

ON GENERALIZED TIMELIKE MANNHEIM CURVES IN MINKOWSKI SPACE-TIME

ALI UÇUM¹, EMILJA NEŠOVIĆ², AND KAZIM İLARSLAN³

¹DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES AND ARTS,
KIRIKKALE UNIVERSITY, KIRIKKALE-TURKEY
E-MAIL: ALIUCUM05@GMAIL.COM

²DEPARTMENT OF MATHEMATICS AND INFORMATICS,
FACULTY OF SCIENCE, UNIVERSITY OF KRAGUJEVAC,
KRAGUJEVAC, SERBIA
E-MAIL: NESOVICKG@SBB.RS

³DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES AND ARTS,
KIRIKKALE UNIVERSITY, KIRIKKALE-TURKEY
E-MAIL: KILARSLAN@YAHOO.COM

(Received: 2 February 2015, Accepted: 13 March 2015)

ABSTRACT. In Minkowski space-time, timelike generalized Mannheim curves are studied by Akyigit and et al in [1]. In that paper, the authors take the Mannheim mate α^* of the timelike curve α as a timelike curve in Minkowski space-time $\mathbb{E}_1^4$. In this statement, the plane spanned by $\{B_1^*, B_2^*\}$ becomes a spacelike plane. Thus the other possible cases such as timelike and null for the plane spanned by $\{B_1^*, B_2^*\}$ is not considered. In this paper, by taking consideration of all possible causal characters of the plane spanned by $\{B_1^*, B_2^*\}$, we give the necessary and sufficient conditions for timelike curves in $\mathbb{E}_1^4$ to be generalized timelike Mannheim curves in terms of their curvature functions. Also, the related examples are given.

AMS Classification: 53C50, 53C40.

Keywords: Mannheim curve; Minkowski space-time; Frenet vector fields

1. INTRODUCTION

In the Euclidean space $\mathbb{E}^3$ there exist some kinds of associated curves whose the Frenet frame fields satisfy certain geometric conditions. The example of such curves are called Mannheim curves, which have the property that their principal normal lines coincide with the binormal lines of the Mannheim mate curve at the corresponding points of the curves ([5],[8],[12],[19]). It is known that the curvature functions of Mannheim curve in $\mathbb{E}^3$ satisfy the equality $\kappa_1 = a(\kappa_1^2 + \kappa_2^2)$ for some positive constant number a and its parametric equation is obtained in ([10], see also [12],[15],[20]). Some characterizations of Mannheim curves in $\mathbb{E}^3$ can be found in ([9],[10]). The notion of Mannheim curves is closely related to the notion of Bertrand curves ([5],[9]). For both of the mentioned curves, there exists the corresponding generalization in Euclidean 4-space ([13],[14]). A regular smooth curve α in $\mathbb{E}^4$ is called *generalized Mannheim curve*, if there exist another regular smooth curve α^* and a bijection $\phi : \alpha \mapsto \alpha^*$ such that the principal normal line of α at each point of α is included in the plane spanned by the first binormal line and the second binormal line of α^* under bijection ϕ ([14]). The curve α^* is called *generalized Mannheim mate (partner) curve* of α . In particular, the pair $\{\alpha, \alpha^*\}$ is called *generalized Mannheim pair* of curves. Explicit parametric equations of generalized Mannheim curves in $\mathbb{E}^4$ are given in ([14]). Some characterizations of Mannheim curves in 3-dimensional space forms are given in ([4]).

In Minkowski space-time, generalized null Mannheim curves are studied in ([7],[16]). They show that every null helix lying fully in $\mathbb{E}_1^4$ is generalized null Mannheim curve whose generalized Mannheim mate curve is a spacelike or a timelike helix. Also, they prove that there are no generalized null Mannheim curves whose generalized Mannheim mate curve is a pseudo null curve.

In Minkowski space-time, generalized spacelike Mannheim curves whose the Frenet frame contains only non-null vectors and generalized timelike Mannheim curves are studied in ([1]) and ([6]) respectively.

Timelike generalized Mannheim curves in Minkowski space-time are studied by Akyigit and et al in ([1]). In that paper, the authors take the Mannheim mate α^* of the timelike curve α as a timelike curve in Minkowski space-time $\mathbb{E}_1^4$. In this statement, the plane spanned by $\{B_1^*, B_2^*\}$ becomes a spacelike plane. Thus the other possible cases such as timelike and null for the plane spanned by $\{B_1^*, B_2^*\}$

is not considered. In this paper, by taking consideration of all possible causal characters of the plane spanned by $\{B_1^*, B_2^*\}$, we give the necessary and sufficient conditions for timelike curves in $\mathbb{E}_1^4$ to be generalized timelike Mannheim curves in terms of their curvature functions. Also, the related examples are given.

2. Preliminaries

The Minkowski space-time $\mathbb{E}_1^4$ is the Euclidean 4-space $\mathbb{E}^4$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}_1^4$. Recall that a vector $v \in \mathbb{E}_1^4 \setminus \{0\}$ can be *spacelike* if $g(v, v) > 0$, *timelike* if $g(v, v) < 0$ and *null (lightlike)* if $g(v, v) = 0$. In particular, the vector $v = 0$ is said to be a spacelike. The norm of a vector v is given by $\|v\| = \sqrt{|g(v, v)|}$. Two vectors v and w are said to be orthogonal, if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^4$, can locally be *spacelike*, *timelike* or *null (lightlike)*, if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null ([17]).

A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$ ([2]). On the other hand, a non-null curve α is parametrized by the arclength parameter s if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along a curve α in $\mathbb{E}_1^4$, consisting of the tangent, the principal normal, the first binormal and the second binormal vector field respectively.

If α is a spacelike or a timelike curve whose the Frenet frame $\{T, N, B_1, B_2\}$ contains only non-null vector fields, the Frenet equations are given by ([9])

$$(2.1) \quad \begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2 k_1 & 0 & 0 \\ -\epsilon_1 k_1 & 0 & \epsilon_3 k_2 & 0 \\ 0 & -\epsilon_2 k_2 & 0 & -\epsilon_1 \epsilon_2 \epsilon_3 k_3 \\ 0 & 0 & -\epsilon_3 k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where $g(T, T) = \epsilon_1$, $g(N, N) = \epsilon_2$, $g(B_1, B_1) = \epsilon_3$, $g(B_2, B_2) = \epsilon_4$, $\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 = -1$, $\epsilon_i \in \{-1, 1\}$, $i \in \{1, 2, 3, 4\}$. In particular, the following conditions hold:

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(N, B_2) = g(B_1, B_2) = 0.$$

If α is a null curve, the Frenet equations are given by ([2])

$$(2.2) \quad \begin{bmatrix} T' \\ N' \\ B'_1 \\ B'_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_2 & 0 & -k_1 & 0 \\ 0 & -k_2 & 0 & k_3 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where the first curvature $k_1(s) = 0$, if $\alpha(s)$ is straight line or $k_1(s) = 1$ in all other cases. In this case, the next conditions hold:

$$g(T, T) = g(B_1, B_1) = 0, \quad g(N, N) = g(B_2, B_2) = 1,$$

$$g(T, N) = g(T, B_2) = g(N, B_1) = g(N, B_2) = g(B_1, B_2) = 0, \quad g(T, B_1) = 1.$$

