

KALUZA-KLEIN TYPE COSMOLOGICAL MODEL WITH Ω -DEPENDENT COSMOLOGICAL CONSTANT AND BIG RIP SINGULARITY

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ABSTRACT. In this paper by using the modified EOS of the form $p = -\rho - f(\rho)$ we investigate the evolution of the scale factor in Kaluza-Klein type cosmological model in which cosmological constant is given by the scalar arisen by the contraction of the scale energy tensor. It is shown that the big rip singularity occurs at finite time $t = t_s$ by violating null, weak, strong energy condition and satisfying the dominant energy condition for $\alpha > 0$. For the general case $f(\rho) = \alpha \rho^{(\beta + \frac{1}{2})}$, it is observed that for $\beta = -\frac{1}{2} > 0$, $R(t)$, $\rho(t)$ and $H(t) \rightarrow \infty$ are reached at a finite time $t_s = \frac{2}{b_0}$, where b_0 is constant. So again take place big rip singularity at a finite time. However, for $\beta < 0$, singularity does not occurs at a finite time in the framework of Kaluza-Klein theory of gravitation.

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1. INTRODUCTION

Today, there are many dark energy candidates in the market. Among them, the famous phantom dark energy [1, 2], is very interesting. Its equation of state (EOS) parameter is smaller than -1 . Although phantom dark energy is consistent with the current observational data [1, 3, 6, 8], it violates all the energy conditions. One of the consequences is that our universe will encounter a singularity at a finite time, namely the so-called Big Rip [2]. At this singularity, the scale factor R , energy density and pressure of our universe are all divergent. In fact, besides the

traditional big bang, big crunch and the big rip, many novel singularities have been considered in the literature, such as sudden singularities, generalized sudden singularities, quiescent singularities, big boost, big brake, big freeze, ω singularities, inaccessible singularities, directional singularities [5, 11]. These singularities arise at the price of violation of one or several energy conditions. As is well known, in [5, 11, 4, 12] the future singularities have been classified into four types, namely

- Type I (Big Rip): $R \rightarrow \infty, \rho \rightarrow \infty, H \rightarrow \infty, |p| \rightarrow \infty$, when $t \rightarrow t_s < \infty$;
- Type II (Sudden singularity): $R \rightarrow R_s, \rho \rightarrow \rho_s, H \rightarrow H_s, |p| \rightarrow \infty, \dot{H} \rightarrow \infty$, when $t \rightarrow t_s < \infty$;
- Type III (Big Freeze) $R \rightarrow R_s, \rho \rightarrow \infty, H \rightarrow \infty, |p| \rightarrow \infty$, when $t \rightarrow t_s < \infty$;
- Type IV (Generalized Sudden singularity): $R \rightarrow R_s, \rho \rightarrow \rho_s, H \rightarrow H_s, |p| \rightarrow p_s, \dot{H} \rightarrow \dot{H}_s$, and higher derivatives of H diverge, when $t \rightarrow t_s < \infty$,

where $t_s, R_s, \rho_s, H_s, p_s, \dot{H}_s$ are all finite constants ($R_s \neq 0$): ρ and p are energy density and pressure respectively; $H \equiv \frac{\dot{R}}{R}$ is the Hubble parameter and a dot denotes a derivative with respect to the cosmic time t . In fact, the above four types can include almost all known future singularities.

As it is well known, the Einstein field equations with a non-zero λ can be rearranged so that their right-hand sides consist of two terms: the stress-energy tensor of the ordinary matter and an additional λ -tensor. A quick look at the λ -tensor $T_{ij}^{(\nu)}$ shows that it consists of two parts: the λ -part, having the dimension of the energy density, which is multiplied by the components of the metric tensor forming its metric part:

$$(1) \quad T_{ij}^{(\nu)} = \left(\frac{\lambda}{8\pi G} \right) g_{ij} = \Lambda g_{ij},$$

where Λ satisfies the requirements: (i) it should have the dimension of energy density, and (ii) it should be invariant under Lorentz transformation. It is also identified with vacuum energy because this quantity satisfies the requirements asked of Λ .

