

TIME EVOLUTION EQUATIONS FOR SURFACES GENERATED VIA BINORMAL SPHERICAL IMAGE IN TERMS OF INEXTENSIBLE FLOWS IN $\mathbb{E}^3$

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ABSTRACT. In this work, we study time-evolution equations that the intrinsic quantities of curves to construct binormal spherical curves as surfaces in terms of inextensible flows in $\mathbb{E}^3$. Using the Frenet frame of the given curve, we present partial differential equations. We give necessary and sufficient condition for a binormal spherical curves as surfaces to be a Bonnet surface. Finally, we draw some pictures of our main results.

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1. INTRODUCTION

A three-dimensional heat transfer model of flows was developed to predict the temperature fields, the weld geometry and the shape of the solidified weld reinforcement surface during hybrid laser-MAG arc welding of fillet joints. Melt pool deformation due to arc pressure was calculated by minimizing the total surface energy. A series of hybrid welding experiments was conducted on S355 steel for different welding speeds and wire feeding rates. A high speed video camera was

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used to measure weld pool depression and surface weld pool geometry. Visualization of the weld pool during welding has also allowed for a better understanding of the interaction between the keyhole and droplets. The various weld bead shapes were explained through these observations. The arc pressure, the surface energy distribution, and arc efficiency were evaluated by comparing experimental data and numerical results for a wide range of welding operating parameters. Good correlation was found between the calculated and experimental weld bead shapes obtained for the hybrid laser-MAG arc welding process as well as for laser or MAG alone, [4,5,9,13].

For centuries, fluid flow researchers have been studying fluid flows in various ways, and today fluid flow is still an important field of research. The areas in which fluid flow plays a role are numerous. Gaseous flows are studied for the development of cars, aircraft and spacecrafts, and also for the design of machines such as turbines and combustion engines. Liquid flow research is necessary for naval applications, such as ship design, and is widely used in civil engineering projects such as harbour design and coastal protection. In chemistry, knowledge of fluid flow in reactor tanks is important; in medicine, the flow in blood vessels is studied. Numerous other examples could be mentioned. In all kinds of fluid flow research, visualization is an key issue, [1,2,5,7,8].

The local theory of space curves are mainly developed by the Frenet–Serret theorem which expresses the derivative of a geometrically chosen basis of $\mathbb{E}^3$ by the aid of itself is proved. Then it is observed that by the solution of some of special ordinary differential equations, further classical topics, for instance spherical curves, Bertrand curves, involutes and evolutes are investigated. One of the mentioned works is spherical images of a regular curve in the Euclidean space. It is a wellknown concept in the local differential geometry of curves. Such curves are obtained in terms of the Frenet–Serret vector fields, [3,6,11,12].

This study is organized as follows: Firstly, we study time-evolution equations that the intrinsic quantities of curves to construct binormal spherical curves as surfaces in terms of inextensible flows in $\mathbb{E}^3$. Using the Frenet frame of the given curve, we present partial differential equations. We give necessary and sufficient condition for a binormal spherical curves as surfaces to be a Bonnet surface. Finally, we draw some pictures of our main results.

2. PRELIMINARIES

The Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$\langle, \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3$. Recall that, the norm of an arbitrary vector $a \in \mathbb{E}^3$ is given by

$$\|a\| = \sqrt{\langle a, a \rangle}.$$

α is called a unit speed curve if velocity vector v of α satisfies $\|v\| = 1$. Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^3$. If the tangent vector of this curve forms a constant angle with a fixed constant vector U , then this curve is called a general helix or an inclined curve. The sphere of radius $r > 0$ and with center in the origin in the space $\mathbb{E}^3$ is defined by

$$\mathbb{S}^2 = \{p = (p_1, p_2, p_3) \in \mathbb{E}^3 : \langle p, p \rangle = r^2\}.$$

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet-Serret frame along the curve α in the space $\mathbb{E}^3$. For an arbitrary curve α with first and second curvature, κ and τ in the space $\mathbb{E}^3$, the following Frenet-Serret formulae are given in [10] written under matrix form

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

where

$$\begin{aligned} \langle \mathbf{T}, \mathbf{T} \rangle &= \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 1, \\ \langle \mathbf{T}, \mathbf{N} \rangle &= \langle \mathbf{N}, \mathbf{B} \rangle = \langle \mathbf{T}, \mathbf{B} \rangle = 0. \end{aligned}$$

Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{E}^3$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be positive oriented natural basis of $\mathbb{E}^3$. Cross product of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Mixed product of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is defined by the determinant

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Torsion of the curve α is given by the aid of the mixed product

$$\tau = \frac{[\alpha', \alpha'', \alpha''']}{\kappa^2}.$$

Definition 2.1. *Let α be a unit speed regular curve in Euclidean 3-space with Frenet vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$. The unit binormal vectors along the curve α generate a curve $(\mathbf{B})$ on the sphere of radius 1 about the origin. The curve $\phi = (\mathbf{B})$ is called the spherical image of $\mathbf{B}$ or more commonly, $(\mathbf{B})$ is called binormal spherical image of the curve α , [3,10].*

3. TIME-EVOLUTION EQUATIONS OF BINORMAL SPHERICAL IMAGES IN $\mathbb{E}^3$

Let ϖ_ϕ be the standard unit normal vector field on a surface $\phi(s, t)$ defined by

$$(3.1) \quad \varpi_\phi(s, t) = \frac{\phi_s \wedge \phi_t}{|g(\phi_s \wedge \phi_t, \phi_s \wedge \phi_t)|^{\frac{1}{2}}}.$$

Then, the first fundamental form $\mathbf{I}$ and the second fundamental form $\mathbf{II}$ of a surface ϕ are defined by, respectively,

$$\begin{aligned} \mathbf{I} &= \mathbf{E}ds^2 + 2\mathbf{F}dsdu + \mathbf{G}dt^2, \\ \mathbf{II} &= \mathbf{e}ds^2 + 2\mathbf{f}dsdu + \mathbf{g}dt^2, \end{aligned}$$

where

$$\begin{aligned} \mathbf{E} &= g(\phi_s, \phi_s), \\ \mathbf{F} &= g(\phi_s, \phi_t), \\ \mathbf{G} &= g(\phi_t, \phi_t), \end{aligned}$$

$$\begin{aligned} \mathbf{e} &= -g(\phi_s, \varpi_s) = g(\phi_{ss}, \varpi), \\ \mathbf{f} &= -g(\phi_s, \varpi_t) = g(\phi_{st}, \varpi), \\ \mathbf{g} &= -g(\phi_t, \varpi_t) = g(\phi_{tt}, \varpi). \end{aligned}$$

Definition 3.1. *A-net on a surface such that, when this net is parametrized, the conditions $\mathbf{E} = \mathbf{G}$, $\mathbf{F} = 0$, $\mathbf{f} = c = \text{const.} = 0$ are satisfied, is called an A-net, where $\mathbf{E}, \mathbf{F}, \mathbf{G}$ are the coefficients of the first fundamental form of the surface and $\mathbf{e}, \mathbf{f}, \mathbf{g}$ are the coefficients of the second fundamental form.*

Theorem 3.2. *Necessary and sufficient condition for a surface to be a Bonnet surface is that the surface can have an A-net, [3].*

On the other hand, the Gaussian curvature $\mathbf{K}$ and the mean curvature $\mathbf{H}$ are

$$\mathbf{K} = \frac{\mathbf{LN} - \mathbf{M}^2}{\mathbf{EG} - \mathbf{F}^2},$$

$$\mathbf{H} = \frac{\mathbf{EN} - 2\mathbf{FM} + \mathbf{GL}}{2(\mathbf{EG} - \mathbf{F}^2)},$$

respectively.

The principal curvatures k_1 and k_2 are given by

$$k_1 = \mathbf{H} + \sqrt{\mathbf{H}^2 - \mathbf{K}}, \quad k_2 = \mathbf{H} - \sqrt{\mathbf{H}^2 - \mathbf{K}}.$$

Let $\alpha(u, t)$ is a one parameter family of smooth curves in $\mathbb{E}^3$.

Any flow of α can be represented as

$$\frac{\partial \alpha}{\partial t} = \mathcal{H}_1 \mathbf{T} + \mathcal{H}_2 \mathbf{N} + \mathcal{H}_3 \mathbf{B},$$

where $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are smooth functions of time and arclength.