If α is a pseudo null curve, the Frenet formulas read ([3],[16])

$$(2.3) \quad \begin{bmatrix} T' \\ N' \\ B'_1 \\ B'_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & -k_2 \\ -k_1 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where the first curvature $k_1(s) = 0$, if α is straight line, or $k_1(s) = 1$ in all other cases. Then the following conditions are satisfied:

$$g(T, T) = g(B_1, B_1) = 1, \quad g(N, N) = g(B_2, B_2) = 0,$$

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(B_1, B_2) = 0, \quad g(N, B_2) = 1.$$

Finally, if α is partially null curve, the Frenet formulas read ([3],[16])

$$(2.4) \quad \begin{bmatrix} T' \\ N' \\ B'_1 \\ B'_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & -k_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where the third curvature $k_3(s) = 0$ for each s . Moreover, the following conditions hold:

$$g(T, T) = g(N, N) = 1, \quad g(B_1, B_1) = g(B_2, B_2) = 0,$$

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(N, B_2) = 0, \quad g(B_1, B_2) = 1.$$

In addition, let P be a subspace of $\mathbb{E}_1^4$. We consider the induced metric $g|_P$. P is called spacelike if the induced metric is positive definite, timelike if the induced metric has index 1 and lightlike if the induced metric is degenerate. On the other

hand, P is called a spacelike subspace if all vector $v \neq 0$ are spacelike on P , a timelike subspace if P has at least one timelike vector, a lightlike subspace for otherwise [11].

3. On Generalized Timelike Mannheim Curves in $\mathbb{E}_1^4$

In this section, we define generalized timelike Mannheim curves in Minkowski space-time (see also [1]) and obtain some explicit parameter equations of such curves. We also give the necessary and sufficient conditions for timelike curves to be generalized timelike Mannheim curves in terms of their curvature functions.

Definition 1. A timelike curve α in Minkowski space-time $\mathbb{E}_1^4$ is called a generalized timelike Mannheim curve if there exists a curve α^* in $\mathbb{E}_1^4$ such that, at the corresponding points of the curves, the principal normal line of α is included in the plane spanned by the first binormal line and the second binormal line of α^* .

By principal normal (binormal) line, we mean the straight line in direction of the principal normal (binormal) vector field. Let $\alpha : I \rightarrow \mathbb{E}_1^4$ be the generalized timelike Mannheim curve in $\mathbb{E}_1^4$ with the Frenet frame $\{T, N, B_1, B_2\}$ and $\alpha^* : I \rightarrow \mathbb{E}_1^4$ the generalized Mannheim mate curve of α with the Frenet frame $\{T^*, N^*, B_1^*, B_2^*\}$. Then the principal normal vector $N(s)$ is a spacelike vector lying in the plane spanned by $\{B_1^*, B_2^*\}$. Hence $N(s)$ is given by $N(s) = a(s)B_1^*(s) + b(s)B_2^*(s)$, where $a(s)$ and $b(s)$ are some differentiable functions. Depending on the causal character of the plane $\text{span}\{B_1^*, B_2^*\}$, we distinguish the following three cases:

Case 1. The plane $\text{span}\{B_1^*, B_2^*\}$ is spacelike;

Case 2. The plane $\text{span}\{B_1^*, B_2^*\}$ is timelike;

Case 3. The plane $\text{span}\{B_1^*, B_2^*\}$ is lightlike;

In what follows, we consider these three cases separately.

Case 1. Let the plane $\text{span}\{B_1^*, B_2^*\}$ be spacelike.

Theorem 1. Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be a generalized timelike Mannheim curve and $\alpha^* : I^* \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be the generalized Mannheim mate curve of α such that the principal normal line of α lies in the spacelike plane spanned by $\{B_1^*, B_2^*\}$. Then α^*

is a non-null Frenet curve such that the curvatures of α and α^* satisfy the relations

$$(3.1) \quad \lambda^2 k_2^2 - (1 + \lambda k_1)^2 \neq 0,$$

$$(3.2) \quad k_1 = -\lambda (k_1^2 - k_2^2),$$

$$(3.3) \quad (k_1^*)^2 = \frac{\lambda^2 [-(k_1')^2 + (k_2')^2 + k_2^2 k_3^2] - \epsilon_1^* (f'')^2}{\epsilon_2^* (f')^4},$$

$$(3.4) \quad k_2^* = \frac{\lambda (k_1 k_1' - k_2 k_2')}{a \epsilon_2^* k_1^* (f')^3},$$

$$(3.5) \quad k_3^* = \frac{-a \epsilon_2^* [2(k_1^*)' k_2^* (f')^3 + 6k_1^* k_2^* (f')^2 f'' + k_1^* (k_2^*)' (f')^3]}{b \epsilon_2^* k_1^* k_2^* (f')^4} + \frac{2\lambda (k_1 k_1'' - k_2 k_2'') + \lambda ((k_1')^2 - (k_2')^2) + \lambda k_2^2 k_3^2}{b \epsilon_2^* k_1^* k_2^* (f')^4}.$$

where $\lambda \in \mathbb{R}_0$, $a = g(N, B_1^*)$, $b = g(N, B_2^*)$ and $f(s) = \int_0^s \|\alpha^{*'}(t)\| dt$.

Proof. Since the principal normal line of α lies in the spacelike plane spanned by $\{B_1^*, B_2^*\}$, α^* is a spacelike or a timelike curve whose the Frenet frame satisfies (2.1) for $\epsilon_1^* = -\epsilon_2^*$, $\epsilon_3^* = \epsilon_4^* = 1$. In particular, the curve α^* can be parameterized by

$$(3.6) \quad \alpha^*(f(s)) = \alpha(s) + \lambda(s)N(s).$$

where s is arc-length parameter of α , $s^* = f(s) = \int_0^s \|\alpha^{*'}(t)\| dt$ is the arc-length parameter of α^* , $f : I \subset \mathbb{R} \rightarrow I^* \subset \mathbb{R}$ and λ are some smooth functions. We distinguish two cases: (A.1) $k_2 = 0$ and (A.2) $k_2 \neq 0$.

Case (A.1) $k_2 = 0$

Differentiating the relation (3.6) with respect to s and applying (2.1), we find

$$(3.7) \quad T^* f' = (1 + \lambda k_1)T + \lambda' N.$$

By taking the scalar product of (3.7) with $N = aB_1^* + bB_2^*$, we get

$$(3.8) \quad \lambda' = 0.$$

Substituting (3.8) in (3.7), we have

$$(3.9) \quad T^* f' = (1 + \lambda k_1)T.$$

If we denote

$$\delta = \frac{(1 + \lambda k_1)}{f'},$$

we get

$$(3.10) \quad T^* = \delta T.$$

Differentiating the relation (3.10) with respect to s and applying (2.1), we obtain

$$(3.11) \quad f' k_1^* N^* = \delta' T + \delta k_1 N.$$

By taking the scalar product of (3.11) with $N = aB_1^* + bB_2^*$, we get $\delta k_1 = 0$ which is a contradiction.

