

DOMAIN THEORY IN GENERAL RELATIVITY

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ABSTRACT. It is proved that k -causal spacetimes, with inner continuous $Int(K^\pm(\cdot))$, are jointly bicontinuous posets whose interval topology is the manifold topology. In particular, using the relation K^+ , causally continuous spacetimes can be reconstructed

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1. INTRODUCTION

It is shown by Martin and Panangaden [5] that it is possible to reconstruct globally hyperbolic spacetimes in a purely order theoretic manner, using the causal relation J^+ . This has the benefit that one can study these spacetimes by domain theory instead of geometry. In this paper the relation K^+ is used instead of J^+ [4]. The relation $K^+ \subseteq M \times M$ is defined as the smallest transitive closed relation which contains I^+ [8, 7, 3]. This definition arose from the fact that the causal relation, J^+ , is transitive but not necessarily closed and $\overline{J^+}$ is closed but not necessarily transitive. The spacetime (M, g) is K -causal if K^+ is antisymmetric. Recently,

it is proved by Minguzzi that stable causality and K -causality coincide [9]. In globally hyperbolic and causally simple spacetimes $K^\pm = J^\pm$ and in causally continuous spacetimes $K^\pm = \overline{J^\pm}$.

We denote with $\text{int}(B)$ and $\overline{B}$, the topological interior and closure of $B \subseteq M$, respectively.

2. PRELIMINARIES

Let us recall some basic definitions in domain theory. A poset is a set together with a reflexive, antisymmetric and transitive relation.

In a poset $(P, \sqsubseteq)$, a nonempty subset $S \subseteq P$ is called directed (filtered) if $(\forall x, y \in S)(\exists z \in S) x, y \sqsubseteq z$ ($(\forall x, y \in S)(\exists z \in S) z \sqsubseteq x, y$). The supremum (infimum) of S is the least of its upper bounds (the greatest of all its lower bounds) provided it exists.

For a subset X of a poset P , set:

$$\uparrow X = \{y \in P : (\exists x \in X) x \sqsubseteq y\}, \downarrow X = \{y \in P : (\exists x \in X) y \sqsubseteq x\}.$$

A dcpo is a poset in which every directed subset has a supremum. The least element in a poset, when it exists, is the unique element $\perp$ with $\perp \sqsubseteq x$ for all x .

Definition 2.1.[1] For elements x, y of a poset, $x \ll y$ if and only if for all directed sets S with a supremum,

$$y \sqsubseteq \bigsqcup S \Rightarrow (\exists s \in S) x \sqsubseteq s.$$

We set $\downarrow x = \{a \in P : a \ll x\}$ and $\uparrow x = \{a \in P : x \ll a\}$.

For symbol “ $\ll$ ”, read “way below”.

Definition 2.2.[1] For elements x, y of a poset, $x \ll_d y$ if and only if for all filtered sets S with an infimum,

$$\bigwedge S \sqsubseteq x \Rightarrow (\exists s \in S) s \sqsubseteq x.$$

We set $\downarrow_d x = \{a \in P : a \ll_d x\}$ and $\uparrow_d x = \{a \in P : x \ll_d a\}$. For symbol “ $\ll_d$ ”, read “way above”.

Definition 2.3.[1] A poset P is (continuous) dual continuous if $(\downarrow x) \uparrow_d x$ is (directed) filtered with (supremum) infimum x for all $x \in P$.

A poset P is bicontinuous if it is both continuous and dual continuous. In addition a poset is called jointly bicontinuous if it is bicontinuous and the way below relation coincides with the way above relation.

Proposition 2.4.[1] If $x \ll y$ in a continuous poset P , then there is $z \in P$ with $x \ll z \ll y$.

Definition 2.5.[1] On a bicontinuous poset P , sets of the form

$$(a, b) := \{x \in P : a \ll x \ll_d b\}$$

form a basis for a topology called the interval topology.

3. CAUSAL STRUCTURE OF A SPACETIME

In this paper we denote with (M, g) a C^r spacetime, $r \in \{2, \dots, \infty\}$, of arbitrary dimension $n \geq 2$ and signature $(-, +, \dots, +)$. I^+ and J^+ are the chronological and causal relations. We recall that if U is an open neighborhood of M , then we denote by J_U^+ the causal relation on the spacetime U with the induced metric. The spacetime M is globally hyperbolic if it is causal and $J^+(x) \cap J^-(y)$ is compact for every $x, y \in M$ [2]. Every event p of a spacetime (M, g) admits arbitrary small globally hyperbolic neighborhoods.

Martin and Panangaden defined an order on the spacetime M in the following manner:

$$p \sqsubseteq q \equiv q \in J^+(p).$$

They proved the following theorem about Globally hyperbolic spacetimes:

Theorem 3.1.[5, 6] If M is a globally hyperbolic spacetime, then $(M, \sqsubseteq)$ is a bicontinuous poset with $I^+ = \ll$ whose interval topology is the manifold topology.

This theorem suggests a formulation of causality independently of geometry.

