TOPOLOGICAL LIMITS AND ω-LIMIT SETS OF THE DENDRITE MAPS

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Abstract. Let (X, d) be a metric space and f be a continuous map from X to X. Denote by $\omega(f)$ and $P(f)$ the ω-limit set and the set of periodic points of f, respectively. It is well known that for an interval map f, the following statements hold: (1) If $P(f) = \{x : f(x) = x\}$, then for any nonempty connected subset A of $[0, 1]$, the topological limit of trajectory of A under f exists. (2) If $x \in \omega(f) - \overline{P(f)}$, then the orbit $O(x, f)$ of x under f is an infinite set. The aim of this note is to show that the above statements do not hold for dendrite maps.

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1. Introduction

Throughout this paper, we assume that $\mathbb{N} = \{1, 2, \cdots\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For a metric space (X, d), denote by $C^0(X)$ the set of all continuous maps from X to X. For $f \in C^0(X)$, let f^0 is the identity map of X and $f^n = f \circ f^{n-1}$. The orbit of $x \in X$ under f is the set $O(x, f) \equiv \{f^n(x) : n \in \mathbb{Z}_+\}$. Write $P(f) = \{x \in X :$ there exists an $n \in \mathbb{N}$ such that $f^n(x) = x\}$, called the set of periodic points of f, and $F(f) = \{x \in X : f(x) = x\}$, called the set of fixed points of f. For any $x \in X$, write $\omega(x, f) = \{y :$ there exists a sequence of positive integers $n_1 < n_2 < \cdots$ such that $\lim_{k \longrightarrow \infty} f^{n_k}(x) = y\}$, called the ω-limit set of x under f. Put $\omega(f) = \cup_{x \in X} \omega(x, f)$. It is obvious that we have $F(f) \subset P(f) \subset \omega(f)$. For any nonempty subset A of X, write

$$limsup\ (A) = limsup\ (A, f) =$$

$$\{y \in X : \text{for any neighborhood } U \text{ of } y \text{ and any } N \in \mathbb{N},$$

$$\text{there exists an } n \geq N \text{ such that } f^n(A) \cap U \neq \emptyset\},$$

called the upper topological limit of trajectory of A under f, and

$$liminf\ (A) = liminf\ (A, f) =$$

$$\{y \in X : \text{for any neighborhood } U \text{ of } y, \text{ there exists an}$$

$$N \in \mathbb{N} \text{ such that } f^n(A) \cap U \neq \emptyset \text{ for any } n \geq N\},$$

called the lower topological limit of trajectory of A under f. It is obvious that $liminf\ (A) \subset limsup\ (A)$. We say that the topological limit of trajectory of A under f exists if $liminf\ (A) = limsup\ (A)$, and $liminf\ (A) = limsup\ (A)$ is said to be the topological limit of trajectory of A under f.

In [8], Fedorenko studied the topological limits of trajectory of nonempty subsets of interval maps and showed that if $f \in C^0([0, 1])$ and $P(f) = F(f)$, then for any nonempty connected subset A of $[0, 1]$, the topological limit of trajectory of A under f exists. In [17], Sun et al. generalized this result to tree maps. In [5], Block and Coppel studied ω-limit set of an interval map f and showed that if $x \in \omega(f) - \overline{P(f)}$, then $O(x, f)$ is an infinite set. Mai and Sun generalized [10] this result to graph maps. In this note, we show that the above results do not hold for dendrite maps.

A space homeomorphic to $[0, 1]$ (resp. to the unit circle S_1 in the complex plane $\mathbb{C}$) is called an arc (resp. a circle). A compact, connected and locally connected

metric space which contains no circle is called a dendrite. Let $(\mathcal{D}, d)$ be a non-degenerate dendrite. Every map $f \in C^0(\mathcal{D})$ is called a dendrite map. For any two different points $P, Q \in \mathcal{D}$, by [12] there is a unique arc in $\mathcal{D}$, denoted by $[P, Q]$ or $[P, Q]_{\mathcal{D}}$, such that the set of endpoints of $[P, Q]$ is $\{P, Q\}$. Write $[P, Q) = (Q, P] = [P, Q] - \{Q\}$, and $(P, Q) = [P, Q) - \{P\}$.