Case (A.2) $k_2 \neq 0$

Differentiating the relation (3.6) with respect to s and applying (2.1), we find

$$(3.12) \quad T^* f' = (1 + \lambda k_1)T + \lambda' N + \lambda k_2 B_1.$$

By taking the scalar product of (3.12) with $N = aB_1^* + bB_2^*$, we have

$$(3.13) \quad \lambda' = 0.$$

Substituting (3.13) in (3.12), we obtain

$$(3.14) \quad T^* f' = (1 + \lambda k_1)T + \lambda k_2 B_1.$$

From (3.14), we get the relation (3.1)

$$(3.15) \quad g(T^* f', T^* f') = \epsilon_1^* (f')^2 = -(1 + \lambda k_1)^2 + (\lambda k_2)^2 \neq 0.$$

Differentiating the relation (3.15) with respect to s and applying (2.1), we find

$$(3.16) \quad T^* f'' + \epsilon_2^* (f')^2 k_1^* N^* = (1 + \lambda k_1)' T + (k_1 + \lambda k_1^2 - \lambda k_2^2) N + \lambda k_2' B_1 + \lambda k_2 k_3 B_2.$$

By taking the scalar product of (3.16) with $N = aB_1^* + bB_2^*$, we have the relation

$$(3.2)$$

$$(3.17) \quad k_1 = -\lambda(k_1^2 - k_2^2).$$

Substituting (3.17) in (3.16), we obtain

$$(3.18) \quad T^* f'' + \epsilon_2^* (f')^2 k_1^* N^* = (1 + \lambda k_1)' T + \lambda k_2' B_1 + \lambda k_2 k_3 B_2.$$

By taking the scalar product of (3.18) with itself, we have the relation (3.3)

$$(3.19) \quad (k_1^*)^2 = \frac{\lambda^2 [-(k_1')^2 + (k_2')^2 + k_2^2 k_3^2] - \epsilon_1^* (f'')^2}{\epsilon_2^* (f')^4}.$$

Differentiating the relation (3.19) with respect to s and taking the scalar product with $N = aB_1^* + bB_2^*$, we find the relation (3.4)

$$(3.20) \quad k_2^* = \frac{\lambda(k_1 k_1' - k_2 k_2')}{a\epsilon_2^* k_1^* (f')^3}.$$

Differentiating the relation (3.20) two times with respect to s and taking the scalar product with $N = aB_1^* + bB_2^*$, we obtain the relation (3.5)

$$(3.21) \quad k_3^* = \frac{-a\epsilon_2^* [2(k_1^*)' k_2^* (f')^3 + 6k_1^* k_2^* (f')^2 f'' + k_1^* (k_2^*)' (f')^3]}{b\epsilon_2^* k_1^* k_2^* (f')^4} + \frac{2\lambda(k_1 k_1'' - k_2 k_2'') + \lambda((k_1')^2 - (k_2')^2) + \lambda k_2^2 k_3^2}{b\epsilon_2^* k_1^* k_2^* (f')^4}$$

Therefore, the proof is complete. $\square$

From theorem 1, we give the following corollaries.

Corollary 1. *If $k_2^* = 0$, then α is a timelike curve with constant k_1 , k_2 and α^* is a timelike curve with constant k_1^* and any function k_3^* where*

$$(3.22) \quad (k_1^*)^2 = \frac{\lambda^2 k_2^2 k_3^2}{(f')^4},$$

$$(3.23) \quad k_3^* = \frac{-(ab)' \pm \sqrt{[(ab)']^2 - [(a')^2 + (b')^2 + k_1^2 - k_2^2]}}{f'}.$$

Proof. From (3.20), we get

$$(3.24) \quad k_1^2 - k_2^2 = c \in \mathbb{R}_0 \text{ (constant)}$$

Using (3.15), (3.20) and (3.24), we obtain that k_1 , k_2 and f' are constant. From (3.18), we have

$$(3.25) \quad \epsilon_2^* (f')^2 k_1^* N^* = \lambda k_2 k_3 B_2.$$

Since B_2 is a spacelike vector, N^* is also a spacelike vector. That is, $\epsilon_2^* = -\epsilon_1^* = 1$. Thus, α^* is a timelike curve. By taking the scalar product of (3.25) with itself, we have the relation (3.22)

$$(3.26) \quad (k_1^*)^2 = \frac{\lambda^2 k_2^2 k_3^2}{(f')^4}.$$

Differentiating $N = aB_1^* + bB_2^*$, we have

$$(3.27) \quad k_1 T + k_2 B_1 = (a' - b f' k_3^*) B_1^* + (b' + a f' k_3^*) B_2^*$$

By taking the scalar product of (3.27) with itself, we have the relation (3.23)

$$(3.28) \quad k_3^* = \frac{-(ab)' \pm \sqrt{[(ab)']^2 - [(a')^2 + (b')^2 + k_1^2 - k_2^2]}}{f'}.$$

Therefore, the proof is complete. $\square$

Theorem 2. Let α be a timelike Frenet curve in $\mathbb{E}_1^4$ such that its first and second curvature functions satisfy the relations $k_1 = -\lambda(k_1^2 - k_2^2)$ and $-(1 + \lambda k_1)^2 + (\lambda k_2)^2 \neq 0$. If the spacelike or timelike curve α^* can be defined by

$$\alpha^*(f(s)) = \alpha(s) + \lambda(s)N(s),$$

then α^* is a generalized Mannheim mate curve of α .

Proof. The proof of the theorem can be found in ([1]). $\square$

Corollary 2. Let α be a generalized timelike Mannheim helix and α^* be the Mannheim mate curve of α in $\mathbb{E}_1^4$. Then $N = bB_2^*$ where $b = \pm 1$ and $\alpha^*(f(s)) = \alpha(s) + \lambda(s)N(s)$ is also a timelike helix with

$$(3.29) \quad (k_1^*)^2 = \frac{\lambda k_2^2 k_3^2}{(f')^4},$$

$$(3.30) \quad (k_2^*)^2 = \frac{\lambda^2 k_2^2 k_3^4 + (k_1^*)^4 (f')^6}{(k_1^*)^2 (f')^6},$$

$$(3.31) \quad k_3^* = \frac{\lambda k_2^2 k_3^2}{b k_1^* k_2^* (f')^4}.$$

Proof. Since k_1 , k_2 and k_3 are constant, $f' = \text{constant}$. From (3.19), we get

$$(k_1^*)^2 = \frac{\lambda k_2^2 k_3^2}{(f')^4} = (\text{constant}) \text{ and } \epsilon_2^* = -\epsilon_1^* = 1.$$

From (3.20), we find $a = 0$, that is $N = bB_2^*$ where $b = \pm 1$. Using (3.18) and that k_1 , k_2 and k_3 are constant, we have

$$(3.32) \quad (f')^2 k_1^* N^* = \lambda k_2 k_3 B_2.$$

Differentiating the relation (3.32) with respect to s and applying (2.1), we find

$$(3.33) \quad (k_1^*)^2 (f')^3 T^* + k_1^* k_2^* (f')^3 B_1^* = -\lambda k_2 k_3^2 B_1.$$

By taking the scalar product of (3.33) with $N = bB_2^*$, we have

$$(3.34) \quad (k_2^*)^2 = \frac{\lambda^2 k_2^2 k_3^4 + (k_1^*)^4 (f')^6}{(k_1^*)^2 (f')^6} = (\text{constant}).$$

From (3.21), we obtain

$$(3.35) \quad k_3^* = \frac{\lambda k_2^2 k_3^2}{b k_1^* k_2^* (f')^4} = (\text{constant}).$$

Thus, α^* is a timelike helix. □

Corollary 3. *Let α and α^* be timelike helix curve mentioned in Theorem 1. It is clear from (3.32) that α and α^* are the Mannheim mate curves of each other.*

