

LYAPUNOV INEQUALITY FOR DYNAMIC EQUATION WITH ORDER $n + 1$ ON TIME SCALES

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ABSTRACT. In this paper, we establish a Lyapunov-type inequality for the higher order dynamic equation

$$S_n^\Delta(t, x(t)) + p(t)x^\alpha(t) = 0$$

on an arbitrary time scale $\mathbb{T}$, where n is a positive integer, $\alpha \geq 1$ is the quotient of two odd positive integers and

$$S_k(t, x(t)) = \begin{cases} x(t), & \text{if } k = 0, \\ a_k(t)S_{k-1}^\Delta(t, x(t)), & \text{if } 1 \leq k \leq n-1, \\ a_n(t)[S_{n-1}^\Delta(t, x(t))]^\alpha, & \text{if } k = n, \end{cases}$$

with $a_k \in C_{rd}(\mathbb{T}, (0, \infty))$ ($1 \leq k \leq n$) and $p \in C_{rd}(\mathbb{T}, \mathbb{R})$.

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1. INTRODUCTION

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Thus, the set $\mathbb{R}$ of the real numbers, the set $\mathbb{Z}$ of the integers and the set $\mathbb{N}$ of the natural numbers are examples of time scales.

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in [7] in order to unify continuous and discrete analysis. The cases when a time scale $\mathbb{T}$ is equal to $\mathbb{R}$ or $\mathbb{Z}$ represent the classical theories of differential and difference equations respectively. Many other interesting time scales exist, and they give rise to plenty of applications (see [9]). Two book on the subject of time scale by Bohner and Peterson [2,3] summarizes and organizes much of the time scale calculus.

Recently, there has been an increasing interest in studying the dynamic inequalities on time scales, especially the Lyapunov inequalities since the Lyapunov inequalities have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications in the theory of differential and difference equations. There are many literatures, see, for example, [1,4-6,8,10,12-14], which improved and extended the classical Lyapunov inequality for second order differential equation in [11].

Motivated by the above studies, in this paper, we shall consider the following higher order nonlinear dynamic equation

$$S_n^\Delta(t, x(t)) + p(t)x^\alpha(t) = 0, \quad (1.1)$$

where n is a positive integer, $\alpha \geq 1$ is the quotient of two odd positive integers and

$$S_k(t, x(t)) = \begin{cases} x(t), & \text{if } k = 0, \\ a_k(t)S_{k-1}^\Delta(t, x(t)), & \text{if } 1 \leq k \leq n-1, \\ a_n(t)[S_{n-1}^\Delta(t, x(t))]^\alpha, & \text{if } k = n, \end{cases}$$

and $a_k \in C_{rd}(\mathbb{T}, (0, \infty))$ ($1 \leq k \leq n$) and $p \in C_{rd}(\mathbb{T}, \mathbb{R})$.

2. MAIN RESULTS AND PROOFS

Lemma 2.1^[2] Let $a, b \in \mathbb{T}$ and $[a, b]_{\mathbb{T}} \equiv \{x : x \in [a, b]\} \cap \mathbb{T}$. For any functions $f, g \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$, we have

$$\int_a^b |f(t)g(t)|\Delta t \leq \left\{ \int_a^b |f(t)|^p \Delta t \right\}^{\frac{1}{p}} \left\{ \int_a^b |g(t)|^q \Delta t \right\}^{\frac{1}{q}},$$

where $p > 1$, and $q = p/(p-1)$.

Lemma 2.2^[2] Let $a, b \in \mathbb{T}$ and $f, g \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$. Then

$$\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g(t)\Delta t,$$

where $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$.

Lemma 2.3 Assume that $u, v \in [a, b]_{\mathbb{T}}$ with $u < v$, $q \in C_{rd}(\mathbb{T}, (0, \infty))$ and $q(t)(x^\Delta(t))^2$ is integrable on $[u, v]$. Then

$$\int_u^v q(t)(x^\Delta(t))^2 \Delta t \geq \frac{[x(v) - x(u)]^2}{\int_u^v \frac{\Delta(t)}{q(t)}}.$$

Proof. Denote $W = \frac{x(v) - x(u)}{\int_u^v \frac{\Delta(t)}{q(t)}}$, then

$$\int_u^v [\sqrt{q(t)}x^\Delta(t) - \frac{W}{\sqrt{q(t)}}]^2 \Delta t \geq 0.$$

It follows that

$$\int_u^v q(t)(x^\Delta(t))^2 \Delta t - 2W \int_u^v x^\Delta(t) \Delta t + \int_u^v \frac{W^2}{q(t)} \Delta t \geq 0.$$

