t)V(t,x)+\beta(t). \tag{17}$$

Let $V(t)=V(t,x)$ and

$$y(t)=V(t)\exp\Big(\int_{t_0}^t \alpha(s)ds\Big).$$

It follows that,

$$\begin{aligned}\dot y(t) &= (\dot V(t)+\alpha(t)V(t))\exp\Big(\int_{t_0}^t\alpha(s)\,ds\Big)\\ &\leq \beta(t)\quad \exp\Big(\int_{t_0}^t\alpha(s)\,ds\Big).\end{aligned}$$

Integrating between t_0 and t, one obtains $\forall t\geq t_0$,

$$y(t)\leq y(t_0)+\int_{t_0}^t\beta(s)\exp\Big(\int_{t_0}^s\alpha(\tau)\,d\tau\Big)\,ds.$$

Then,

$$\begin{aligned} V(t) &\leq V(t_0) \ \exp\Big(-\int_{t_0}^{t}\alpha(s)\,ds\Big) \\ &+ \Big[\int_{t_0}^{t}\beta(s)\ \exp\Big(\int_{t_0}^{s}\alpha(\tau)\,d\tau\Big)\,ds\Big]\exp\Big(-\int_{t_0}^{t}\alpha(s)\,ds\Big). \end{aligned}$$

Therefore,

$$\int_{t_0}^{t}\alpha(s)\,ds = \frac{c_3}{c_2}(t-t_0) - \frac{c_4}{c_2}\int_{t_0}^{t}\gamma(s)\,ds$$

which implies that, using condition (16), for all $t \geq t_0 \geq 0$,

$$\exp\Big(\int_{t_0}^{t}\alpha(s)\,ds\Big) \leq \exp\Big(\frac{c_3}{c_2}(t-t_0)\Big),$$

and

$$\exp\Big(-\int_{t_0}^{t}\alpha(s)\,ds\Big) \leq \exp\Big(\frac{c_4}{c_2}M_\gamma\Big)\ \exp\Big(-\frac{c_3}{c_2}(t-t_0)\Big).$$

Moreover, we have

$$\begin{aligned} \int_{t_0}^{t}\beta(s)\exp\Big(\int_{t_0}^{s}\alpha(\tau)\,d\tau\Big)\,ds &= \int_{t_0}^{t}\Big(\Big(\frac{c_3 - c_4\gamma(s)}{c_2}\Big)a + K\Big)\exp\Big(\int_{t_0}^{s}\alpha(\tau)\,d\tau\Big)\,ds \\ &\leq \int_{t_0}^{t}\Big(\Big(\frac{c_3 - c_4\gamma(s)}{c_2}\Big)a + K\Big)\exp\Big(\frac{c_3}{c_2}(s-t_0)\Big)\,ds \\ &\leq \int_{t_0}^{t}\Big(\frac{c_3}{c_2}\ a + K\Big)\ \exp\Big(\frac{c_3}{c_2}(s-t_0)\Big)\,ds \\ &\leq \Big(\ a + K\ \tfrac{c_2}{c_3}\Big)\exp\Big(\tfrac{c_3}{c_2}(t-t_0)\Big). \end{aligned}$$

It follows that, by using the above expressions, $\forall t \geq t_0 \geq 0$,

$$\Big(\int_{t_0}^{t}\beta(s)\ \exp\Big(\int_{t_0}^{s}\alpha(\tau)\,d\tau\Big)\,ds\Big)\exp\Big(-\int_{t_0}^{t}\alpha(s)\,ds\Big) \quad\leq\quad \Big(\ a + K\ \tfrac{c_2}{c_3}\Big)\ \exp\Big(\tfrac{c_4}{c_2}M_\gamma\Big).$$

Thus, we obtain

$$V(t) \quad\leq\quad V(t_0)\ \exp\Big(\tfrac{c_4}{c_2}M_\gamma\Big)\ \exp\Big(-\tfrac{c_3}{c_2}(t-t_0)\Big) + \Big(\ a + K\ \tfrac{c_2}{c_3}\Big)\ \exp\Big(\tfrac{c_4}{c_2}M_\gamma\Big).$$