Example 1. *Let α be the following timelike helix in $\mathbb{E}_1^4$*

$$\alpha(s) = \left(\sqrt{\frac{3}{2}} \sinh s, \sqrt{\frac{3}{2}} \cosh s, \frac{1}{\sqrt{6}} \sin(\sqrt{3}s), -\frac{1}{\sqrt{6}} \cos(\sqrt{3}s) \right)$$

with the Frenet vector fields and the curvature functions

$$\begin{aligned} T(s) &= \left(\sqrt{\frac{3}{2}} \cosh s, \sqrt{\frac{3}{2}} \sinh s, \frac{1}{\sqrt{2}} \cos(\sqrt{3}s), \frac{1}{\sqrt{2}} \sin(\sqrt{3}s) \right), \\ N(s) &= \left(\frac{1}{\sqrt{2}} \sinh s, \frac{1}{\sqrt{2}} \cosh s, -\frac{1}{\sqrt{2}} \sin(\sqrt{3}s), \frac{1}{\sqrt{2}} \cos(\sqrt{3}s) \right), \\ B_1(s) &= \left(-\frac{1}{\sqrt{2}} \cosh s, -\frac{1}{\sqrt{2}} \sinh s, -\sqrt{\frac{3}{2}} \cos(\sqrt{3}s), -\sqrt{\frac{3}{2}} \sin(\sqrt{3}s) \right), \\ B_2(s) &= \left(\frac{1}{\sqrt{2}} \sinh s, \frac{1}{\sqrt{2}} \cosh s, \frac{1}{\sqrt{2}} \sin(\sqrt{3}s), -\frac{1}{\sqrt{2}} \cos(\sqrt{3}s) \right) \end{aligned}$$

and

$$k_1 = \sqrt{3}, \quad k_2 = 2, \quad k_3 = 1.$$

It is clear that k_1 and k_2 satisfy the relations (3.1) and (3.2). So, for $\lambda = \sqrt{3}$, we can define α^ by*

$$\alpha^*(f(s)) = \alpha(s) + \lambda(s)N(s)$$

that is,

$$\alpha^*(s) = \left(\sqrt{6} \sinh s, \sqrt{6} \cosh s, -\sqrt{\frac{2}{3}} \sin(\sqrt{3}s), -\sqrt{\frac{2}{3}} \cos(\sqrt{3}s) \right).$$

It can be easily calculated that

$$\begin{aligned} T^*(s) &= \left(\sqrt{\frac{3}{2}} \cosh s, \sqrt{\frac{3}{2}} \sinh s, -\frac{1}{\sqrt{2}} \cos(\sqrt{3}s), -\frac{1}{\sqrt{2}} \sin(\sqrt{3}s) \right), \\ N^*(s) &= \left(\frac{1}{\sqrt{2}} \sinh s, \frac{1}{\sqrt{2}} \cosh s, \frac{1}{\sqrt{2}} \sin(\sqrt{3}s), -\frac{1}{\sqrt{2}} \cos(\sqrt{3}s) \right), \\ B_1^*(s) &= \left(-\frac{1}{\sqrt{2}} \cosh s, -\frac{1}{\sqrt{2}} \sinh s, \sqrt{\frac{3}{2}} \cos(\sqrt{3}s), \sqrt{\frac{3}{2}} \sin(\sqrt{3}s) \right), \\ B_2^*(s) &= \left(\frac{1}{\sqrt{2}} \sinh s, \frac{1}{\sqrt{2}} \cosh s, -\frac{1}{\sqrt{2}} \sin(\sqrt{3}s), \frac{1}{\sqrt{2}} \cos(\sqrt{3}s) \right) \end{aligned}$$

and

$$k_1^* = \frac{\sqrt{3}}{2}, k_2^* = 1, k_3^* = \frac{1}{2}.$$

It is clear that α^* is a timelike helix and also is the Mannheim mate curve of α .

Case 2. Let the plane $\text{span}\{B_1^*, B_2^*\}$ be timelike.

It is known that a timelike plane can be spanned by the spacelike and the timelike mutually orthogonal unit vectors or else by two linearly independent null vectors. Thus, in this case, we have two subcases:

(2.1) B_1^* is a timelike (spacelike) and B_2^* is spacelike (timelike),

(2.1) B_1^* and B_2^* are null vectors.

In what follows, we consider these two subcases separately.

Case (2.1). Let B_1^* be a timelike (spacelike) and B_2^* be spacelike (timelike). Then the curve α^* is a spacelike Frenet curve satisfying (2.1). In this case, we omit the proof of the following theorems since they are similar to the proof of the theorem in Case 1.

Theorem 3. Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be a generalized timelike Mannheim curve and $\alpha^* : I^* \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be the generalized Mannheim mate curve of α such that the principal normal line of α lies in the timelike plane spanned by $\{B_1^*, B_2^*\}$. Then α^* is a spacelike Frenet curve such that the curvatures of α and α^* satisfy the relations

$$(3.36) \quad \lambda^2 k_2^2 - (1 + \lambda k_1)^2 > 0,$$

$$(3.37) \quad k_1 = -\lambda(k_1^2 - k_2^2),$$

$$(3.38) \quad (k_1^*)^2 = \frac{\lambda^2 [-(k_1')^2 + (k_2')^2 + k_2^2 k_3^2] - (f'')^2}{\epsilon_2^*(f')^4},$$

$$(3.39) \quad k_2^* = \frac{\lambda(k_1 k_1' - k_2 k_2')}{a k_1^* (f')^3},$$

$$(3.40) \quad k_3^* = \frac{a[2(k_1^*)' k_2^* (f')^3 + 6k_1^* k_2^* (f')^2 f'' + k_1^* (k_2^*)' (f')^3]}{\epsilon_4^* b k_1^* k_2^* (f')^4} - \frac{2\lambda(k_1 k_1'' - k_2 k_2'') + \lambda((k_1')^2 - (k_2')^2) + \lambda k_2^2 k_3^2}{\epsilon_4^* b k_1^* k_2^* (f')^4}.$$

where $\lambda \in \mathbb{R}_0$, $a = \epsilon_3^* g(N, B_1^*)$, $b = \epsilon_4^* g(N, B_2^*)$ and $f(s) = \int_0^s \|\alpha^*{}'(t)\| dt$.

From theorem 3, we have the following corollaries.

Corollary 4. *If $k_2^* = 0$, then α is a timelike curve with constant k_1 , k_2 and α^* is a timelike curve with constant k_1^* and any function k_3^* where*

$$(3.41) \quad (k_1^*)^2 = \frac{\lambda^2 k_2^2 k_3^2}{(f')^4},$$

$$(3.42) \quad k_3^* = \frac{-(a'b - ab') \pm \sqrt{(a'b - ab')^2 - [k_2^2 - k_1^2 + \epsilon_3^* ((b')^2 - (a')^2)]}}{f'}.$$

Theorem 4. *Let α be a timelike Frenet curve $\mathbb{E}_1^4$ such that its first and second curvature functions satisfy the relations $k_1 = -\lambda(k_1^2 - k_2^2)$ and $-(1 + \lambda k_1)^2 + (\lambda k_2)^2 > 0$. If the spacelike curve α^* can be defined*

$$\alpha^*(f(s)) = \alpha(s) + \lambda(s)N(s),$$

then α^ is a generalized Mannheim mate curve of α .*

Corollary 5. *Let α be a generalized timelike Mannheim and α^* be the Mannheim mate curve of α in $\mathbb{E}_1^4$. Then $N = bB_2^*$ where $b = \pm 1$ and $\alpha^*(f(s)) = \alpha(s) + \lambda(s)N(s)$ is also a spacelike helix with*

$$(3.43) \quad (k_1^*)^2 = \frac{\lambda k_2^2 k_3^2}{(f')^4},$$

$$(3.44) \quad (k_2^*)^2 = \frac{\lambda^2 k_2^2 k_3^4 - (k_1^*)^4 (f')^6}{\epsilon_3^* (k_1^*)^2 (f')^6},$$

$$(3.45) \quad k_3^* = \frac{-\lambda k_2^2 k_3^2}{\epsilon_4^* b k_1^* k_2^* (f')^4}.$$