Instead of identified Λ with the vacuum energy Majernik [9] has identified Λ with stress energy scalar $T = T_i^i$ which has the dimension of energy density and is Lorentz Invariant. Majernik [10] assume the cosmological constant Λ of the form

$$(2) \quad \Lambda = \frac{\lambda}{8\pi G} = \kappa T_i^i = \kappa T,$$

or

$$(3) \quad \lambda = 8\pi G \kappa T,$$

where κ is a dimensionless constant to be determined.

Inspired with the earlier work of Huang and Wang [7], in this paper we discuss the evolution of the scale factor with Ω -dependent cosmological constant which is given by the scalar arising from the contraction of the stress energy tensor by using the modified equation of state of the form $p = -\rho - f(\rho)$ in the framework of Kaluza-Klein theory of gravitation. It is also shown that the big rip singularity occurs at a finite time by violating null, weak, strong energy condition, and satisfying the dominant energy condition.

The paper is organized as follows. In section 2, we obtain Einstein's field equations for the Kaluza-Klein type cosmological model. Section 3 deals with the general formalism of the modified equation of state of the form $p = -\rho - f(\rho)$. In section 4, we discuss the evolution of the scale factor for Kaluza-Klein type model with Ω -dependent cosmological constant by using $f(\rho) = \alpha\rho$ in the modified equation of state with additional cosmic term $\Lambda = \kappa T$. It is observe that for $\Omega = \frac{1}{2}$, big rip singularity occurs at a finite time. It is also shows that for $\alpha > 0$ by imposing the standard energy condition on the mathematical fluid, satisfying the dominant energy condition but violating null, weak and strong energy condition. In next subsection 4.1, we again discuss the evolution of the scale factor and it is observe that for the more general case $f(\rho) = \alpha\rho^{\beta+\frac{1}{2}}$, the big rip singularity can occurs when $\beta > 0$ and for $\beta < 0$ we don't have any singularity at a finite time. We summaries the results in the last section.

2. MODEL AND FIELD EQUATIONS

Let us consider the Kaluza-Klein type metric

$$(4) \quad ds^2 = dt^2 - R^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + (1 - Kr^2)d\psi^2 \right],$$

where $R(t)$ is a scale factor, $K = 0, \pm 1$ is the curvature parameter for flat, open and closed universe, respectively..

The Einstein field equations with cosmological constant may be written as

$$(5) \quad R_{ij} - \frac{Rg_{ij}}{2} - \lambda g_{ij} = 8\pi GT_{ij},$$

where T_{ij} is the energy-momentum tensor for the perfect fluid

$$(6) \quad T_{ij} = (\rho + p)u_i u_j - pg_{ij}.$$

The stress-energy tensor of the cosmic medium T_j^i has only five non-zero components $T_0^0 = \rho, T_1^1 = T_2^2 = T_3^3 = T_4^4 = -p$. Therefore,

$$(7) \quad T = T_0^0 + T_1^1 + T_2^2 + T_3^3 + T_4^4 = \rho - 4p.$$

Einstein's field equations with non-zero cosmological term λ for the model (4) with energy momentum tensor (6) can be written as

$$(8) \quad 6\frac{\dot{R}^2}{R^2} + 6\frac{K}{R^2} = 8\pi G\rho + \lambda,$$

and

$$(9) \quad 3\frac{\ddot{R}}{R} + 3\frac{\dot{R}^2}{R^2} + 3\frac{K}{R^2} = -8\pi Gp + \lambda.$$

From the conservation of energy, $T_{;j}^{ij} = 0$, we get

$$(10) \quad \dot{\rho} + 4(\rho + p)H = \frac{-\dot{\lambda}}{8\pi G}.$$

We now impose standard energy conditions on the mathematical fluid. The local energy conditions considered here are the null energy condition (NEC), weak energy condition (WEC), strong energy condition (SEC) and dominant energy condition (DEC). In the present context we are always dealing with a formal perfect fluid and in that case the conditions are given by

$$(11) \quad NEC \Leftrightarrow (\rho + p \geq 0),$$

$$(12) \quad WEC \Leftrightarrow (\rho \geq 0) \text{ and } (\rho + p \geq 0),$$

$$(13) \quad SEC \Leftrightarrow (\rho + 2p \geq 0) \text{ and } (\rho + p \geq 0),$$

and

$$(14) \quad DEC \Leftrightarrow (\rho \geq 0) \text{ and } (\rho \pm p \geq 0).$$

