

MEAN EQUICONTINUITY, SENSITIVITY, EXPANSIVENESS AND DISTALITY OF PRODUCT DYNAMICAL SYSTEMS

KHUNDRAKPAM BINOD MANGANG
DEPARTMENT OF MATHEMATICS,
MANIPUR UNIVERSITY,
CANCHIPUR, IMPHAL-795003
E-MAIL: KHUBIM@GMAIL.COM

(Received: 1 September 2014, Accepted: 27 January 2015)

ABSTRACT. The mean equicontinuity, mean sensitivity, mean expansiveness and mean distality of the product dynamical system $(\prod X_i, \prod f_i)$ have been studied when each of the constituent dynamical system (X_i, f_i) has these properties and vice versa.

AMS Classification: 54H20, 37B05, 58F08, 26A18

Keywords: Mean Equicontinuity; Mean Sensitivity; Mean Expansiveness

1. INTRODUCTION

In this paper we study the mean equicontinuity, mean sensitivity and mean expansiveness of the product dynamical system $(\prod X_i, \prod f_i)$ when each dynamical system (X_i, f_i) has these properties and vice versa. In [5], there is detailed study of equicontinuity, sensitivity, distality, expansiveness, pseudo orbit tracing property (POTP) of the product dynamical system $(\prod X_i, \prod f_i)$ when each dynamical system (X_i, f_i) has these properties and vice versa. In this set up, we want to investigate the mean equicontinuity, mean sensitivity, mean expansiveness and mean distality of product dynamical systems when each individual dynamical system has these properties and vice versa.

A dynamical system is a pair (X, f) in which X is a compact metrizable space with metric d and f is continuous self map on X . A dynamical system (X, f) is said to be equicontinuous if for each $x \in X$ and for each $\epsilon > 0$, there exists $\delta > 0$ such that $d(f^j(y), f^j(x)) < \epsilon$ for all $j \geq 0$ and for all y such that $d(x, y) < \delta$. A dynamical system (X, f) is said to be sensitive if there exists $\epsilon > 0$ such that for all $x \in X$ and for all $\delta > 0$, there exists $y \in B_d(x, \delta)$ such that $d(f^j(y), f^j(x)) \geq \epsilon$ for some $j \geq 0$. A dynamical system (X, f) is said to be expansive if there exists $\delta > 0$ such that for any two distinct points $x, y \in X$ there exists some integer $j \geq 0$ such that $d(f^j(x), f^j(y)) \geq \delta$. A dynamical system (X, f) is said to be distal if for $x, y \in X$, $\inf\{d(f^j(x), f^j(y)) : j \in \mathbb{N} \cup \{0\}\} = 0$, then $x = y$. In [1],[3], [2], one can find the detail discussion of equicontinuity, expansiveness, sensitivity, distality. In this paper we use the notation $\mathcal{N}$ for the set of positive integers.

Let $(X_i, f_i), i = 1, 2, \dots, n$ be a finite sequence of dynamical systems with respective metrics d_i . Let $X = \prod_{i=1}^n X_i$, $F = \prod_{i=1}^n f_i$. Then F is a self map on X defined by $F(x) = f_1 \times f_2 \times \dots \times f_n(x_1, x_2, \dots, x_n) = (f_1(x_1), f_2(x_2), \dots, f_n(x_n))$ for all $x = (x_1, x_2, \dots, x_n) \in X$. (X, d) is the product space of

$(X_i, d_i), 1 \leq i \leq n$ and the corresponding product metric d is defined by $d(x, y) = d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \sum_{i=1}^n d_i(x_i, y_i)$. It is well known that (X, d) is compact metric space and F is continuous if and only if each f_i is continuous. Therefore (X, F) is a dynamical system and it is called finite product dynamical system.