Example 2. Let α be the following timelike helix in $\mathbb{E}_1^4$

$$\alpha(s) = \left(\sqrt{2} \sinh s, \sqrt{2} \cosh s, \sin s, \cos s \right)$$

with the Frenet vector fields and the curvature functions

$$\begin{aligned} T(s) &= \left(\sqrt{2} \cosh s, \sqrt{2} \sinh s, \cos s, -\sin s \right), \\ N(s) &= \left(\frac{\sqrt{6}}{3} \sinh s, \frac{\sqrt{6}}{3} \cosh s, -\frac{\sqrt{3}}{3} \sin s, -\frac{\sqrt{3}}{3} \cos s \right), \\ B_1(s) &= \left(-\cosh s, -\sinh s, -\sqrt{2} \cos s, \sqrt{2} \sin s \right), \\ B_2(s) &= \left(\frac{1}{\sqrt{3}} \sinh s, \frac{1}{\sqrt{3}} \cosh s, \frac{\sqrt{6}}{3} \sin s, \frac{\sqrt{6}}{3} \cos s \right) \end{aligned}$$

and

$$k_1 = \sqrt{3}, k_2 = 2\frac{\sqrt{6}}{3}, k_3 = \frac{1}{\sqrt{3}}.$$

It is clear that k_1 and k_2 satisfy the relations (3.36) and (3.37). So, for $\lambda = -3\sqrt{3}$, we can define α^* by

$$\alpha^*(f(s)) = \alpha(s) + \lambda(s)N(s)$$

that is,

$$\alpha^*(s) = \left(-2\sqrt{2} \sinh s, -2\sqrt{2} \cosh s, 4 \sin s, 4 \cos s \right).$$

It can be easily calculated that

$$\begin{aligned} T^*(s) &= \left(-\cosh s, -\sinh s, \sqrt{2} \cos s, -\sqrt{2} \sin s \right), \\ N^*(s) &= \left(-\frac{1}{\sqrt{3}} \sinh s, -\frac{1}{\sqrt{3}} \cosh s, -\frac{\sqrt{6}}{3} \sin s, -\frac{\sqrt{6}}{3} \cos s \right), \\ B_1^*(s) &= \left(-\sqrt{2} \cosh s, -\sqrt{2} \sinh s, \cos s, -\sin s \right), \\ B_2^*(s) &= \left(\frac{\sqrt{6}}{3} \sinh s, \frac{\sqrt{6}}{3} \cosh s, -\frac{\sqrt{3}}{3} \sin s, -\frac{\sqrt{3}}{3} \cos s \right) \end{aligned}$$

and

$$k_1^* = \frac{\sqrt{6}}{4}, k_2^* = -\frac{1}{\sqrt{3}}, k_3^* = \frac{1}{2\sqrt{6}}.$$

It is clear that α^* is a spacelike helix with timelike B_1^* and also is the Mannheim mate curve of α .

Case (2.2). B_1^* and B_2^* are null vectors. Then the curve α^* is a partially null Frenet curve satisfying (2.4).

Theorem 5. *Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be a generalized timelike Mannheim curve and $\alpha^* : I^* \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be the generalized Mannheim mate curve of α such that the principal normal line of α lies in the timelike plane spanned by null vectors $\{B_1^*, B_2^*\}$. Then α^* is a partially null Frenet curve such that the curvatures of α and α^* satisfy the following relations*

$$(3.46) \quad \lambda^2 k_2^2 - (1 + \lambda k_1)^2 > 0,$$

$$(3.47) \quad k_1 = -\lambda (k_1^2 - k_2^2),$$

$$(3.48) \quad (k_1^*)^2 = \frac{\lambda^2 [-(k_1')^2 + (k_2')^2 + k_2^2 k_3^2] - (f'')^2}{(f')^4},$$

$$(3.49) \quad k_2^* = \frac{\lambda(k_1 k_1' - k_2 k_2')}{b k_1^* (f')^3},$$

$$(3.50) \quad \begin{aligned} & \frac{-b[2(k_1^*)' k_2^* (f')^3 + 6k_1^* k_2^* (f')^2 f'' + k_1^* (k_2^*)' (f')^3]}{b k_1^* k_2^* (f')^4} \\ & + \frac{2\lambda(k_1 k_1'' - k_2 k_2'') + \lambda((k_1')^2 - (k_2')^2) + \lambda k_2^2 k_3^2}{b k_1^* k_2^* (f')^4} = 0 \end{aligned}$$

where $\lambda \in \mathbb{R}_0$, $f(s) = \int_0^s \|\alpha^{*'}(t)\| dt$ and $b = g(N, B_1^*)$.

Proof. Since the principal normal line of α lies in the timelike plane spanned by null vectors $\{B_1^*, B_2^*\}$, α^* is a partially null curve whose the Frenet frame satisfies (2.4). In particular, the curve α^* can be parameterized by

$$(3.51) \quad \alpha^*(f(s)) = \alpha(s) + \lambda(s)N(s).$$

where s is arclength parameter of α , $s^* = f(s) = \int_0^s \|\alpha^{*'}(t)\| dt$ is the arc-length parameter of α^* , $f : I \subset \mathbb{R} \rightarrow I^* \subset \mathbb{R}$ and λ are some smooth functions. We distinguish two cases: (2.2.1) $k_2 = 0$ and (2.2.2) $k_2 \neq 0$.

Case (2.2.1) $k_2 = 0$

Differentiating the relation (3.51) with respect to s and applying (2.1), we find

$$(3.52) \quad T^* f' = (1 + \lambda k_1)T + \lambda' N.$$

By taking the scalar product of (3.52) with $N = aB_1^* + bB_2^*$, we get

$$(3.53) \quad \lambda' = 0.$$

Substituting (3.53) in (3.52), we have

$$(3.54) \quad T^* f' = (1 + \lambda k_1)T.$$

From (3.54), $g(T^* f', T^* f') = (f')^2 = -(1 + \lambda k_1)^2$ which is a contradiction.

Case (2.2.2) $k_2 \neq 0$

Differentiating the relation (3.51) with respect to s and applying (2.1), we find

$$(3.55) \quad T^* f' = (1 + \lambda k_1)T + \lambda' N + \lambda k_2 B_1.$$

By taking the scalar product of (3.55) with $N = aB_1^* + bB_2^*$, we have

$$(3.56) \quad \lambda' = 0.$$

Substituting (3.56) in (3.55), we obtain

$$(3.57) \quad T^* f' = (1 + \lambda k_1)T + \lambda k_2 B_1.$$

From (3.57), we get the relation (3.46)

$$(3.58) \quad g(T^* f', T^* f') = (f')^2 = -(1 + \lambda k_1)^2 + (\lambda k_2)^2 > 0.$$

Differentiating the relation (3.57) with respect to s and applying (2.1), we find

$$(3.59) \quad T^* f'' + (f')^2 k_1^* N^* = (1 + \lambda k_1)' T + (k_1 + \lambda k_1^2 - \lambda k_2^2) N + \lambda k_2' B_1 + \lambda k_2 k_3 B_2.$$

By taking the scalar product of (3.59) with $N = aB_1^* + bB_2^*$, we have the relation

$$(3.47) \quad (3.60) \quad k_1 = -\lambda(k_1^2 - k_2^2).$$

Substituting (3.60) in (3.59), we obtain

$$(3.61) \quad T^* f'' + (f')^2 k_1^* N^* = (1 + \lambda k_1)' T + \lambda k_2' B_1 + \lambda k_2 k_3 B_2.$$