For this energy conditions from the field equations (8) to (9), we find

$$(15) \quad (\rho + 2p) \geq 0 \Leftrightarrow \lambda \geq \frac{6\ddot{R}}{R},$$

similarly

$$(16) \quad \rho + p \geq 0 \Leftrightarrow k + \dot{R}^2 \geq R\ddot{R},$$

$$(17) \quad \rho - p \geq 0 \Leftrightarrow \lambda \leq \frac{9}{2R^2}(k + \dot{R}^2) + \frac{3\ddot{R}}{2R},$$

and

$$(18) \quad \rho \geq 0 \Leftrightarrow \lambda \leq \frac{6}{R^2}(K + \dot{R}^2).$$

3. MODIFIED EQUATION OF STATE

We consider the modified equation of state of the form

$$(19) \quad p = -\rho - f(\rho),$$

where $f(\rho)$ being an arbitrary function.

By using Eq. (19), Eq. (7) becomes $T = 5\rho + 4f(\rho)$.

From Eq. (19) and Eq. (10), with the help of Eq. (3), we obtain

$$(20) \quad \dot{\rho} - 4f(\rho)H = -\kappa\dot{T}.$$

From Eq. (8) for flat universe i.e., ($K = 0$), we get

$$6H^2 = 8\pi G\rho + \lambda,$$

for the value of $\lambda = 8\pi G\kappa T$, we get

$$\begin{aligned} 6H^2 &= 8\pi G\rho + 8\pi G\kappa T, \\ \Rightarrow H &= \sqrt{\left(\frac{8\pi G}{6}\right) [\rho + 5\kappa\rho + 4\kappa f(\rho)]}. \end{aligned}$$

For the above value of H and T , Eq. (20) becomes

$$(21) \quad t = \frac{\sqrt{6}}{4\sqrt{8\pi G}} \int_{\rho_0}^{\rho} \frac{(1+5\kappa)d\rho}{f(\rho)\sqrt{(1+5\kappa)\rho+4\kappa f(\rho)}} + \frac{\sqrt{6}}{4\sqrt{8\pi G}} \int_{f(\rho_0)}^{f(\rho)} \frac{4\kappa d[f(\rho)]}{f(\rho)\sqrt{(1+5\kappa)\rho+4\kappa f(\rho)}}.$$

This is the relation between the cosmological time $t(>0)$ and the density ρ .

4. COSMOLOGICAL MODEL WITH A Ω -DEPENDENT COSMOLOGICAL CONSTANT

Let us take $f(\rho) = \alpha\rho$ then the Eq. (19) becomes

$$(22) \quad p = -(1+\alpha)\rho,$$

where α is a constant.

Huang and Wang [7] discussed the evolution of scale factor in higher dimensional space time for $p = 0$. In our case we discuss the evolution of scale factor in 5D Kaluza-Klein model for $p \neq 0$.

By using Eq. (22), from the field equations (8) and (9) gives

$$(23) \quad 3\frac{\ddot{R}}{R} - 3(1+2\alpha) \left[\frac{\dot{R}^2}{R^2} + \frac{K}{R^2} \right] + \alpha\lambda = 0.$$

This is the evolution equation of the scale factor in five dimensional Kaluza-Klein theory in the presence of λ term.

We determine the values of dimensionless constant κ by assuming the universe is flat i.e., ($\Omega_{tot} = 1$). Since $\Omega = \frac{\rho}{\rho_{crit}} < 1$, we suppose that the remaining energy of cosmological constant required to produce a geometrical flat universe is given by the equation [Majernik (2003)].