Recently, there has been a lot of work on dynamical systems on dendrites (see [1,2,3,6,7,9,11]), as this kind of research is motivated in part by the fact that dendrites appear as Julia sets in complex dynamics (see [4] or [14]). For some other papers in the area, see also [13,15,16] and the references cited therein.

2. Main results and proofs

In this section, we show that the main results for interval maps in [5,8] do not hold for dendrite maps.

Theorem 1. There exist a dendrite $\mathcal{D}$ and a continuous map $f \in C^0(\mathcal{D})$ with $P(f) = F(f)$ such that for some nonempty connected subset $\mathcal{M}$ of $\mathcal{D}$, the topological limit of the trajectory of $\mathcal{M}$ under f does not exist.

Proof In $\mathbb{R}^3$, let $X_k = \{(x, \frac{1}{k}, 0) : |x| \leq \frac{1}{k}\}$ and $Z_k = \{(0, \frac{1}{k}, z) : 0 \leq z \leq \frac{1}{k}\}$ for any $k \in \mathbb{N}$. Write $Y = \{(0, y, 0) : 0 \leq y \leq 2\}$. Then $\mathcal{D} = Y \bigcup (\cup_{k=1}^{\infty} Z_k) \bigcup (\cup_{k=1}^{\infty} X_k)$ is a dendrite. For any $k, n \in \mathbb{N}$ with $n \geq k$, write $P_k^n = (-\frac{1}{2k} - \frac{1}{2n}, \frac{1}{k}, 0), R_k^n = (-\frac{1}{2k} + \frac{1}{2n}, \frac{1}{k}, 0), P_k = (-\frac{1}{2k}, \frac{1}{k}, 0), Q_k^n = (\frac{1}{k} - \frac{1}{n+1}, \frac{1}{k}, 0), Q_k = (\frac{1}{k}, \frac{1}{k}, 0) \in X_k$, and $M_k^n = (0, \frac{1}{k}, \frac{1}{k} - \frac{1}{n+1}), M_k = (0, \frac{1}{k}, \frac{1}{k}) \in Z_k$, and $A_k = (0, \frac{3}{2} + \frac{1}{k+1}, 0), B_k = (0, \frac{3}{2} - \frac{1}{k+2}, 0) \in Y$. Put $C = (0, \frac{3}{2}, 0)$ and $O = (0, 0, 0)$. (see **Figure 1.1**)

Now we define continuous map $f : \mathcal{D} \longrightarrow \mathcal{D}$ as following (see **Figure 1.2**):

(1) $f(A_n) = P_1^n$, and $f(B_n) = R_1^n$, and $f(M_1^n) = B_n$ for any $n \in \mathbb{N}$.

(2) $f(R_k^n) = R_{k+1}^n$, and $f(P_k^n) = P_{k+1}^n$, and $f(M_{k+1}^n) = Q_k^n$ for any $n, k \in \mathbb{N}$ with $k \leq n - 1$.

(3) $f(Q_k^n) = M_k^n$ for any $n, k \in \mathbb{N}$ with $k \leq n$.

(4) $f(P_n^n) = Q_n^n$, and $f(Q_n) = M_n$, and $f(M_{n+1}) = Q_n$ and $f(P_n) = P_{n+1}$ for any $n \in \mathbb{N}$.

(5) $f(M_1) = C$, and $f(C) = P_1$ and $f(P) = P$ for any $P \in [R_1^1, O]$.

(6) f is linear elsewhere.

Write $\mathcal{M} = [A_1, B_1]$. In the following, we show that $P(f) = F(f)$ and $C \in limsup\ (\mathcal{M}) - liminf\ (\mathcal{M})$.

Claim 1 $C \in limsup\ (\mathcal{M})$.