That is,

$$\begin{aligned} \int_u^v q(t)(x^\Delta(t))^2 \Delta t &\geq 2W \int_u^v x^\Delta(t) \Delta t - \int_u^v \frac{W^2}{q(t)} \Delta t \\ &= 2W \int_u^v x^\Delta(t) \Delta t - W \frac{x(v) - x(u)}{\int_u^v \frac{\Delta(t)}{q(t)}} \int_u^v \frac{1}{q(t)} \Delta t \\ &= W[x(v) - x(u)] \\ &= \frac{[x(v) - x(u)]^2}{\int_u^v \frac{\Delta(t)}{q(t)}}. \end{aligned}$$

The proof is completed.

Now we state and prove our main results.

Theorem 2.1 Let $n = 1$ and $x(t)$ be a solution of (1.1) with $x(a) = x(b) = 0$ and $x(t) \not\equiv 0$ for $t \in [a, b]_{\mathbb{T}}$. Then we have

$$\int_a^b |p(t)| \Delta t \geq \left[\frac{\int_a^b \frac{\Delta t}{a_1(t)}}{f(d)} \right]^{\frac{1+\alpha}{2}} \left[\frac{1}{\int_a^b a_1(t) \Delta t} \right]^{\frac{\alpha-1}{2}}, \quad (2.1)$$

where $f(d) = \max \left\{ \int_a^c \frac{\Delta t}{a_1(t)} \int_c^b \frac{\Delta t}{a_1(t)} : c \in (a, b)_{\mathbb{T}} \right\}$.

Proof. For $n = 1$, (1.1) reduces to the following second order boundary value problem:

$$(a_1(t)(x^\Delta(t))^\alpha)^\Delta + p(t)x^\alpha(t) = 0$$

and

$$x(a) = x(b) = 0.$$

Let M be the maximum of $|x(t)|$ on $[a, b]_{\mathbb{T}}$, we see that there is $c \in (a, b)_{\mathbb{T}}$ such that $|x(c)| = M > 0$.

case 1. If $\alpha > 1$, then

$$\begin{aligned} M^{1+\alpha} \int_a^b |p(t)| \Delta t &= \int_a^b M M^\alpha |p(t)| \Delta t \\ &\geq \int_a^b |x(\sigma(t))| |x^\alpha(t)| |p(t)| \Delta t \\ &\geq \int_a^b x(\sigma(t)) x^\alpha(t) p(t) \Delta t \\ &= - \int_a^b x(\sigma(t)) (a_1(t) (x^\Delta(t))^\alpha)^\Delta \Delta t \\ &= -x(t) (a_1(t) (x^\Delta(t))^\alpha) \Big|_a^b + \int_a^b a_1(t) (x^\Delta(t))^{1+\alpha} \Delta t \quad (\text{Lemma 2.2}) \\ &= \int_a^b a_1(t) (x^\Delta(t))^{1+\alpha} \Delta t. \end{aligned}$$

By Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned} & \left[\int_a^b (a_1(t))^{\frac{2}{1+\alpha}} (x^\Delta(t))^2)^{\frac{1+\alpha}{2}} \Delta t \right]^{\frac{2}{1+\alpha}} \left[\int_a^b [(a_1(t))^{\frac{\alpha-1}{1+\alpha}}]^{\frac{1+\alpha}{\alpha-1}} \Delta t \right]^{\frac{\alpha-1}{1+\alpha}} \\ & \geq \int_a^b a_1(t)^{\frac{2}{1+\alpha}} (x^\Delta(t))^2 (a_1(t))^{\frac{\alpha-1}{1+\alpha}} \Delta t \\ & = \int_a^b a_1(t) (x^\Delta(t))^2 \Delta t \\ & = \int_a^c a_1(t) (x^\Delta(t))^2 \Delta t + \int_c^b a_1(t) (x^\Delta(t))^2 \Delta t \\ & \geq \frac{M^2}{\int_a^c \frac{\Delta(t)}{a_1(t)}} + \frac{M^2}{\int_c^b \frac{\Delta(t)}{a_1(t)}} \\ & = \frac{M^2 \int_a^b \frac{\Delta(t)}{a_1(t)}}{\int_a^c \frac{\Delta(t)}{a_1(t)} \int_c^b \frac{\Delta(t)}{a_1(t)}}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \int_a^b a_1(t)(x^\Delta(t))^{1+\alpha} \Delta t &\geq \left[\frac{M^2 \int_a^b \frac{\Delta(t)}{a_1(t)}}{\int_a^c \frac{\Delta(t)}{a_1(t)} \int_c^b \frac{\Delta(t)}{a_1(t)} (\int_a^b a_1(t) \Delta t)^{\frac{\alpha-1}{\alpha+1}}} \right]^{\frac{1+\alpha}{2}} \\ &= \frac{M^{1+\alpha} [\int_a^b \frac{\Delta(t)}{a_1(t)}]^{\frac{1+\alpha}{2}}}{[\int_a^c \frac{\Delta(t)}{a_1(t)} \int_c^b \frac{\Delta(t)}{a_1(t)}]^{\frac{1+\alpha}{2}} [\int_a^b a_1(t) \Delta t]^{\frac{\alpha-1}{2}}}. \end{aligned}$$