From the Eq. (8), we get

$$\Omega + \kappa \Omega(5+4\alpha) = 1,$$

this gives

$$(24) \quad \kappa = \frac{1}{5+4\alpha} \left(\frac{1}{\Omega} - 1 \right).$$

From Eqs. (7) and (22), we can obtain $T = (5+4\alpha)\rho$.

Thus

$$(25) \quad \lambda = 8\pi G \kappa T = \frac{8\pi G}{(5+4\alpha)} \left(\frac{1}{\Omega} - 1 \right) (5+4\alpha)\rho,$$

$$\lambda = 8\pi G(\rho_{crit} - \rho).$$

Inserting Eq. (25) into Eq. (8), we obtain

$$(26) \quad \rho_{crit} = \frac{3}{4\pi G} \frac{\dot{R}^2}{R^2}.$$

Substituting Eqs. (25) and (26) into Eq. (23) immediately yields the equation for the evolution of $R(t)$

$$(27) \quad \ddot{R}(t) - (2\alpha\Omega + 1) \frac{\dot{R}^2(t)}{R(t)} = 0.$$

General solution of the Eq. (27) gives

$$(28) \quad R(t) = C_2 \exp \left(\int \frac{dt}{\int -2\alpha\Omega dt + C_1} \right),$$

where C_1 and C_2 are the integration constants.

To obtain the solution of Eq. (27) we consider $\Omega = \frac{1}{2}$, then we get

$$(29) \quad R(t) = \frac{1}{[\alpha C_1(t_s - t)]^{\frac{1}{\alpha}}},$$

where $t_s = \frac{C_2}{C_1}$.

Hence, the Hubble parameter and energy density in the following form

$$(30) \quad H(t) = \frac{1}{\alpha(t_s - t)},$$

$$(31) \quad \rho(t) = \frac{3}{8\pi G[\alpha(t_s - t)]^2}.$$

Thus, at the finite time $t = t_s$, all the quantities $R(t)$, $H(t)$ and $\rho(t)$ diverges at big rip. For the positive value of α it is observed that big rip singularity occurs at a finite time by violating null, weak, strong energy condition but satisfying dominant energy condition.

4.1. General case. Let us take $f(\rho) = \alpha\rho^{\beta+\frac{1}{2}}$, then the Eq. (19) becomes

$$(32) \quad p = - \left(\rho + \alpha\rho^{\beta+\frac{1}{2}} \right).$$

From Eq. (7) and (32), we get the stress energy tensor

$$(33) \quad T = 5\rho + 4\alpha\rho^{\beta+\frac{1}{2}}.$$

With the help of Eqs. (8) and (33), we get

$$(34) \quad H = \sqrt{\left(\frac{8\pi G}{6} \right) [(1 + 5\kappa)\rho + 4\kappa\alpha\rho^{\beta+\frac{1}{2}}]}.$$

After substituting the value of $f(\rho)$ from Eq. (32) in Eq. (20), we obtain

$$\dot{\rho} - 4\alpha\rho^{\beta+\frac{1}{2}}H = -\kappa\dot{T}.$$

By using Eq. (33) and (34), above equations becomes

$$\left[\frac{[a_0 + a_1\rho^{\beta-\frac{1}{2}}]}{4\alpha\sqrt{\frac{8\pi G}{6}} [a_0\rho + a_2\rho^{\beta+\frac{1}{2}}]^{\frac{1}{2}}\rho^{\beta+\frac{1}{2}}} \right] d\rho = dt,$$

where $a_0 = 1 + 5\kappa$, $a_1 = 4\alpha\kappa(\beta + \frac{1}{2})$, $a_2 = 4\kappa\alpha$.