Theorem 3.3. *Let ϕ be binormal spherical image of α . Then,*

$$\begin{aligned} \nabla_t \mathbf{T}^\phi &= -\nabla_t \mathbf{N} = (\mathcal{H}_1 \kappa - \mathcal{H}_3 \tau + \frac{\partial \mathcal{H}_2}{\partial s}) \mathbf{T} - \mathcal{C} \mathbf{B}, \\ \nabla_t \mathbf{N}^\phi &= [\frac{\partial}{\partial t} [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}] + (\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}]] \mathbf{T} \\ &\quad + [\mathcal{C} [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] + (\mathcal{H}_1 \kappa - \mathcal{H}_3 \tau + \frac{\partial \mathcal{H}_2}{\partial s}) [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}]] \mathbf{N} \\ &\quad + [(\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}] - \frac{\partial}{\partial t} [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}]] \mathbf{B}, \\ \nabla_t \mathbf{B}^\phi &= [\frac{\partial}{\partial t} [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] - (\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}]] \mathbf{T} \\ &\quad + [(\mathcal{H}_1 \kappa - \mathcal{H}_3 \tau + \frac{\partial \mathcal{H}_2}{\partial s}) [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] - \mathcal{C} [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}]] \mathbf{N} \\ &\quad + [\frac{\partial}{\partial t} [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}] + (\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}]] \mathbf{B}, \end{aligned} \tag{3.2}$$

where $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are smooth functions of time and arc length and

$$\mathcal{C} = \langle \nabla_t \mathbf{N}, \mathbf{B} \rangle.$$

Proof. By using Frenet-Serret equation and definition 2.1 we easily obtain (3.2), which completes the proof.

Proposition 3.4.

$$\frac{\partial \phi}{\partial t} = -(\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) \mathbf{T} - \mathcal{C} \mathbf{N},$$

where $\mathcal{H}_2, \mathcal{H}_3$ are smooth functions of time and arclength.

Any flow of ϕ can be represented as

$$\frac{\partial \phi}{\partial t} = \mathcal{H}_1^\phi \mathbf{T}^\phi + \mathcal{H}_2^\phi \mathbf{N}^\phi + \mathcal{H}_3^\phi \mathbf{B}^\phi,$$

where $\mathcal{H}_1^\phi, \mathcal{H}_2^\phi, \mathcal{H}_3^\phi$ are smooth functions of time and arclength.

Lemma 3.5. Let ϕ be binormal spherical image of α . Then,

$$\mathbf{E} = g(\phi_s, \phi_s) = 1,$$

$$\mathbf{F} = g(\phi_s, \phi_t) = \mathcal{H}_1^\phi,$$

$$\mathbf{G} = g(\phi_t, \phi_t) = (\mathcal{H}_1^\phi)^2 + (\mathcal{H}_2^\phi)^2 + (\mathcal{H}_3^\phi)^2,$$

where $\mathcal{H}_1^\phi, \mathcal{H}_2^\phi, \mathcal{H}_3^\phi$ are smooth functions of time and arclength.

Lemma 3.6. Let ϕ be binormal spherical image of α . Then, ϕ is developable if and only if

$$\mathcal{H}_2^\phi = \mathcal{H}_3^\phi = 0.$$

Proof. By using Lemma 3.5 and definition of Gaussian curvature, we obtain above equation.

Moreover, (3.1) imply that

$$\begin{aligned} \varpi_\phi(s, t) &= \frac{\mathbf{T}^\phi \wedge (\mathcal{H}_1^\phi \mathbf{T}^\phi + \mathcal{H}_2^\phi \mathbf{N}^\phi + \mathcal{H}_3^\phi \mathbf{B}^\phi)}{|g(\phi_s \wedge \phi_t, \phi_s \wedge \phi_t)|^{\frac{1}{2}}} \\ &= \frac{\mathcal{H}_2^\phi \mathbf{B}^\phi - \mathcal{H}_3^\phi \mathbf{N}^\phi}{\left| (\mathcal{H}_2^\phi)^2 + (\mathcal{H}_3^\phi)^2 \right|^{\frac{1}{2}}}. \end{aligned} \quad (3.3)$$

Then, we have

$$\varpi_\phi(s, t) = \mathcal{P}^\phi \mathcal{H}_2^\phi \mathbf{B}^\phi - \mathcal{P}^\phi \mathcal{H}_3^\phi \mathbf{N}^\phi,$$

where

$$\mathcal{P}^\phi = \frac{1}{\left| (\mathcal{H}_2^\phi)^2 + (\mathcal{H}_3^\phi)^2 \right|^{\frac{1}{2}}}.$$

Theorem 3.7. *Let ϕ be binormal spherical image of α . Then,*

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{H}_1^\phi &= 0, \\ \mathcal{H}_1^\phi \frac{\partial \mathcal{H}_1^\phi}{\partial t} + \mathcal{H}_2^\phi \frac{\partial \mathcal{H}_2^\phi}{\partial t} + \mathcal{H}_3^\phi \frac{\partial \mathcal{H}_3^\phi}{\partial t} &= 0, \end{aligned}$$

where $\mathcal{H}_1^\phi, \mathcal{H}_2^\phi, \mathcal{H}_3^\phi$ are smooth functions of time and arclength.