Hence in view of condition (11), it follows that

$$V(t_0) \leq c_2\|x_0\|^p + a$$

$$\|x(t)\| \leq \left[\frac{V(t)}{c_1}\right]^{\frac{1}{p}}.$$

This implies that,

$$\begin{aligned}
\|x(t)\| &\leq \left[\left(\tfrac{c_2}{c_1}\ \|x_0\|^p + \tfrac{a}{c_1}\right)\ \exp\left(\tfrac{c_4}{c_2}M_\gamma\right)\exp\left(-\tfrac{c_3}{c_2}(t-t_0)\right) + \left(\tfrac{a}{c_1} + \tfrac{Kc_2}{c_1c_3}\right)\ \exp\left(\tfrac{c_4}{c_2}M_\gamma\right)\right]^{\frac{1}{p}} \\
&\leq \left(\tfrac{c_2}{c_1}\right)^{\frac{1}{p}}\ \exp\left(\frac{c_4}{c_2p}M_\gamma\right)\|x_0\|\ \exp\left(-\tfrac{c_3}{c_2p}(t-t_0)\right) \\
&+ \left(\tfrac{a}{c_1}\right)^{\frac{1}{p}}\ \exp\left(\tfrac{c_4}{c_2p}M_\gamma\right)\ \exp\left(-\tfrac{c_3}{c_2p}(t-t_0)\right) + \left(\tfrac{a}{c_1} + \tfrac{Kc_2}{c_1c_3}\right)^{\frac{1}{p}}\ \exp\left(\tfrac{c_4}{c_2p}M_\gamma\right).
\end{aligned}$$

Furthermore,

$$\forall t \geq T > 0 \text{ and } \varepsilon \in [0,1],$$

(T large enough)

$$\|x(t)\| \leq \left(\frac{c_2}{c_1}\right)^{\frac{1}{p}}\|x_0\|\ \exp\left(\frac{c_4}{c_2p}M_\gamma\right)\ \exp\left(-\frac{c_3}{c_2p}(t-t_0)\right)+(1+\varepsilon)\left(\frac{a}{c_1}+\frac{Kc_2}{c_1c_3}\right)^{\frac{1}{p}}\ \exp\left(\frac{c_4}{c_2p}M_\gamma\right).$$

Letting

$$r = (1+\varepsilon)\left(\frac{a}{c_1} + \frac{Kc_2}{c_1c_3}\right)^{\frac{1}{p}}\ \exp\left(\frac{c_4}{c_2p}M_\gamma\right),$$

we deduce that system (10) in closed-loop with $u = u(t,x)$ is practically globally uniformly exponentially stable.

In the sequel, the perturbation term will satisfies other condition. Hence, (H_2) can be replaced by the following assumption:

(H_3) There exists a positive constant $\tilde{\gamma}$ such that

$$\|g(t,x)\| \leq \tilde{\gamma}\|x\| \tag{18}$$

$$\tilde{\gamma} < \frac{c_3}{c_4}. \tag{19}$$

Then, according to assumption (H_1), we can state the next result.

Corollary 3.1. *If (H_1) and (H_3) are satisfied. Then the system* (10) *in closed-loop with the controller $u = u(t,x)$ is globally practically uniformly exponentially stable.*

Proof. By the same arguments used in the proof of Theorem 3.4, we have

$$\dot{V}(t,x) \leq V(t_0,x_0) \quad \exp\Big(-\Big(\frac{c_3-c_4\tilde{\gamma}}{c_2}\Big)(t-t_0)\Big) + \Big(a + \frac{Kc_2}{c_3-c_4\ \tilde{\gamma}}\Big), \quad \forall t \geq t_0 \geq 0.$$

Taking condition (11) into account, we have

$$V(t_0,x_0) \leq c_2\|x_0\|^p + a$$

$$\|x(t)\| \leq \left[\frac{V(t,x(t))}{c_1}\right]^{\frac{1}{p}},$$

into the last inequalities, we obtain

$$\begin{aligned}
\|x(t)\| \quad &\leq \quad \Big[\Big(\tfrac{c_2}{c_1}\ \|x_0\|^p + \tfrac{a}{c_1}\Big)\ \exp\Big(-\tfrac{c_3-c_4\tilde{\gamma}}{c_2}(t-t_0)\Big) + \Big(\tfrac{a}{c_1} + \tfrac{Kc_2}{c_1(c_3-c_4\tilde{\gamma})}\Big)\Big]^{\frac{1}{p}} \\
&\leq \quad \Big(\tfrac{c_2}{c_1}\Big)^{\frac{1}{p}}\|x_0\|\ \exp\Big(-\tfrac{c_3-c_4\tilde{\gamma}}{c_2 p}(t-t_0)\Big) + \Big(\tfrac{a}{c_1}\Big)^{\frac{1}{p}}\ \exp\Big(-\tfrac{c_3-c_4\tilde{\gamma}}{c_2 p}(t-t_0)\Big) \\
&+ \quad \Big(\tfrac{a}{c_1} + \tfrac{Kc_2}{c_1(c_3-c_4\tilde{\gamma})}\Big)^{\frac{1}{p}}.
\end{aligned}$$