After integrating, we get

$$(35) \quad t = \frac{\sqrt{6}}{4\alpha\sqrt{8\pi G}} \int_{\rho_0}^{\rho} \frac{[a_0\rho^{-(\beta+\frac{1}{2})} + a_1\rho^{-1}]}{\sqrt{a_0\rho + a_2\rho^{\beta+\frac{1}{2}}}} d\rho.$$

This expression can be integrated generically in terms of an hypergeometric function, the result being

$$(36) \quad t = \frac{\sqrt{6}}{4\alpha\sqrt{8\pi G}} \left(\frac{4\rho^{-\beta}}{a_2(1+2\beta)(6\beta-1)\sqrt{a_0\rho + a_2\rho^{\frac{1}{2}+\beta}}} \right) \times \\ \left[(1-6\beta)a_1(a_0\rho^{\frac{1}{2}} + a_2\rho^{\beta}) - a_0(4\beta a_1 + a_2 + 2\beta a_3) \right] \times \\ \sqrt{\rho \left(1 + \frac{a_0\rho^{\frac{1}{2}-\beta}}{a_2} \right)} F_1 \left[\frac{1-6\beta}{2-4\beta}, \frac{1}{2}, \frac{3-10\beta}{2-4\beta}, \frac{a_0\rho^{\frac{1}{2}-\beta}}{a_2} \right].$$

For $\beta = \frac{1}{2}$, from Eq. (35), we get

$$(37) \quad t = \frac{2}{b_0} \left[\frac{1}{\sqrt{\rho_0}} - \frac{1}{\sqrt{\rho}} \right],$$

where $b_0 = \frac{4\alpha\sqrt{\frac{4\pi G}{3}[1+(5+4\alpha)\kappa]}}{1+\kappa(5+4\alpha)}$.

Thus, if $\alpha > 0$ a finite value of the time t is compatible with a big rip singularity ($\rho = \infty$).

The variation of the density with time is

$$(38) \quad \rho(t) = \rho_0 \left(1 - \frac{1}{2}b_0 t \right)^{-2}.$$

Similarly, the Hubble parameter is

$$(39) \quad H(t) = H_0 \left(1 - \frac{1}{2}b_0 t \right)^{-1},$$

and the scale factor is

$$(40) \quad R(t) = R_0 \left(1 - \frac{1}{2}b_0 t \right)^{-\frac{2(a_0+a_2)}{\alpha}}.$$

When $\beta = \frac{1}{2} > 0$, one obtains that the value $\rho \rightarrow \infty$ is reached at a finite and positive time, namely

$$(41) \quad t_s = \frac{2}{b_0}.$$

Here we note that all the quantities $\rho(t)$, $R(t)$ and $H(t)$ diverge at big rip. However, when $\beta < 0$, no singularity can occurs at finite time.

5. CONCLUSION

In this paper, we considered the modified equation of state of the form $p = -\rho - f(\rho)$ and discussed the evolution of the scale factor for $f(\rho) = \alpha\rho$ and $f(\rho) = \alpha\rho^{\beta+\frac{1}{2}}$ in Kaluza-Klein type cosmological model with the additional cosmic term $\Lambda = \kappa T$. From the evolution Eq. (27), for particular value of $\Omega = \frac{1}{2}$, we obtained the evolution of $R(t)$ which is given in Eq. (29). It is observed that from the solutions (29) to (31), all the quantities $R(t)$, $H(t)$ and $\rho(t)$ diverges at big rip and it is found that relation between energy density ρ and scale factor R of the form $\rho(t) = \frac{3}{8\pi G} R^{2\alpha}$. It is also shown that the big rip singularity occurs at a finite time by violating null, weak, strong energy condition but satisfying dominant energy condition for $(\alpha > 0)$. Again from Eq. (27) for $\Omega = 0$, we obtained the typical inflationary solution and hence in this case no singularity can occurs at finite time.

In general case we have considered $f(\rho) = \alpha\rho^{\beta+\frac{1}{2}}$ and found that for $\beta = \frac{1}{2} > 0$, one obtains that the value $\rho \rightarrow \infty$ is reached at a finite and positive time, namely, $t_s = \frac{2}{b_0}$. So that we are back to the big rip situation as we have discussed in the case $f(\rho) = \alpha\rho$. However, when $\beta < 0$, in which circumstance no singularity can be ever reached in finite time.

The Big Rip singularity problem is more involved and, in order to address and solve it one needs to work harder, it could turn out to be that the problem of time is associated with the avoidance of singularities. Our results hint toward this direction. Another way to deal with the Big Rip singularity problem is loop quantum cosmology. Work along these lines is currently in progress.

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