Since $B_n = f^{3n+1}(A_n)$ for any $n \in \mathbb{N}$, it follows $[A_n, B_n] \cap f^{3r+1}(\mathcal{M}) \neq \emptyset$ for any $r \geq n$, which implies $C \in limsup\ (\mathcal{M})$.

Claim 2 $P(f) = F(f) = [R_1^1, O]$.

(i) It is easy to see that for any $k, n \in \mathbb{N}$, we have

$$\begin{aligned} f^n([P_k, R_k^k]) &= [P_{n+k}, R_{n+k}^{n+k}] \cup [R_{n+k}^{n+k}, R_k^k], \\ f^n([C, R_1^1]) &= [P_n, R_n^n] \cup [R_n^n, R_1^1], \\ f^{2k+n}([R_k^k, Q_k]) &= [P_n, R_n^n] \cup [R_n^n, R_k^k] \cup [R_k^k, R_1^1], \\ f^{2k-1+n}([R_k^k, M_k]) &= [P_n, R_n^n] \cup [R_n^n, R_k^k] \cup [R_k^k, R_1^1]. \end{aligned}$$

Thus $P(f) \cap \{[C, R_1^1) \bigcup [\bigcup_{k=1}^{\infty}([P_k, R_k^k) \cup [Q_k, R_k^k) \cup [M_k, R_k^k))]\} = \emptyset$.

(ii) For any $k, n \in \mathbb{N}$ with $k \leq n$ and $x \in [P_k^n, P_k^{n+1}]$, we have $f^{n-k}(x) \in [P_n^n, P_n^{n+1}]$ and $f^{n-k+1}(x) \in [P_{n+1}^{n+1}, P_{n+1}] \cup [P_{n+1}, R_{n+1}^{n+1}] \cup [R_{n+1}^{n+1}, R_n^n] \cup [R_n^n, Q_n^n]$. Thus we have

i) For any $m \geq n$, $f^{m-k}(x) \in [P_m^m, P_m]$ or

ii) For some $m \geq n$, $f^{m-k+1}(x) \in [P_{m+1}, R_{m+1}^{m+1}] \cup [R_{m+1}^{m+1}, R_m^m] \cup [R_m^m, Q_m^m]$ and $f^{r-k}(x) \in [P_r^r, P_r]$ for any $k \leq r \leq m$.

If case i) holds, then it is obvious that $x \notin P(f)$. If case ii) holds, then it follows from (i) That $x \notin P(f)$.

Since $f([A_n, A_{n+1}]) = [P_1^n, P_1^{n+1}]$ for any $n \in \mathbb{N}$, it follows that $[A_n, A_{n+1}] \cap P(f) = \emptyset$ for any $n \in \mathbb{N}$. Thus we have $P(f) \cap \{[A_1, C] \bigcup (\cup_{n=1}^{\infty}[P_n^n, P_n])\} = \emptyset$.

Claim 3 $C \notin liminf\ (\mathcal{M})$.

It follows from the above (i) that $f^n([C, B_1]) \cap \mathcal{M} = \emptyset$ for any $n \in \mathbb{N}$.

Now take $x \in [A_1, C)$. Let $x \in [A_n, A_{n+1}]$ for some $n \in \mathbb{N}$. Then we have $f^k(x) \in [P_k^n, P_k^{n+1}]$ for any $1 \leq k \leq n$ and $f^{n+1}(x) \in [P_{n+1}^{n+1}, P_{n+1}] \cup [P_{n+1}, R_{n+1}^{n+1}] \cup [R_{n+1}^{n+1}, R_n^n] \cup [R_n^n, Q_n^n]$.

If $f^m(x) \in [P_m^m, P_m]$ for any $m \geq n$, then $f^r(x) \notin \mathcal{M}$ for any $r \in \mathbb{N}$.