Then, $\forall t \geq T > 0$ and $\varepsilon \in [0,1]$, (T large enough)

(20)

$$\|x(t)\| \leq \Big(\frac{c_2}{c_1}\Big)^{\frac{1}{p}}\|x_0\|\ \exp\Big(-\frac{c_3-c_4\tilde{\gamma}}{c_2 p}(t-t_0)\Big) + (1+\varepsilon)\Big(\frac{a}{c_1} + \frac{Kc_2}{c_1(c_3-c_4\tilde{\gamma})}\Big)^{\frac{1}{p}}.$$

Hence we obtain the global practical uniform exponential stability of system (10) in closed-loop with the feedback controller $u = u(t,x)$.

4. Application to a DC motor

The dynamic of the separately excited DC motor may be expressed by the following equations (see [6]):

$$\begin{cases} K\omega(t) = -R_a i_a(t) - L_a \frac{di_a(t)}{dt} + U_a(t) \\[2ex] K i_a(t) = J\frac{d\omega}{dt} + f\omega(t) + T_l(t), \end{cases}$$

equivalently

$$\begin{cases} \frac{di_a(t)}{dt} = -\frac{R_a}{L_a} i_a(t) - \frac{K}{L_a}\omega(t) + \frac{1}{L_a} U_a(t) \\ \\ \frac{d\omega}{dt} = \frac{K}{J} i_a(t) - \frac{f}{J}\omega(t) - \frac{1}{J} T_l(t) \end{cases}$$

where K, R_a, L_a, J and f are respectively, the torque and back-EMF, the armature resistance, the armature inductance, the rotor mass moment of inertia and the viscous friction coefficient with:

$$R_a > \max\Big(K, \frac{J}{2f}, \frac{KL_a}{J}, \frac{L_a}{2f}\Big) \text{ and } K < \min\Big(f, \frac{L_a f}{J}, \frac{L_a}{2K}, \frac{J}{2K}\Big). \tag{21}$$

$\omega(t), i_a(t), U_a(t)$ and $T_l(t)$ respectively denote the rotor angular speed, the armature current, the terminal voltage and the load torque.

For ease of presentation, we denote the state variables x_1, x_2 such that:

$$x_1 = i_a,\ x_2 = \omega.$$

Therefore, the equations describing a separately excited DC motor can be written as:

$$\begin{cases} \dot{x}_1 = -\frac{R_a}{L_a} x_1 - \frac{K}{L_a} x_2 + \frac{1}{L_a} U_a \\ \\ \dot{x}_2 = \frac{K}{J} x_1 - \frac{f}{J} x_2 - \frac{1}{J} T_l. \end{cases}$$

In these new coordinates $x = (x_1, x_2)$, the system is represented by

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{R_a}{L_a} x_1 - \frac{K}{L_a} x_2 + \frac{1}{L_a} U_a \\ \\ \frac{K}{J} x_1 - \frac{f}{J} x_2 - \frac{1}{J} T_l \\ \\ \end{pmatrix}. \tag{22}$$

The state-model (22) is equivalent to a system (10) where $f : \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ are defined by

$$f(t, x, u) = \begin{pmatrix} -\dfrac{R_a}{L_a} x_1 + \dfrac{1}{L_a} U_a \\ -\dfrac{f}{J} x_2 - \dfrac{1}{J} T_l \end{pmatrix},$$

and

$$g(t, x) = \begin{pmatrix} -\frac{K}{L_a} x_2 \\ \frac{K}{J} x_1 \end{pmatrix}.$$

Therefore

$$\|g(t,x)\| \leq \max\Big(\frac{K}{L_a}, \frac{K}{J}\Big)\|x\|,$$

so, we deduce that condition (18) is satisfied with:

$$\tilde{\gamma} = \max\Big(\frac{K}{L_a}, \frac{K}{J}\Big)$$

and $p = 2$.