From above, it is easy to prove that

$$\begin{aligned} \int_a^b |p(t)| \Delta t \geq \frac{\int_a^b a_1(t)(x^\Delta(t))^{1+\alpha} \Delta t}{M^{1+\alpha}} &\geq \frac{M^{1+\alpha} [\int_a^b \frac{\Delta(t)}{a_1(t)}]^{\frac{1+\alpha}{2}}}{M^{1+\alpha} [\int_a^c \frac{\Delta(t)}{a_1(t)} \int_c^b \frac{\Delta(t)}{a_1(t)}]^{\frac{1+\alpha}{2}} [\int_a^b a_1(t) \Delta t]^{\frac{\alpha-1}{2}}} \\ &= \left[\frac{\int_a^b \frac{\Delta t}{a_1(t)}}{f(d)} \right]^{\frac{1+\alpha}{2}} \left[\frac{1}{\int_a^b a_1(t) \Delta t} \right]^{\frac{\alpha-1}{2}}. \end{aligned}$$

case 2. If $\alpha = 1$, then from **case 1**, we have

$$\begin{aligned} M^2 \int_a^b |p(t)| \Delta t &\geq \int_a^b a_1(t)(x^\Delta(t))^2 \Delta t \\ &= \int_a^c a_1(t)(x^\Delta(t))^2 \Delta t + \int_c^b a_1(t)(x^\Delta(t))^2 \Delta t \\ &\geq \frac{M^2}{\int_a^c \frac{\Delta(t)}{a_1(t)}} + \frac{M^2}{\int_c^b \frac{\Delta(t)}{a_1(t)}} \\ &\geq \frac{M^2 \int_a^b \frac{\Delta(t)}{a_1(t)}}{f(d)}, \end{aligned}$$

which implies $\int_a^b p(t) \Delta t \geq \frac{\int_a^b \frac{\Delta(t)}{a_1(t)}}{f(d)}$. This completes the proof of Theorem 2.1.

Theorem 2.2 Let $n > 1$ and $x(t)$ be a solution of (1.1) with $S_n(a, x(a)) = \dots = S_2(a, x(a)) = x(a) = x(b) = 0$ and $x(t) \not\equiv 0$ for $t \in [a, b]_{\mathbb{T}}$. Then we have

$$\int_a^b \frac{\Delta t}{a_2(t)} \int_a^t \frac{\Delta \tau_3}{a_3(\tau_3)} \int_a^{\tau_3} \frac{\Delta \tau_4}{a_4(\tau_4)} \dots \int_a^{\tau_{n-1}} \left[\frac{\Delta \tau_n}{a_n(\tau_n)} \int_a^{\tau_n} |p(\tau)| \Delta \tau \right]^{\frac{1}{\alpha}} \geq \frac{\int_a^b \frac{\Delta t}{a_1(t)}}{f(d)}, \quad (2.2)$$

where $f(d) = \max \left\{ \int_a^c \frac{\Delta t}{a_1(t)} \int_c^b \frac{\Delta t}{a_1(t)} : c \in (a, b)_{\mathbb{T}} \right\}$.