In this paper, using the relation K^+ , we try to generalize theorem 3.1.

3.1. Inner and outer continuity of K^+ . Let F be a function which assigns to each point $p \in M$ an open set $F(p) \subseteq M$. We say that F is inner continuous (outer continuous) if for any p and any compact set $C \subseteq F(p)$ ($C \subseteq M - \overline{F(p)}$), there exists a neighborhood U of p with $C \subseteq F(q)$ ($C \subseteq M - \overline{F(q)}$), for every $q \in U$.

Remark. 1) $I^\pm(\cdot)$ are inner continuous but not necessarily outer continuous [2]. The spacetime (M, g) is causally continuous if the relations $I^\pm(\cdot)$ are outer continuous.

2) In globally hyperbolic spacetimes, $K^+ = J^+$ and $\text{int}(K^\pm(\cdot)) = I^\pm(\cdot)$ which are inner continuous.

Lemma 3.1.1.[10] In a K -causal spacetime (M, g) , $\text{int}(K^+(\cdot))$ and $\text{int}(K^-(\cdot))$ are inner continuous if and only if for every $p, q \in M$, $p \in \text{int}(K^-(q)) \Leftrightarrow q \in \text{int}(K^+(p))$.

Lemma 3.1.2. $\text{int}(K^+(\cdot))$ and $\text{int}(K^-(\cdot))$ are outer continuous.

3.2. K - Alexandrov topology. The following topology plays an important role in general relativity.

Definition 3.2.1. Let (M, g) be a spacetime. Alexandrov topology on M is the one which admits as a base,

$$B_A = \{I^+(p) \cap I^-(q) : p, q \in M\}.$$

Theorem 3.2.1.[2] For a spacetime (M, g) , the following properties are equivalent:

- (a) (M, g) is strongly causal.
- (b) Alexandrov topology is equal to the original topology on M .

The spacetime M is strongly K -causal at p if it contains arbitrary small K -convex neighborhoods of p , and it is strongly K -causal if it is strongly K -causal for all $p \in M$. K -causality implies strong K -causality [8, 7]. The converse is trivial.

Using the relation K^+ , we define K -Alexandrov topology with the base,

$$B_K = \{\text{int}(K^+(p)) \cap \text{int}(K^-(q)) : p, q \in M\}.$$

Theorem 3.2.2.[4] Let (M, g) be a spacetime, with inner continuous $\text{int}(K^\pm(\cdot))$.

The following are equivalent:

- (a) (M, g) is K -causal.
- (b) K -Alexandrov topology is equal to the original topology on M .

Proof. Assume that (M, g) is K -causal. K -causality implies strong K -causality. Definition of strong K -causality implies that each point has arbitrary small K -convex neighborhoods. Let V be an open neighborhood of p in the manifold topology. There exists a causally K -convex neighborhood $U \subseteq V$ of p , contained in a globally hyperbolic neighborhood. Indeed, $K_U^+(p) = J_U^+(p)$ and $\text{int}(K_U^+(p)) = I_U^+(p)$.

Thus K - Alexandrov topology on $(U, g|_U)$ agrees with Alexandrov topology. Using theorem 3.2.1 and the fact that a strongly K - causal spacetime is strongly causal demonstrate that the manifold topology on U agrees with K - Alexandrov topology. Hence K - Alexandrov topology agrees with the manifold topology.

Conversely, suppose that M is not strongly K - causal at p . There is a neighborhood V of p such that if W is any neighborhood of p with $W \subseteq V$, there exist points $p', q' \in W$, $K^+(p') \cap K^-(q') \not\subseteq W$. Thus all open sets in K - Alexandrov topology contain points outside of V . Indeed, if for $c, c' \in M$, $W = \text{int}(K^+(c)) \cap \text{int}(K^-(c')) \subseteq V$ then by assumption, there is a point $d \in K^+(p') \cap K^-(q')$ that $d \notin W$. But since $\text{int}(K^\pm(\cdot))$ are inner continuous, $d \in W$. This is a contradiction. As a consequence, K - Alexandrov topology is different from the given manifold topology.

4. SPACETIME AND DOMAIN THEORY

Let M be a K - causal spacetime. We write the relation K^+ as:

$$p \sqsubseteq q \equiv (p, q) \in K^+.$$

Lemma 4.1. Let p, q and $r \in M$. Then:

- i) $p \sqsubseteq q$ and $r \in \text{int}(K^+(q)) \Rightarrow r \in \text{int}(K^+(p))$.
- ii) $p \in \text{int}(K^-(q))$ and $q \sqsubseteq r \Rightarrow p \in \text{int}(K^-(r))$.

Lemma 4.2. Let y_n be a sequence in M with $y_n \sqsubseteq y$ ($y_n \sqsupseteq y$) for all n and $\lim_{n \rightarrow \infty} y_n = y$; then $\bigsqcup y_n = y$ ($\bigwedge y_n = y$).