Proof. It is obvious from Theorem 3.3.

Corollary 3.8. *Let ϕ be binormal spherical image of α . Then,*

$$\frac{\partial \mathcal{H}_2^\phi}{\partial t} = -\frac{\mathcal{H}_3^\phi}{\mathcal{H}_2^\phi} \frac{\partial \mathcal{H}_3^\phi}{\partial t},$$

where $\mathcal{H}_1^\phi, \mathcal{H}_2^\phi, \mathcal{H}_3^\phi$ are smooth functions of time and arclength.

Lemma 3.9. *Let ϕ be binormal spherical image of α . Then,*

$$\begin{aligned} \mathbf{L} &= -\kappa^\phi \mathcal{P}^\phi \mathcal{H}_3^\phi, \\ \mathbf{M} &= -[(\mathcal{P}^\phi \mathcal{H}_2^\phi)[(\mathcal{H}_1\kappa - \mathcal{H}_3\tau + \frac{\partial \mathcal{H}_2}{\partial s})[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] - \mathcal{C}[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}]] \\ &\quad - (\mathcal{P}^\phi \mathcal{H}_3^\phi)[\mathcal{C}[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] + (\mathcal{H}_1\kappa - \mathcal{H}_3\tau + \frac{\partial \mathcal{H}_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}]], \\ \mathbf{N} &= [\frac{\partial}{\partial t} \mathcal{H}_3^\phi \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} + \mathcal{H}_3^\phi [\frac{\partial}{\partial t} [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] - (\mathcal{H}_2\tau + \frac{\partial \mathcal{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}]] \\ &\quad + \mathcal{H}_1^\phi (\mathcal{H}_1\kappa - \mathcal{H}_3\tau + \frac{\partial \mathcal{H}_2}{\partial s}) + \frac{\partial}{\partial t} \mathcal{H}_2^\phi \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} + \mathcal{H}_2^\phi [\frac{\partial}{\partial t} [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}] \\ &\quad + (\mathcal{H}_2\tau + \frac{\partial \mathcal{H}_3}{\partial s})[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}]] [\mathcal{P}^\phi \mathcal{H}_2^\phi \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} - \mathcal{P}^\phi \mathcal{H}_3^\phi \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}] \\ &\quad + [\mathcal{H}_2^\phi [(\mathcal{H}_2\tau + \frac{\partial \mathcal{H}_3}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}] - \frac{\partial}{\partial t} [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}]] - \frac{\partial}{\partial t} \mathcal{H}_2^\phi \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \\ &\quad - \mathcal{H}_1^\phi \mathcal{C} + \frac{\partial}{\partial t} \mathcal{H}_3^\phi \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} + \mathcal{H}_3^\phi [\frac{\partial}{\partial t} [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}] + (\mathcal{H}_2\tau + \frac{\partial \mathcal{H}_3}{\partial s}) \\ &\quad [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}]]] [\mathcal{P}^\phi \mathcal{H}_2^\phi \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} + \mathcal{P}^\phi \mathcal{H}_3^\phi \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}], \end{aligned}$$

where $\mathcal{H}_1^\phi, \mathcal{H}_2^\phi, \mathcal{H}_3^\phi$ are smooth functions of time and arclength.