Proof. By integrating (1.1) from a to t and taking into account that $S_n(a, x(a)) = \dots = S_2(a, x(a)) = x(a) = x(b) = 0$, one has

$$\int_a^t S_n^\Delta(\tau, x(\tau)) \Delta \tau + \int_a^t p(\tau) x^\alpha(\tau) \Delta \tau = 0.$$

That is

$$S_{n-1}^\Delta(t, x(t)) = -[\frac{1}{a_n(t)} \int_a^t p(\tau) x^\alpha(\tau) \Delta\tau]^\frac{1}{\alpha}, \quad (2.3)$$

Integrating (2.3) from a to t , we similarly have

$$S_{n-1}(t, x(t)) = - \int_a^t [\frac{1}{a_n(\tau)} \int_a^\tau p(s) x^\alpha(s) \Delta s]^\frac{1}{\alpha} \Delta\tau.$$

By induction, it is obvious that

$$(a_1(t)x^\Delta(t))^\Delta + \frac{1}{a_2(t)} \int_a^t \frac{\Delta\tau_3}{a_3(\tau_3)} \int_a^{\tau_3} \frac{\Delta\tau_4}{a_4(\tau_4)} \cdots \int_a^{\tau_{n-1}} [\frac{\Delta\tau_n}{a_n(\tau_n)} \int_a^{\tau_n} p(\tau) x^\alpha(\tau) \Delta\tau]^\frac{1}{\alpha} = 0. \quad (2.4)$$

Then

$$\begin{aligned} & M^2 \int_a^b \frac{\Delta t}{a_2(t)} \int_a^t \frac{\Delta\tau_3}{a_3(\tau_3)} \int_a^{\tau_3} \frac{\Delta\tau_4}{a_4(\tau_4)} \cdots \int_a^{\tau_{n-1}} [\frac{\Delta\tau_n}{a_n(\tau_n)} \int_a^{\tau_n} |p(\tau)| \Delta\tau]^\frac{1}{\alpha} \\ &= \int_a^b M \frac{\Delta t}{a_2(t)} \int_a^t \frac{\Delta\tau_3}{a_3(\tau_3)} \int_a^{\tau_3} \frac{\Delta\tau_4}{a_4(\tau_4)} \cdots \int_a^{\tau_{n-1}} [\frac{\Delta\tau_n}{a_n(\tau_n)} \int_a^{\tau_n} |p(\tau)| M^\alpha \Delta\tau]^\frac{1}{\alpha} \\ &\geq \int_a^b |x(\sigma(t))| \frac{\Delta t}{a_2(t)} \int_a^t \frac{\Delta\tau_3}{a_3(\tau_3)} \int_a^{\tau_3} \frac{\Delta\tau_4}{a_4(\tau_4)} \cdots \int_a^{\tau_{n-1}} [\frac{\Delta\tau_n}{a_n(\tau_n)} \int_a^{\tau_n} |p(\tau)| |x(\tau)|^\alpha \Delta\tau]^\frac{1}{\alpha} \\ &\geq \int_a^b x(\sigma(t)) \frac{\Delta t}{a_2(t)} \int_a^t \frac{\Delta\tau_3}{a_3(\tau_3)} \int_a^{\tau_3} \frac{\Delta\tau_4}{a_4(\tau_4)} \cdots \int_a^{\tau_{n-1}} [\frac{\Delta\tau_n}{a_n(\tau_n)} \int_a^{\tau_n} p(\tau) x(\tau)^\alpha \Delta\tau]^\frac{1}{\alpha} \\ &= - \int_a^b x(\sigma(t)) (a_1(t)x^\Delta(t))^\Delta \Delta t = -x(t)a_1(t)x^\Delta(t)|_a^b + \int_a^b a_1(t)(x^\Delta(t))^2 \Delta t \\ &= \int_a^c a_1(t)(x^\Delta(t))^2 \Delta t + \int_c^b a_1(t)(x^\Delta(t))^2 \Delta t \geq \frac{M^2}{\int_a^c \frac{\Delta t}{a_1(t)}} + \frac{M^2}{\int_c^b \frac{\Delta t}{a_1(t)}} = \frac{M^2 \int_a^b \frac{\Delta t}{a_1(t)}}{\int_a^c \frac{\Delta t}{a_1(t)} \int_c^b \frac{\Delta t}{a_1(t)}}, \end{aligned}$$

where M and c are defined as the proof of Theorem 2.1. From which we obtain

$$\int_a^b \frac{\Delta t}{a_2(t)} \int_a^t \frac{\Delta\tau_3}{a_3(\tau_3)} \int_a^{\tau_3} \frac{\Delta\tau_4}{a_4(\tau_4)} \cdots \int_a^{\tau_{n-1}} [\frac{\Delta\tau_n}{a_n(\tau_n)} \int_a^{\tau_n} |p(\tau)| \Delta\tau]^\frac{1}{\alpha} \geq \frac{\int_a^b \frac{\Delta t}{a_1(t)}}{f(d)}.$$

This completes the proof of Theorem 2.2.

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