If $f^{m+1}(x) \in [P_{m+1}, R_{m+1}^{m+1}] \cup [R_{m+1}^{m+1}, R_m^m] \cup [R_m^m, Q_m^m]$ for some $m \geq n$ and $f^r(x) \in [P_r^r, P_r]$ for any $1 \leq r \leq m$, then by the above (i) we have

$$f^{m+r}(x) \in [P_{m+r}, R_{m+r}^{m+r}] \cup [R_{m+r}^{m+r}, R_{m-(r+1)/2+1}^{m-(r+1)/2+1}] \cup [R_{m-(r+1)/2+1}^{m-(r+1)/2+1}, Q_{m-(r+1)/2+1}^{m}]$$

if $1 \leq r \leq 2m$ is odd, and

$$f^{m+r}(x) \in [P_{m+r}, R_{m+r}^{m+r}] \cup [R_{m+r}^{m+r}, R_{m-r/2+1}^{m-r/2+1}] \cup [R_{m-r/2+1}^{m-r/2+1}, M_{m-r/2+1}^{m}]$$

if $1 \leq r \leq 2m$ is even and

$$f^{3m+1}(x) \in [P_{3m+1}, R^{3m+1}_{3m+1}] \cup [R^{3m+1}_{3m+1}, B_m].$$

Thus for any $1 \leq k \leq m$, we have

$$f^{3m+1+k}(x) \in [P_{3m+1+k}, R^{3m+1+k}_{3m+1+k}] \cup [R^{3m+1+k}_{3m+1+k}, R^1_1] \cup [R^m_k, R^k_k],$$

and $f^k(x) \in [P_k, R^k_k] \cup [R^k_k, R^1_1]$ for any $k \geq 4m+1$. Which implies $f^{3r}(x) \notin \mathcal{M}$ for any $r \in \mathbb{N}$.

Thus $f^{3r}(\mathcal{M}) \cap \mathcal{M} = \emptyset$ for any $r \in \mathbb{N}$. That is $C \notin liminf\ (\mathcal{M})$.

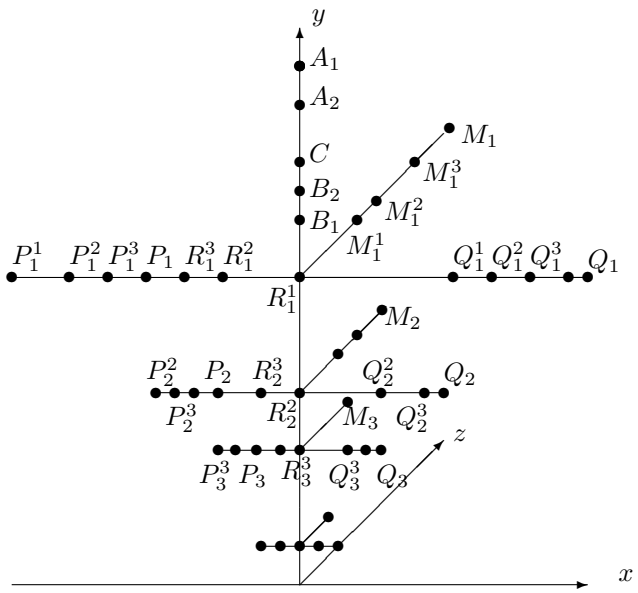


Figure 1.1

$A_1 \longrightarrow P^1_1 \longrightarrow Q^1_1 \longrightarrow M^1_1 \longrightarrow B_1 \longrightarrow R^1_1;$

$A_2 \longrightarrow P^2_1 \longrightarrow P^2_2 \longrightarrow Q^2_2 \longrightarrow M^2_2 \longrightarrow Q^2_1 \longrightarrow M^2_1 \longrightarrow B_2 \longrightarrow R^2_1 \longrightarrow R^2_2;$

. . . . . .