Remark 4.1. *Let $i_a > 0$ (i.e $x_1 > 0$). This constraint represents a reasonable assumption since when i_a the current of the circuit, equal zero there is no power being supplied to the engine and therefore there is nothing to control.*

Let

$$u(x) = \frac{1}{L_a}U_a = \frac{x_2 T_l}{2Kfx_1J}$$

and choose a Lyapunov function

$$V(t,x) = \sin t\ \exp(-t) + Kx_1^2 + \frac{J}{2f}x_2^2.$$

One obtains

$$\min\Big(K, \frac{J}{2f}\Big)\|x\|^2 \leq V(t,x) \leq \max\Big(K, \frac{J}{2f}\Big)\|x\|^2 + \sin t\ \exp(-t)$$

Then condition (11) holds with

$$c_1 = \min\left(K, \frac{J}{2f}\right),$$

$$c_2 = \max\Big(K, \frac{J}{2f}\Big),$$

and

$$a(t) = \sin t \exp(-t).$$

On the other hand, The derivative of V along the trajectories of system is given by

$$\begin{aligned}
\dot{V}(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\ f(t,x,u(x)) \\
&= (\cos t - \sin t)\exp(-t) + 2Kx_1\dot{x}_1 + \tfrac{J}{f}x_2\dot{x}_2 \\
&= -2\tfrac{R_aK}{L_a}x_1^2 - x_2^2 + \cos t\ \exp(-t) \\
&\leq -\min\left(2\tfrac{R_aK}{L_a},1\right)\|x\|^2 + \cos t\ \exp(-t) \\
&\leq -c_3\|x\|^2 + K(t)\exp(-\delta t),
\end{aligned}$$

where

$$c_3 = \min\left(2\frac{R_aK}{L_a},1\right), K(t) = \cos t$$

and $\delta = 1$ satisfies (12). In addition, one can easily verify that condition (13) is satisfied with

$$c_4 = 2c_2 = 2\max\left(K,\frac{J}{2f}\right).$$

Therefore, by (21), we deduce that condition (19) of (H_3) is satisfied. Hence, the equivalent state-model (22) in closed-loop is uniformly practically exponentially stable.

5. **Conclusion**

Practical exponential stabilizability of a class of nonlinear time-varying control systems in closed-loop by feedback controller using the second Lyapunov method has been studied. Numerical examples are given to illustrate the applicability of the main result.

Acknowledgement The authors wish to thank the reviewer for his valuable and careful comments.

References

[1] A. Ben Abdallah, I. Ellouze and M. A. Hammami, *Practical stability of nonlinear time-varying cascade systems, Journal of Dynamical and Control Systems,* Vol. 15, No. 1, (2009) 45-62.

[2] A. Ben Abdallah and I. Ellouze, M. A. Hammami, *Practical exponential stability of perturbed Triangular systems and a separation principle, Asian Journal of Control,* Vol. 13, No. 3, (2011) 445-448.

[3] A. Ben Abdallah, M. Dlala, M.A. Hammami, *A new Lyapunov function for perturbed nonlinear systems, Systems and Control letters,* Vol. 56, No. 3, (2007) 179-187.

[4] A. Chaillet, A. Loria, *Necessary and sufficient conditions for uniform semiglobal practical asymptotic stability : Application to cascaded systems, Automatica,* Vol. 42, (2006) 1899-1906.

[5] M. Corless, G. Leitmann, *Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems, IEEE Transactions on Automatic Control.,* Vol 26, No. 5, (1981) 1139-1143.

[6] M.A. Hammami, M. Zribi and J. Kallel, *An exponential observer and a controller for a class of nonlinear systems, Archives of Control Sciences,* Vol. 23, No. 1, (2013) 79-92.

[7] H.K. Khalil, *Nonlinear systems, 2nd edition* Mac-Millan, (1996).

[8] I. Konstantopoulos and P. Antsaklis, *New bounds for robust stability of continuous and discrete-time systems under parametric uncertainty, Kybernetika*, Vol. 31, No. 6, (1995) 623-636.

[9] E. Lee, L. Markus, *Foundation of Optimal Control Theory, John Wiley,* New York, (1967).

[10] A.G. Soldatos, M. Corless, *Stabilizing uncertain systems with bounded control, Dynam. Control,* Vol. 1, (1991) 227-238.

[11] Z. Qu, *Global stabilization of nonlinear systems with a class of unmatched uncertainties, Systems Control Letters,* Vol. 18, (1992) 301-307.

[12] P. Niamsup, V.N. Phat, *Asymptotic stability of nonlinear control systems described by differences equations with multiple delays. Elect. J. of Diff. Equations,* No. 11, (2000) 1-17.

[13] V.Yu. Slyusarchuk, *Conditions for the existence of bounded solutions of nonlinear differential and functional differential equations, Ukrainian Mathematical Journal*, Vol. 62, N (2010) 970-981.

[14] J. Wang and X. Li, *New Razumikhin type stability theorem for impulsive functional differential equations with infinite delays, Analysis and Applications*, Vol. 9, No. 3, (2011) 315-327.

[15] J. Zabczyk, *Mathematical Control Theory: An Introduction,* Bikhauser, Berlin, 1992.