Proof. Firstly, we have

$$\nabla_t \varpi_\phi(s, t)$$

$$\begin{aligned}
&= [(\mathcal{P}^\phi \mathcal{H}_2^\phi) \left[\frac{\partial}{\partial t} \left[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right] \right. \\
&\quad - (\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) \left[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \right] \\
&\quad - (\mathcal{P}^\phi \mathcal{H}_3^\phi) \left[\frac{\partial}{\partial t} \left[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \right] \right. \\
&\quad \left. \left. + (\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) \left[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right] \right] \right. \\
&\quad \left. - \frac{\partial}{\partial t} (\mathcal{P}^\phi \mathcal{H}_3^\phi) \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \right. \\
&\quad \left. + \frac{\partial}{\partial t} (\mathcal{P}^\phi \mathcal{H}_2^\phi) \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right] \mathbf{T} \\
&+ [(\mathcal{P}^\phi \mathcal{H}_2^\phi) \left[(\mathcal{H}_1 \kappa - \mathcal{H}_3 \tau + \frac{\partial \mathcal{H}_2}{\partial s}) \left[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right] - \mathcal{C} \left[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \right] \right. \right. \\
&\quad \left. \left. - (\mathcal{P}^\phi \mathcal{H}_3^\phi) \left[\mathcal{C} \left[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right] \right. \right. \right. \\
&\quad \left. \left. + (\mathcal{H}_1 \kappa - \mathcal{H}_3 \tau + \frac{\partial \mathcal{H}_2}{\partial s}) \left[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \right] \right] \right] \mathbf{N} + [(\mathcal{P}^\phi \mathcal{H}_2^\phi) \\
&\quad \left[\frac{\partial}{\partial t} \left[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \right] \right. \\
&\quad \left. + (\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) \left[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right] \right] \\
&\quad - (\mathcal{P}^\phi \mathcal{H}_3^\phi) \left[(\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) \left[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \right] \right. \\
&\quad \left. \left. - \frac{\partial}{\partial t} \left[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right] \right] \right. \\
&\quad \left. + \frac{\partial}{\partial t} (\mathcal{P}^\phi \mathcal{H}_2^\phi) \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} + \frac{\partial}{\partial t} (\mathcal{P}^\phi \mathcal{H}_3^\phi) \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right] \mathbf{B}.
\end{aligned}$$

Also,

$$\begin{aligned}
\phi_{tt} &= \left[\frac{\partial}{\partial t} \mathcal{H}_3^\phi \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} + \mathcal{H}_3^\phi \left[\frac{\partial}{\partial t} \left[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right] - (\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) \left[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \right] \right. \right. \\
&\quad \left. \left. + \mathcal{H}_1^\phi (\mathcal{H}_1 \kappa - \mathcal{H}_3 \tau + \frac{\partial \mathcal{H}_2}{\partial s}) \right. \right. \\
&\quad \left. + \frac{\partial}{\partial t} \mathcal{H}_2^\phi \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} + \mathcal{H}_2^\phi \left[\frac{\partial}{\partial t} \left[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \right] + (\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) \left[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right] \right] \right] \mathbf{T} \\
&\quad + [\mathcal{H}_2^\phi \left[\mathcal{C} \left[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right] \right. \\
&\quad \left. + (\mathcal{H}_1 \kappa - \mathcal{H}_3 \tau + \frac{\partial \mathcal{H}_2}{\partial s}) \left[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \right] - \frac{\partial}{\partial t} \mathcal{H}_1^\phi + \mathcal{H}_3^\phi [(\mathcal{H}_1 \kappa - \mathcal{H}_3 \tau \right. \\
&\quad \left. \left. + \frac{\partial \mathcal{H}_2}{\partial s}) \left[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \right] - \mathcal{C} \left[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \right] \right] \right] \mathbf{N}
\end{aligned}$$

$$\begin{aligned}
& + [\mathcal{H}_2^\phi (\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}] - \frac{\partial}{\partial t} [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}]] - \frac{\partial}{\partial t} \mathcal{H}_2^\phi \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \\
& - \mathcal{H}_1^\phi \mathcal{C} + \frac{\partial}{\partial t} \mathcal{H}_3^\phi \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \\
& + \mathcal{H}_3^\phi [\frac{\partial}{\partial t} [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}] + (\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}]]] \mathbf{B}.
\end{aligned}$$

Corollary 3.10.