$A_n \longrightarrow P^n_1 \longrightarrow P^n_2 \longrightarrow \ldots \longrightarrow P^n_n \longrightarrow Q^n_n \longrightarrow M^n_n \longrightarrow Q^n_{n-1} \longrightarrow M^n_{n-1} \longrightarrow \ldots$
$\longrightarrow Q^n_2 \longrightarrow M^n_2 \longrightarrow Q^n_1 \longrightarrow M^n_1 \longrightarrow B_n \longrightarrow R^n_1 \longrightarrow R^n_2 \longrightarrow \ldots \longrightarrow R^n_n;$

. . . . . .

and

$$\ldots \longrightarrow Q_n \longrightarrow M_n \longrightarrow Q_{n-1} \longrightarrow M_{n-1} \longrightarrow \ldots \longrightarrow Q_1 \longrightarrow M_1 \longrightarrow C$$

$$\longrightarrow P_1 \longrightarrow \ldots \longrightarrow P_{n-1} \longrightarrow P_n \longrightarrow \ldots.$$

Figure 1.2

Theorem 2. There exist a dendrite $\mathcal{D}$ and a continuous map $f \in C^0(\mathcal{D})$ such that for some $M \in \omega(f) - \overline{P(f)}$, we have $f(M) \in P(f)$.

Proof In $\mathbb{R}^2$, let $X_k = \{(x, \frac{1}{k}) : |x| \leq \frac{1}{k}\}$ for any $k \in \mathbb{N}$ and $Y = \{(0, y) : 0 \leq y \leq 2\}$. Then $\mathcal{D} = Y \bigcup (\cup_{k=1}^{\infty} X_k)$ is a dendrite. For any $k, n \in \mathbb{N}$ with $n \geq k$, write $Q_k^n = (\frac{1}{2k} + \frac{1}{2n}, \frac{1}{k}), Q_k = (\frac{1}{2k}, \frac{1}{k}), P_k = (-\frac{1}{k}, \frac{1}{k}) \in X_k$. For any $n \in \mathbb{N}$, write $A_n = (0, 1 + \frac{1}{n}) \in Y$. Put $O = (0, 0)$ and $Y \cap X_k = O_k$ for any $k \in \mathbb{N}$. (see **Figure 2.1**)

Now we define continuous map $f : \mathcal{D} \longrightarrow \mathcal{D}$ as following (see **Figure 2.2**):

(1) $f(A_n) = P_n$ and $f(P_n) = Q_n^n$ for any $n \in \mathbb{N}$.

(2) $f(Q_k^n) = Q_{k-1}^n$ for any $2 \leq k \leq n$ and $f(Q_1^n) = A_{n+1}$ for any $n \in \mathbb{N}$.

(3) $f(Q_1) = O_1$, $f(O_1) = O$ and $f(Q_{n+1}) = Q_n$ for any $n \in \mathbb{N}$.

(4) $f(P) = P$ for any $P \in [O_2, O]$.

(5) f is linear elsewhere.

In the following, we show that $O_1 \in \omega(f) - \overline{P(f)}$ but $f(O_1) = O \in P(f)$.

By (1) and (2) we have $A_n \in O(A_1, f)$ for any $n \in \mathbb{N}$ and $\lim_{n \longrightarrow \infty} A_n = O_1$. Then $O_1 \in \omega(f)$. Now we show $\{[P_1, Q_1] \cup [A_1, O_2)\} \cap P(f) = \emptyset$, which implies $O_1 \notin \overline{P(f)}$.

If $x \in [O_1, O_2]$, then by (3), (4) and (5) we have $f^n(x) \in [O_2, O]$ for any $n \in \mathbb{N}$. Thus $[O_1, O_2) \cap P(f) = \emptyset$.

If $x \in [O_1, Q_1]$, then by (3), (4) and (5) we have $f(x) \in [O_1, O]$ and $f^n(x) \in [O_2, O]$ for any $n \geq 2$. Thus $[O_1, Q_1] \cap P(f) = \emptyset$.