By taking the scalar product of (3.61) with itself, we have the relation (3.48)

$$(3.62) \quad (k_1^*)^2 = \frac{\lambda^2 [-(k_1')^2 + (k_2')^2 + k_2^2 k_3^2] - (f'')^2}{(f')^4}.$$

Differentiating the relation (3.61) with respect to s and taking the scalar product with $N = aB_1^* + bB_2^*$, we find the relation (3.49)

$$(3.63) \quad k_2^* = \frac{\lambda(k_1 k_1' - k_2 k_2')}{b k_1^* (f')^3}.$$

Differentiating the relation (3.61) two times with respect to s and taking the scalar product with $N = aB_1^* + bB_2^*$, we obtain the relation (3.50)

$$(3.64) \quad \frac{-b[2(k_1^*)'k_2^*(f')^3 + 6k_1^*k_2^*(f')^2f'' + k_1^*(k_2^*)'(f')^3]}{bk_1^*k_2^*(f')^4} + \frac{2\lambda(k_1k_1'' - k_2k_2'') + \lambda((k_1')^2 - (k_2')^2) + \lambda k_2^2k_3^2}{bk_1^*k_2^*(f')^4} = 0$$

Therefore, the proof is complete. $\square$

From theorem 5, we get the following corollaries.

Corollary 6. *If $k_2^* = 0$, then α is a timelike curve with constant k_1 , k_2 and α^* is a partially null curve with constant k_1^* where*

$$(3.65) \quad \left(\left(\frac{1}{k_3} \right)' \right)^2 = \frac{(1 + \lambda k_1)^2}{(\lambda k_2)^2 - (1 + \lambda k_1)^2},$$

$$(3.66) \quad (k_1^*)^2 = \frac{\lambda^2 k_2^2 k_3^2}{(f')^4},$$

$$(3.67) \quad b^{\pm 1} = t_0 e^{\sqrt{k_1^2 - k_2^2} s}, \quad t_0 \in \mathbb{R}_0, \quad k_1^2 > k_2^2,$$

$$(3.68) \quad ab = \frac{1}{2}$$

where $\lambda \in \mathbb{R}_0$, $N = aB_1^* + bB_2^*$ and $f(s) = \int_0^s \|\alpha^{*'}(t)\| dt$.

Proof. Differentiating the relation (3.61) with respect to s and taking the scalar product with $N = aB_1^* + bB_2^*$, we have

$$(3.69) \quad k_1^2 - k_2^2 = t_1 \in \mathbb{R}_0.$$

From (3.58), (3.69) and (3.60), we have that k_1 , k_2 and f' are constant. Then, (3.61) can be rewritten as

$$(3.70) \quad (f')^2 k_1^* N^* = \lambda k_2 k_3 B_2.$$

By taking the scalar product of (3.70) with itself, we have the relation (3.66)

$$(3.71) \quad (k_1^*)^2 = \frac{\lambda^2 k_2^2 k_3^2}{(f')^4}$$

Differentiating the relation (3.70) with respect to s , we obtain

$$(3.72) \quad -(k_1^*)^2 (f')^3 T^* = -\lambda k_2 k_3^2 B_1 + \lambda k_2 k_3' B_2.$$

By taking the scalar product of (3.72) with itself and using (3.71), we have the relation (3.65)

$$(3.73) \quad \left(\left(\frac{1}{k_3} \right)' \right)^2 = \frac{(1 + \lambda k_1)^2}{(\lambda k_2)^2 - (1 + \lambda k_1)^2},$$

Here, $k_3 \neq \text{constant}$. Differentiating $N = aB_1^* + bB_2^*$ with respect to s , we get

$$(3.74) \quad k_1 T + k_2 B_1 = a' B_1^* + b' B_2^*,$$

$$(3.75) \quad ab = \frac{1}{2}.$$

By taking the scalar product of (3.74) with itself and using (3.75), we have the relation (3.67)

$$(3.76) \quad b^{\pm 1} = t_0 e^{\sqrt{k_1^2 - k_2^2} s}, \quad t_0 \in \mathbb{R}_0, \quad k_1^2 > k_2^2.$$

□

Corollary 7. *There exists no generalized timelike Mannheim helix such that its Mannheim mate curve is a partially null curve.*

Proof. In theorem 5, let α be a generalized timelike Mannheim helix. Then, from (3.58), $f' = \text{constant}$. Rewriting (3.61), we get

$$(3.77) \quad (f')^2 k_1^* N^* = \lambda k_2 k_3 B_2.$$

Differentiating the relation (3.77) with respect to s , we obtain

$$(3.78) \quad -(k_1^*)^2 (f')^3 T^* + k_1^* k_2^* (f')^3 B_1^* = -\lambda k_2 k_3^2 B_1.$$

y taking the scalar product of (3.78) with $N = aB_1^* + bB_2^*$, we get

$$(3.79) \quad b k_1^* k_2^* (f')^3 = 0.$$

Assume that $b = 0$. Then $N = aB_1^*$ where N is spacelike and B_1^* is null, which is contradiction. If $k_1^* = 0$, then from (3.77) which is a contradiction. Lastly, if $k_2^* = 0$, then (3.65) and (3.60) lead to a contradiction. Thus, There exists no generalized timelike Mannheim helix such that its Mannheim mate curve is a partially null curve. □

Case 3. Let the plane $\text{span}\{B_1^*, B_2^*\}$ be lightlike.

In this case, we obtain two theorems depending on the causal character of the basis vectors of a lightlike plane $\text{span}\{B_1^*, B_2^*\}$ which can be spanned by a null

vector B_1^* and a spacelike vector B_2^* , or else by a null vector B_2^* and a spacelike vector B_1^* .

Theorem 6. *Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be the generalized timelike Mannheim curve with curvatures k_1, k_2, k_3 and $\alpha^* : I^* \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be the generalized Mannheim mate curve of α such that principal normal N lies in the lightlike plane spanned by a null vector B_1^* and a spacelike vector B_2^* . Then the following conditions hold:*

(i) α is a timelike curve with $k_1 \neq 0, k_2 = 0, k_3 \neq 0$ and α^* is a Cartan null curve with $k_1^* = 1$ such that the curvatures of α and α^* satisfy the relations

$$(3.80) \quad (\lambda')^2 = (1 + \lambda k_1)^2$$

$$(3.81) \quad k_2^* = \frac{\lambda''' + k_1' + 3\lambda'k_1^2 + 3\lambda k_1 k_1' - a f'''}{a(f')^3},$$

and

$$(3.82) \quad k_3^* = \frac{a[6(f')^2 f'' k_2^* + (f')^3 (k_2^*)' + f^{(4)}] - k_1'' - \lambda^{(4)} - 6\lambda'' k_1^2 - 12k_1 k_1' \lambda'}{b(f')^4} - \frac{4\lambda k_1 k_1'' + 3\lambda(k_1')^2 + k_1^3 + \lambda k_1^4}{b(f')^4}.$$

where

$$\begin{aligned} a &= e^{[\pm(\int k_1(s)ds) + t_2]}, \quad t_2 \in \mathbb{R}, \\ b &= \pm 1, \\ f' &= e^{\left[\left(\int \frac{k_1 + \lambda k_1^2 + \lambda''}{\lambda'} ds\right) + t_3\right]}, \quad t_3 \in \mathbb{R}. \end{aligned}$$