We use the notation $F^m(x_1, x_2, \dots, x_n) = (f_1^m(x_1), f_2^m(x_2), \dots, f_n^m(x_n))$. The open ball of radius ϵ and center $(x_1, x_2, \dots, x_n)$ in the product metric space (X, d) is defined by

$$B_d((x_1, x_2, \dots, x_n), \epsilon) = B_{d_1}(x_1, \epsilon/n) \times B_{d_2}(x_2, \epsilon/n) \times \dots \times B_{d_n}(x_n, \epsilon/n),$$

where $B_{d_i}(x_i, \epsilon/n)$, for each i , is an open ball of radius $\frac{\epsilon}{n}$ and center x_i in the space (X_i, d_i) . If $(X_i, f_i)_{i=1}^\infty$ is an infinite sequence of dynamical systems, we put $X = \prod_{i=1}^\infty X_i$, $F = \prod_{i=1}^\infty f_i$. F is a self map on X defined by $F(x) = f_1 \times f_2 \times \dots(x_1, x_2, \dots) = (f_1(x_1), f_2(x_2), \dots)$ for all $x = (x_1, x_2, \dots) \in X$. The corresponding countable product metric d is defined by

$d(x, y) = d((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sum_{i=1}^\infty \frac{d_i(x_i, y_i)}{2^i}$. Without lost of generality we assume that $d_i(s, t) \leq 1$ for all $s, t \in X_i$ and for each positive integer i . It is well known that (X, d) is a compact metric space and the metric topology T_d is identical

with the product topology T on X . It is well known that F is continuous if and only if each f_i is continuous and therefore (X, F) is a dynamical system. We call this dynamical system a countable product dynamical system.

A dynamical system (Y, g) is said to be a factor of the dynamical system (X, f) if there exists a continuous surjection $h : X \rightarrow Y$ such that $h \circ f = g \circ h$. The dynamical system (X, f) is a factor of its product dynamical system $(X \times X, f \times f)$ in which the factor map $h : X \times X \rightarrow X$ is the projection map $h(x, y) = x$. If the factor map h is a homeomorphism then we say that the two dynamical system (X, f) and (Y, g) are conjugate to each other and the map h is a conjugacy. Then it is well known that the two dynamical systems (X, f) and (Y, g) have the same dynamical properties [3].

A dynamical system (X, f) is said to be mean equicontinuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$,

$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i x, f^i y) < \epsilon$. In [4], there is a detailed study of mean

equicontinuity. They have investigated the existing results of an equicontinuous dynamical system when the equicontinuity is replaced by mean equicontinuity. The concept of mean distality has been introduced in [6]. In this paper we investigate the dynamical properties such as mean equicontinuity, mean sensitivity, mean distality, mean expansiveness of a finite as well as countable product dynamical system (X, F) vis a vis the dynamical systems (X_i, f_i) for each i . We use the notation (X, F) both for finite product and countable product. For finite product we take $X = \prod_{i=1}^n X_i$, $F = \prod_{i=1}^n f_i$ and the corresponding metric d is given by

$d(x, y) = d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \sum_{i=1}^n d_i(x_i, y_i)$. For countable product,

we take $X = \prod_{i=1}^{\infty} X_i$, $F = \prod_{i=1}^{\infty} f_i$. The corresponding metric d is given by

$d(x, y) = d((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$.

2. DEFINITIONS AND THEOREMS

The countable product dynamical system (X, F) is equicontinuous if and only if (X_i, f_i) is equicontinuous for each positive integer i [5]. We investigate the similar result for mean equicontinuity. The following is a result for mean equicontinuity.

Theorem 2.1. *The countable product dynamical system (X, F) is mean equicontinuous if and only if (X_i, f_i) is mean equicontinuous for each positive integer i .*

Proof. Suppose (X, F) is mean equicontinuous. For each k , let $x_k \in X_k$ be a point and $\epsilon > 0$. Let $x = (x_1, x_2, \dots)$, then by the mean equicontinuity of (X, F) , there exists $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(F^i x, F^i y) < \frac{\epsilon}{2^k},$$

$$\text{i.e. } \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=1}^{\infty} \frac{d_j(f_j^i x_j, f_j^i y_j)}{2^j} < \frac{\epsilon}{2^k}.$$

$$\text{This implies that } \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \frac{d_k(f_k^i(x_k), f_k^i(y_k))}{2^k} < \frac{\epsilon}{2^k}$$

$\text{i.e. } \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d_k(f_k^i(x_k), f_k^i(y_k)) < \epsilon$. This shows that (X_k, f_k) is mean equicontinuous.