If $x \in [P_1, O_1)$, then we have $f(x) \in [O, O_1] \cup [O_1, Q_1^1]$, and $f^2(x) \in [A_2, O]$, and $f^3(x) \in [O_2, O] \bigcup (\cup_{n=2}^{\infty} [P_n, O_n])$. On the other hand, for any $n \geq 2$ and $1 \leq k \leq n$, we have

$$f^k([P_n, O_n]) = [O_n, O_{n-k+1}] \cup [O_{n-k+1}, Q_{n-k+1}^n],$$

and

$$\begin{aligned} f^{n+1}([P_n, O_n]) &= [A_{n+1}, O], \\ f^{n+2}([P_n, O_n]) &= [O_2, O] \bigcup (\cup_{k=n+1}^{\infty} [P_k, O_k]). \end{aligned}$$

Thus, $f^n(x) \notin [P_1, O_1)$ for any $n \in \mathbb{N}$, which implies $[P_1, O_1) \cap P(f) = \emptyset$.

If $x \in [A_n, A_{n+1})$ for some $n \geq 2$, then we have $f(x) \in [P_n, O_n] \cup [O_n, O_{n+1}] \cup [O_{n+1}, P_{n+1})$, and for any $1 \leq k \leq n$,

$$f^{k+1}(x) \in [O_{n-k+1}, Q^n_{n-k+1}] \cup [O_{n-k+1}, O_{n+1}] \cup [O_{n-k+2}, Q^{n+1}_{n-k+2}).$$

Thus $f^{n+2}(x) \in [A_{n+1}, O] \cup [O_1, Q_1^{n+1})$ and $f^{n+3}(x) \in f([A_{n+1}, O]) \cup [A_{n+2}, O)$. Thus $f^r(x) \notin [A_n, A_{n+1})$ for any $r \in \mathbb{N}$, from which it follows that $[A_n, A_{n+1}) \cap P(f) = \emptyset$. That is $[A_1, O_1] \cap P(f) = \emptyset$.

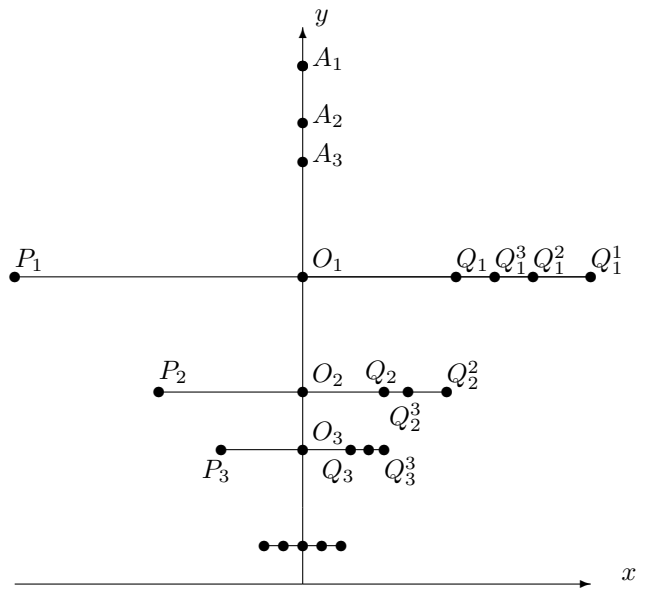


Figure 2.1

$$
\begin{array}{lll}
 & A_1 & \longrightarrow P_1 \longrightarrow Q_1^1 \\
\longrightarrow & A_2 & \longrightarrow P_2 \longrightarrow Q_2^2 \longrightarrow Q_1^2 \\
\longrightarrow & A_3 & \longrightarrow P_3 \longrightarrow Q_3^3 \longrightarrow Q_2^3 \longrightarrow Q_1^3 \\
 & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\longrightarrow & A_n & \longrightarrow P_n \longrightarrow Q_n^n \longrightarrow Q_{n-1}^n \longrightarrow \ldots \longrightarrow Q_2^n \longrightarrow Q_1^n \\
\longrightarrow & A_{n+1} & \longrightarrow P_{n+1} \longrightarrow Q_{n+1}^{n+1} \longrightarrow Q_n^{n+1} \longrightarrow \ldots \longrightarrow Q_2^{n+1} \longrightarrow Q_1^{n+1} \\
 & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}
$$

and

$$\ldots \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \ldots \longrightarrow Q_1 \longrightarrow O_1 \longrightarrow O.$$

Figure 2.2

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