(ii) α is a timelike curve with non-zero curvature k_1, k_2, k_3 and its Mannheim mate α^* is a Cartan null curve with $k_1^* = 1$ such that the curvatures of α and α^* satisfy the relations

$$(3.83) \quad a f'' = \lambda'' + k_1 + \lambda(k_1^2 - k_2^2)$$

$$(3.84) \quad k_2^* = \frac{\lambda''' + k_1' + 3\lambda'(k_1^2 - k_2^2) + 3\lambda(k_1 k_1' - k_2 k_2') - a f'''}{a(f')^3}$$

and

$$(3.85) \quad k_3^* = \frac{a[6(f')^2 f'' k_2^* + (f')^3 (k_2^*)' + f^{(4)}] - \lambda^{(4)} - k_1''}{b(f')^4} - \frac{12\lambda'(k_1' k_1 - k_2' k_2) + 4\lambda(k_1 k_1'' - k_2 k_2'')}{b(f')^4} - \frac{5\lambda''(k_1^2 - k_2^2) + 3\lambda((k_1')^2 - (k_2')^2) + \lambda k_2^2 k_3^2}{b(f')^4} - \frac{(\lambda'' + k_1 + \lambda(k_1^2 - k_2^2))(k_1^2 - k_2^2)}{b(f')^4}$$

where

$$\begin{aligned} a &= e^{\left[-\left(\int \left(\frac{k_1 + \lambda(k_1^2 - k_2^2)}{\lambda'}\right) ds\right) + t_2\right]}, \quad t_4 \in \mathbb{R}, \\ b &= \pm 1, \\ f' &= e^{\left[\left(\int \frac{k_1 + \lambda(k_1^2 - k_2^2) + \lambda''}{\lambda'} ds\right) + t_3\right]}, \quad t_5 \in \mathbb{R}. \end{aligned}$$

Proof. By assumption, the principal normal line of α lies in the lightlike plane spanned by a null vector B_1^* and a spacelike vector B_2^* . Therefore, α^* is a null Cartan curve whose the frame satisfies (2.2). The curve α^* has parametrization of the form

$$(3.86) \quad \alpha^*(f(s)) = \alpha(s) + \lambda(s)N(s).$$

where s is arclength parameter of α , $s^* = f(s) = \int_0^s \|\alpha^{*'}(t)\| dt$ is the arc-length parameter of α^* , $f : I \subset \mathbb{R} \rightarrow I^* \subset \mathbb{R}$ and λ are some smooth functions. We distinguish two cases: (C.1) $k_2 = 0$ and (C.2) $k_2 \neq 0$.

Case (C.1) $k_2 = 0$

Differentiating the relation (3.86) with respect to s and applying (2.1), we find

$$(3.87) \quad T^* f' = (1 + \lambda k_1)T + \lambda' N.$$

By taking the scalar product of (3.87) with $N = aB_1^* + bB_2^*$, we get

$$(3.88) \quad a f' = \lambda'.$$

From (3.87), $g(T^* f', T^* f') = 0 = -(1 + \lambda k_1)^2 + (\lambda')^2$. Then

$$(3.89) \quad (\lambda')^2 = (1 + \lambda k_1)^2$$

Differentiating (3.87) with respect to s and applying (2.1), we have

$$(3.90) \quad T^* f'' + (f')^2 N^* = \left((1 + \lambda k_1)' + \lambda' k_1\right) T + (k_1 + \lambda k_1^2 + \lambda'') N.$$

By taking the scalar product of (3.90) with $N = aB_1^* + bB_2^*$, we obtain

$$(3.91) \quad af'' = \lambda'' + \lambda k_1^2 + k_1$$

Using (3.88) and (3.91), we find

$$(3.92) \quad \begin{aligned} f' &= e^{\left[\left(\int \frac{k_1 + \lambda k_1^2 + \lambda''}{\lambda'} ds\right) + t_3\right]}, t_3 \in \mathbb{R}, \\ a &= e^{[\pm(\int k_1(s) ds) + t_2]}, t_2 \in \mathbb{R}. \end{aligned}$$

Differentiating (3.90) with respect to s and taking the scalar product with $N = aB_1^* + bB_2^*$, we get

$$(3.93) \quad k_2^* = \frac{\lambda''' + k_1' + 3\lambda'k_1^2 + 3\lambda k_1 k_1' - af'''}{a(f')^3}.$$

Differentiating (3.90) two times with respect to s and taking the scalar product with $N = aB_1^* + bB_2^*$, we have

$$(3.94) \quad k_3^* = \frac{a[6(f')^2 f'' k_2^* + (f')^3 (k_2^*)' + f^{(4)}] - k_1'' - \lambda^{(4)} - 6\lambda'' k_1^2 - 12k_1 k_1' \lambda'}{(f')^4} - \frac{4\lambda k_1 k_1'' + 3\lambda(k_1')^2 + k_1^3 + \lambda k_1^4}{(f')^4}.$$

This completes the proof of (i).

Case (C.2) $k_2 \neq 0$

Differentiating the relation (3.86) with respect to s and applying (2.1), we find

$$(3.95) \quad T^* f' = (1 + \lambda k_1)T + \lambda' N + \lambda k_2 B_1.$$

By taking the scalar product of (3.55) with $N = aB_1^* + bB_2^*$, we have

$$(3.96) \quad af' = \lambda'.$$

From (3.96), we get

$$(3.97) \quad g(T^* f', T^* f') = 0 = -(1 + \lambda k_1)^2 + (\lambda')^2 + (\lambda k_2)^2$$

Differentiating the relation (3.95) with respect to s and applying (2.1), we find

$$(3.98) \quad \begin{aligned} T^* f'' + f' k_1^* N^* &= \left((1 + \lambda k_1)' + \lambda' k_1\right) T + (\lambda'' + k_1 + \lambda k_1^2 - \lambda k_2^2) N \\ &\quad (2\lambda' k_2 + \lambda k_2') B_1 + \lambda k_2 k_3 B_2. \end{aligned}$$

By taking the scalar product of (3.98) with $N = aB_1^* + bB_2^*$, we have

$$(3.99) \quad af'' = \lambda'' + k_1 + \lambda(k_1^2 - k_2^2)$$

Using (3.96) and (3.99), we obtain

$$\begin{aligned} a &= e^{\left[-\left(\int \left(\frac{k_1 + \lambda(k_1^2 - k_2^2)}{\lambda'}\right) ds\right) + t_2\right]}, \quad t_2 \in \mathbb{R} \\ f' &= e^{\left[\left(\int \frac{k_1 + \lambda(k_1^2 - k_2^2) + \lambda''}{\lambda'} ds\right) + t_3\right]}, \quad t_3 \in \mathbb{R} \end{aligned}$$

Differentiating the relation (3.98) with respect to s and taking the scalar product with $N = aB_1^* + bB_2^*$, we find

$$(3.100) \quad k_2^* = \frac{\lambda''' + k_1' + 3\lambda'(k_1^2 - k_2^2) + 3\lambda(k_1k_1' - k_2k_2') - af'''}{a(f')^3}.$$

Differentiating the relation (3.61) two times with respect to s and taking the scalar product with $N = aB_1^* + bB_2^*$, we obtain

$$\begin{aligned} k_3^* &= \frac{a[6(f')^2 f'' k_2^* + (f')^3 (k_2^*)' + f^{(4)}] - \lambda^{(4)} - k_1''}{(f')^4} \\ &\quad - \frac{12\lambda'(k_1' k_1 - k_2' k_2) + 4\lambda(k_1 k_1'' - k_2 k_2'')}{(f')^4} \\ &\quad - \frac{5\lambda''(k_1^2 - k_2^2) + 3\lambda((k_1')^2 - (k_2')^2) + \lambda k_2^2 k_3^2}{(f')^4} \\ &\quad - \frac{(\lambda'' + k_1 + \lambda(k_1^2 - k_2^2))(k_1^2 - k_2^2)}{(f')^4} \end{aligned}$$