Conversely suppose for each k , (X_k, f_k) is mean equicontinuous. Let $\epsilon > 0$, then for each k there exists $\delta_k > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta_k$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d_k(f_k^i(y_k), f_k^i(x_k)) < \frac{\epsilon}{2^k}.$$

It follows that $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(F^i(y), F^i(x)) < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$. Hence (X, F) is mean equicontinuous. $\square$

Definition 2.2. A dynamical systems (X, f) is said to be mean sensitive if there exists $\delta > 0$ such that for every $x \in X$ and every $\epsilon > 0$ there exists $y \in X$ with $d(x, y) < \epsilon$ satisfying $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i x, f^i y) > \delta$.

As in [5], we prove that product dynamical system is mean sensitive if each of the constituent dynamical system is mean sensitive.

Theorem 2.3. The countable product dynamical system (X, F) is mean sensitive if (X_i, f_i) is mean sensitive for each positive integer i .

Proof. Suppose (X_k, f_k) for each k , is mean sensitive with sensitive constant δ_k .

Let $x_k \in X_k$ and $\epsilon > 0$. Then there exists y_k with $d_k(y_k, x_k) < \epsilon$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d_k(f_k^i x_k, f_k^i y_k) > \delta_k.$$

It follows that $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \sum_{k=1}^{\infty} \frac{d_k(f_k^i x_k, f_k^i y_k)}{2^k} > \sum_{k=1}^{\infty} \frac{\delta_k}{2^k} = \delta$, where $\delta = \inf\{\delta_i : i \in \mathcal{N}\}$. It follows that $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(F^i(x), F^i(y)) > \delta$ for some $y \in X$ with $d(x, y) < \epsilon$. Hence (X, F) is mean sensitive with sensitive constant δ . $\square$

The result is true for finite product dynamical system so we have the following corollary and the proof is similar to the above theorem.

Corollary 2.4. The finite product dynamical system (X, F) is mean sensitive if (X_i, f_i) is mean sensitive for each positive integer $i \leq n$

The converse, both for finite as well as infinite product, is not true. The following is a counter example.

Example 2.5. Let (X_1, f_1) be a mean sensitive dynamical system with sensitive constant $\epsilon > 0$. $f_i : X_i \rightarrow X_i$ for $i \neq 1$ are constant functions. Then each of them is not a mean sensitive function whereas $F : X \rightarrow X$ is mean sensitive with sensitive constant 2ϵ .

The converse of the above theorem is still an open problem.

Definition 2.6. A dynamical systems (X, f) is said to be mean expansive if there exists $\delta > 0$ such that for any two distinct points $x, y \in X$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i x, f^i y) \geq \delta.$$

The product dynamical system (X, F) is expansive if and only if (X_i, f_i) is expansive for each positive integer i [5]. We prove the similar result for mean expansiveness.

Theorem 2.7. The product dynamical system (X, F) is mean expansive if and only if (X_i, f_i) is mean expansive for each positive integer i .

Proof. Suppose (X, F) is mean expansive with expansive constant δ . For each positive integer k , let x_k, y_k be two distinct points of X_k . Let $z_j \in X_j$, $j \neq k$ and put $x = (z_1, z_2, \dots, z_{k-1}, x_k, z_{k+1}, \dots), y = (z_1, z_2, \dots, z_{k-1}, y_k, z_{k+1}, \dots)$. By the mean expansiveness of (X, F) we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(F^i x, F^i y) \geq \delta.$$
 This implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \frac{d_k(f_k^i x_k, f_k^i y_k)}{2^k} \geq \delta.$$
 This shows that (X_k, f_k) is mean expansive with expansive constant $2^k \delta$.