Proof. Let $y_n \sqsubseteq x$ for every $n \in N$. Since K^+ is closed and $y_n \in K^-(x)$, $y = \lim_{n \rightarrow \infty} y_n \in K^-(x)$. Thus $y \sqsubseteq x$ and this proves $y = \bigsqcup y_n$. The proof for the dual part is similar to this.

Note that the above lemma is true for every causal closed relation.

Lemma 4.3.[5] For any $x \in M$, $I^-(x)$ ($I^+(x)$) contains an increasing (decreasing) sequence with supremum (infimum) x .

Lemma 4.4.[7] Let B be an open set with compact boundary $\overline{B}$. Let $(x, z) \in K^+$, if $x \in B$ and $z \notin B$, then there is $y \in \overline{B}$ such that $(x, y) \in K^+$ and $(y, z) \in K^+$.

Lemma 4.5. Let S be a directed set in the K - causal spacetime (M, g) with supremum $\bigsqcup S$. Then there is an increasing sequence $\{s_n\}$ in S such that $\lim_{n \rightarrow \infty} s_n = \bigsqcup S$.

Proof. Let $A = \{\{s_n\} : s_n \in S, s_n \sqsubseteq s_{n+1} \ \forall n \in N\}$. We define an equivalence relation on A in the following manner:

$$\{s_n\} \sim \{s'_n\} \Leftrightarrow \exists m \in N : s_n = s'_n \ \forall n > m.$$

Now we define a partial order on $A/\sim$.

$$[\{s_n\}] \sqsubseteq_1 [\{s'_n\}] \Leftrightarrow \exists m \in N : s_n \sqsubseteq s'_n, \forall n \geq m.$$

Suppose that $\{a_m\}_{m \in N} = \{[\{s_{m,n}\}_{n \in N}] : m \in N\}$ is a chain in $A/\sim$. We show that it has an upper bound. We define the sequence $\{b_m\}$ in the following manner:

$$b_1 = s_{1,n_1} : s_{1,n} \sqsubseteq s_{2,n} \ \forall n > n_1,$$

$$b_i = s_{i,n_i} : s_{i,n} \sqsubseteq s_{i+1,n} \ \forall n > m \text{ and } n_i = \max\{m, n_1, \dots, n_{i-1}\}.$$

It is easy to show that $[\{b_m\}]$ is an upper bound of $\{a_m\}$. Hence by Zorn's lemma $A/\sim$ has a maximum element $c = [\{c_m\}]$. Suppose by contradiction that there is a neighborhood U of $\bigsqcup S$ with compact closure such that $S \cap U = \emptyset$. Let $\{c_m\}$ be a representation of $[\{c_m\}]$. Since $c_m \sqsubseteq \bigsqcup S$, using lemma 4.4, there is $d_m \in \partial U$, such that $c_m \sqsubseteq d_m$ and $d_m \sqsubseteq \bigsqcup S$. $\{d_m\}$ has an accumulation point like d since ∂U is compact. $c_n \sqsubseteq c_{n+1}, \forall n \in N$ and K^+ is closed. Hence $c_i \sqsubseteq d_j, \forall i, j \in N$ and consequently $c_i \sqsubseteq d, \forall i \in N$. But $[\{c_m\}]$ is a maximal element of $A/\sim$ and this implies that d is an upper bound of S which is a contradiction to the fact that $d \sqsubseteq \bigsqcup S$ and $d \neq \bigsqcup S$.

Theorem 4.6. Let M be a K -causal spacetime. Then

$$x \ll y \Leftrightarrow y \in \text{int}(K^+(x)) \Leftrightarrow x \ll_d y.$$

Proof. Let $y \in \text{int}(K^+(x))$. If for the directed set $S, y \sqsubseteq \bigsqcup S$, then by assumption and lemma 3.1.1, $\bigsqcup S \in \text{int}(K^+(x))$. By lemma 4.5 and the fact that $\text{int}(K^+(x))$ is open, there exists $s \in S$ such that $s \in \text{int}(K^+(x))$. Consequently, $x \ll y$.

If $x \ll y$, by lemma 4.3 there exists an increasing sequence y_n in $I^-(y)$ such that $\bigsqcup y_n = y$. Thus $x \sqsubseteq y_n$, for some n . Since I^+ is an open relation, $x \in \text{int}(K^-(y))$. The proof of the other part is similar.

Now we are ready to obtain the main result of this paper.

Theorem 4.7. If M is a K -causal spacetime, with $\text{int}(K^\pm(\cdot))$ inner continuous, then $(M, \sqsubseteq)$ is a jointly bicontinuous poset with $\ll = \text{int}(K^-(\cdot))$ whose interval topology is equal to the manifold topology.

Proof. By lemma 4.6, $\Downarrow x = \text{int}(K^-(x))$. In addition, Using lemma 4.3, for every $x \in M$ there is an increasing sequence $x_n \subseteq I^-(x) \subseteq \text{int}(K^-(x)) = \Downarrow x$ with $\bigsqcup x_n = x$. Hence M is continuous. In a similar way we can prove that it is dually continuous. In addition, Using theorem 4.6 and 3.2.2, the interval topology is equal to the manifold topology.

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