Also, $g(N, N) = 1 = b^2$. Therefore, the proof is complete. $\square$

Theorem 7. Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be the generalized timelike Mannheim curve with curvatures k_1, k_2, k_3 and $\alpha^* : I^* \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be the generalized Mannheim mate curve of α such that principal normal N lies in the lightlike plane spanned by a spacelike vector B_1^* and a null vector B_2^* . Then α^* is a pseudo null Frenet curve such that the curvatures of α and α^* satisfy the relations

$$-(1 + \lambda k_1)^2 + (\lambda k_2)^2 > 0$$

$$(3.101) \quad k_2^* = \frac{k_1' + 3\lambda(k_1 k_1' - k_2 k_2') - 3f' f'' b}{a(f')^3}$$

$$(3.102) \quad k_3^* = \frac{k_1'' + 4\lambda(k_1 k_1'' - k_2 k_2'') + \lambda(k_1^4 + k_2^4)}{b(f')^4 k_2^*}$$

$$(3.103) \quad + \frac{3\lambda[(k_1')^2 - (k_2')^2] - 2\lambda k_1^2 k_2^2}{b(f')^4 k_2^*}$$

$$(3.104) \quad + \frac{\lambda k_2^2 k_3^2 - k_1 k_2^2 + k_1^3 - b(3(f'')^2 + 4f' f''')}{b(f')^4 k_2^*}$$

$$(3.105) \quad - \frac{a[6(f')^2 f'' k_2^* + (f')^3 (k_2^*)']}{b(f')^4 k_2^*}$$

where

$$\begin{aligned} a^2 &= 1 \\ b &= \frac{k_1 + \lambda(k_1^2 - k_2^2)}{(\lambda k_2)^2 - (1 + \lambda k_1)^2} \end{aligned}$$

Proof. We omit the proof of the theorem because it is so similar to other proofs. $\square$

Corollary 8. *There exists no generalized timelike Mannheim helix such that its Mannheim mate is a pseudo null curve.*

Proof. Assume that there exists a generalized timelike Mannheim helix such that its Mannheim mate is a pseudo null curve. Let α be a generalized timelike Mannheim helix and its Mannheim mate curve α^* be a pseudo null curve satisfying (2.3). In particular, the curve α^* can be parameterized by

$$(3.106) \quad \alpha^*(f(s)) = \alpha(s) + \lambda(s)N(s).$$

Differentiating the relation (3.106) with respect to s and using the relation (2.1), we get

$$(3.107) \quad T^*f' = (1 + \lambda k_1)T + \lambda'N + \lambda k_2 B_1.$$

By taking the scalar product of (3.107) with $N = aB_1^* + bB_2^*$, we have

$$(3.108) \quad \lambda' = 0.$$

Substituting (3.108) in (3.107), we find

$$(3.109) \quad T^*f' = (1 + \lambda k_1)T + \lambda k_2 B_1.$$

By taking the scalar product of (3.109) with itself, we have

$$(3.110) \quad (f')^2 = -(1 + \lambda k_1)^2 + (\lambda k_2)^2 = \text{constant}$$

Differentiating the relation (3.110) with respect to s and using the relation (2.1), we get

$$(3.111) \quad (f')^2 N^* = [k_1 + \lambda(k_1^2 - k_2^2)]N + \lambda k_2 k_3 B_2.$$

Differentiating the relation (3.111) with respect to s and using the relation (2.1), we find

$$(3.112) \quad (f')^3 k_2^* B_1^* = k_1 [k_1 + \lambda(k_1^2 - k_2^2)]T + [k_2(k_1 + \lambda(k_1^2 - k_2^2)) - \lambda k_2 k_3^2]B_1$$

By taking the scalar product of (3.112) with $N = aB_1^* + bB_2^*$, we have $k_2^* = 0$. Then, from (3.112), we see that $k_2 = 0$ or $k_3 = 0$ which is a contradiction. Thus, there exists no generalized timelike Mannheim helix such that its Mannheim mate is a pseudo null curve. $\square$

REFERENCES

- [1] M. Akyiğit, S. Ersoy, I. Özgür, M. Tosun, Generalized Timelike Mannheim curves in Minkowski space-time. Mathematical Problems in Engineering, Vol. 2011, Article ID 539378, 19 pages.
- [2] W. B. Bonnor, Null curves in a Minkowski space-time, Tensor 20, 229-242 (1969).
- [3] W. B. Bonnor, Curves with null normals in Minkowski space-time, A random walk in relativity and cosmology, Wiley Easten Limited, 33-47 (1985).
- [4] J. H. Choi, T. H. Kang, Y. H. Kim, Mannheim curves in 3-dimensional space forms, Bull. Korean Math. Soc., Vol. 50, Number 4, 1099-1108 (2013).
- [5] L. P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces (Dover Edition), Dover Publication, 1960.
- [6] S. Ersoy, M. Tosun, H. Matsuda, Generalized Mannheim curves in Minkowski space-time $\mathbb{E}_1^4$, Hokkaido Mathematical Journal, Vol. 41, Number 3, 441-461 (2012).
- [7] M. Grbović, K. İlarslan, E. Nešović, On null and pseudo null Mannheim curves in Minkowski 3-space, J. Geom., Vol. 105, Number 1, 177-183 (2014).
- [8] C. C. Hsiung, A First Course in Differential Geometry, International press, 1997.
- [9] W. Kuhnel, Differential geometry: curves-surfaces-manifolds, Braunschweig, Wiesbaden, 1999.
- [10] H. Liu, F. Wang, Mannheim partner curves in 3-space, J. Geom., Vol. 88, 120-126 (2008).
- [11] R. Lopez, Differential geometry of curves and surfaces in Lorentz-Minkowski space, Int. Electron. J. Geom., Vol. 7, Number 1, 108-125 (2014).
- [12] G. Loria, Curve Sghembe Speciali Algebriche E Trascendenti, Volume II, Bologna, 1925.
- [13] H. Matsuda, S. Yorozu, Notes on Bertrand curves, Yokohama Math. J., Vol. 50, Number 1-2, 41-58 (2003).
- [14] H. Matsuda, S. Yorozu, On generalized Mannheim curves in Euclidean 4-space, Nihonkai Math. J., Vol. 20, 33-56 (2009).
- [15] J. Miller, Note on Tortuous Curves, Proceedings of the Edinburgh Mathematical Society, Vol. 24, 51-55 (1905).
- [16] E. Nešović, K. İlarslan, M. Grbović, On Generalized Null Mannheim curves in Minkowski space-time, preprint, 2014.
- [17] B. O'Neill, Semi-Riemannian geometry with applications to relativity, Academic Press, New York, 1983.
- [18] K. Orbay, E. Kasap, On Mannheim partner curves in $\mathbb{E}^3$, International Journal of Physical Sciences, Number 4(5), 261-264 (2009).

- [19] T. Otsuki, Differential Geometry (Japanese), Asakura Shoten, Tokyo, 1961.
- [20] O. Tigano, Sulla determinazione delle curve di Mannheim, *Matematiche Catania*, Vol. 3, 25-29 (1948).
- [21] J. Walrave, Curves and surfaces in Minkowski space, Doctoral thesis, Faculty of Science, Leuven, 1995.