Conversely suppose for each positive integer k , (X_k, f_k) is mean expansive with expansive constant δ_k . Let $x = (x_1, x_2, \dots, x_k, \dots), y = (y_1, y_2, \dots, y_k, \dots)$ be two distinct points. Now by the mean expansiveness of (X_k, f_k) , we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d_k(f_k^i x_k, f_k^i y_k) \geq \delta_k$$
 and it implies

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \frac{d_k(f_k^i x_k, f_k^i y_k)}{2^k} \geq \frac{\delta_k}{2^k}.$$

Therefore
$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \sum_{k=1}^{\infty} \frac{d_k(f_k^i x_k, f_k^i y_k)}{2^k} \geq \sum_{k=1}^{\infty} \frac{\delta_k}{2^k} \geq \delta,$$
 where $\delta \in \{\delta_1, \delta_2, \dots\}$. It follows that
$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(F^i(x), F^i(y)) \geq \delta.$$
 Hence (X, F) is mean expansive. $\square$

The result is true for finite product dynamical systems and it can be shown in the similar line as in theorem 1.7. We have the following corollary.

Corollary 2.8. *The finite product dynamical system (X, F) is mean expansive if and only if (X_i, f_i) is mean expansive for each positive integer $i \leq n$.*

Definition 2.9. *A dynamical system (X, f) is said to be mean distal if for each pair of distinct points $x, y \in X$, we have $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(x), f^i(y)) > 0$*

The product dynamical system (X, F) is distal if and only if (X_i, f_i) is distal for each positive integer i [5]. The following is a result for mean distality.

Theorem 2.10. *The product dynamical system (X, F) is mean distal if and only if (X_i, f_i) is mean distal for each $i \in \mathcal{N}$.*

Proof. Suppose (X, F) is mean distal. Let k be a positive integer and $x_k, y_k \in X_k$ be any two point. Let $z_j \in X_j$ where $j \neq k$ then $x = (z_1, z_2, \dots, z_{k-1}, x_k, z_{k+1}, \dots), y = (z_1, z_2, \dots, z_{k-1}, y_k, z_{k+1}, \dots)$ are two distinct points of X . By the mean distality of (X, F) , we have $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(F^i(x), F^i(y)) > 0$.

It follows that $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f_k^i(x_k), f_k^i(y_k)) > 0$. Hence (X_k, f_k) is mean distal.

Conversely suppose (X_k, f_k) is mean distal for each positive integer k .

Let $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots)$ be two points such that $x_k \neq y_k$, then by mean distality of (X_k, f_k) , we have $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f_k^i(x_k), f_k^i(y_k)) > 0$. It follows that $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(F_k^i(x_k), F_k^i(y_k)) > 0$. Hence (X, F) is mean distal. $\square$

The result holds true for finite product dynamical system and it can be shown in the similar line as in above theorem. We have the following corollary.

Corollary 2.11. *The product dynamical system (X, F) is mean distal if and only if (X_i, f_i) is mean distal for each $i \leq n$.*

REFERENCES

- [1] M. Brin and Garrett Stuck, Introduction to Dynamical Systems, Cambridge University Press, 2002.
- [2] M. Denker, C. Grillenberger and K. Sigmund, Ergodic theory on compact spaces, Lecture Notes in Math., 527, Springer, 1977.
- [3] P. Kurka, Topological and Symbolic Dynamics, Societe Mathematique de France, 2003.

- [4] J. Li, Siming Tu and Ziangdong Ye, Mean equicontinuity and mean sensitivity, *Ergodic Theory and Dynamical Systems* doi:10.1017/etds.2014.41, (2014), 1-26.
- [5] K. B. Mangang, Product Dynamical Systems, *Far East Journal of Dynamical Systems*, 24, 1-2, (2014), 1-13.
- [6] D. Ornstein and B. Weiss, Mean distality and tightness, *Proc. Steklov Inst.* 244, 1, (2004), 36-51.