$$\begin{aligned}
\nabla_t \varpi_\phi(s, t) = & [(\mathcal{P}^\phi \mathcal{H}_2^\phi) [\frac{\partial}{\partial t} [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] - (\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}]] \\
& - (\mathcal{P}^\phi \mathcal{H}_3^\phi) [\frac{\partial}{\partial t} [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}] \\
& + (\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}]] - \frac{\partial}{\partial t} (\mathcal{P}^\phi \mathcal{H}_3^\phi) \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \\
& + \frac{\partial}{\partial t} (\mathcal{P}^\phi \mathcal{H}_2^\phi) \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] \mathbf{T} \\
& + [(\mathcal{P}^\phi \mathcal{H}_2^\phi) [(\mathcal{H}_1 \kappa - \mathcal{H}_3 \tau + \frac{\partial \mathcal{H}_2}{\partial s}) [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] - \mathcal{C} [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}]] \\
& - (\mathcal{P}^\phi \mathcal{H}_3^\phi) [\mathcal{C} [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] + (\mathcal{H}_1 \kappa - \mathcal{H}_3 \tau + \frac{\partial \mathcal{H}_2}{\partial s}) [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}]]] \mathbf{N} \\
& + [(\mathcal{P}^\phi \mathcal{H}_2^\phi) [\frac{\partial}{\partial t} [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}] + (\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}]] \\
& - (\mathcal{P}^\phi \mathcal{H}_3^\phi) [(\mathcal{H}_2 \tau + \frac{\partial \mathcal{H}_3}{\partial s}) [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}] - \frac{\partial}{\partial t} [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}]] \\
& + \frac{\partial}{\partial t} (\mathcal{P}^\phi \mathcal{H}_2^\phi) \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \\
& + \frac{\partial}{\partial t} (\mathcal{P}^\phi \mathcal{H}_3^\phi) \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] \mathbf{B}.
\end{aligned}$$

Theorem 3.11. *Let ϕ be binormal spherical image of α . If ϕ have an A-net, then*

$$\begin{aligned}
& (\mathcal{H}_2^\phi)^2 + (\mathcal{H}_3^\phi)^2 = 1, \mathcal{H}_1^\phi = 0, \\
& -[(\mathcal{P}^\phi \mathcal{H}_2^\phi) [(\mathcal{H}_1 \kappa - \mathcal{H}_3 \tau + \frac{\partial \mathcal{H}_2}{\partial s}) [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] - \mathcal{C} [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}]] \\
& - (\mathcal{P}^\phi \mathcal{H}_3^\phi) [\mathcal{C} [\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] + (\mathcal{H}_1 \kappa - \mathcal{H}_3 \tau + \frac{\partial \mathcal{H}_2}{\partial s}) [\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}]]] = \text{const.},
\end{aligned}$$

where $\mathcal{H}_1^\phi, \mathcal{H}_2^\phi, \mathcal{H}_3^\phi$ are smooth functions of time and arclength.

Proof. It is obvious from Theorem 3.9.

Corollary 3.12. *Let ϕ be binormal spherical image of α . ϕ is a Bonnet surface if and only if*

$$\begin{aligned} &(\mathcal{H}_2^\phi)^2 + (\mathcal{H}_3^\phi)^2 = 1, \\ & -[(\mathcal{P}^\phi \mathcal{H}_2^\phi)[(-\mathcal{H}_3\tau + \frac{\partial \mathcal{H}_2}{\partial s})[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] - \mathcal{C}[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}]] \\ & -(\mathcal{P}^\phi \mathcal{H}_3^\phi)[\mathcal{C}[\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}] + (-\mathcal{H}_3\tau + \frac{\partial \mathcal{H}_2}{\partial s})[\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}]] = \text{const.}, \end{aligned}$$

where $\mathcal{H}_1^\phi, \mathcal{H}_2^\phi, \mathcal{H}_3^\phi$ are smooth functions of time and arclength.

Proof. By using Theorem 3.11 and Definition 3.1, we obtain above equations.

4. SOME PICTURES

In this section we draw some pictures corresponding to different cases of $A(t)$ that appeared in the following example.

Example 4.1. The time-helix is parametrized by

$$\alpha(u, t) = (A(t) \cos(u), A(t) \sin(u), B(t)u),$$

where A, B are functions only of time. The arc-length derivative is

$$(A^2 + B^2)^{\frac{1}{2}} \frac{\partial}{\partial s} = \frac{\partial}{\partial u}$$

and the evolution of γ is explicitly given by

$$(\frac{\partial A}{\partial t} \cos(u), \frac{\partial A}{\partial t} \sin(u), \frac{\partial B}{\partial t} u) = \frac{1}{A^2 + B^2} (-A \cos(u), -A \sin(u), 0).$$

By using above equations for binormal spherical curves, we obtain

5. CONCLUSION

Physically, inextensible curve and surface flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of physical applications.

In this article, we construct time-evolution equations that the intrinsic quantities of curves to construct binormal spherical curves as surfaces in terms of inextensible flows in $\mathbb{E}^3$. Using the Frenet frame of the given curve, we present partial differential

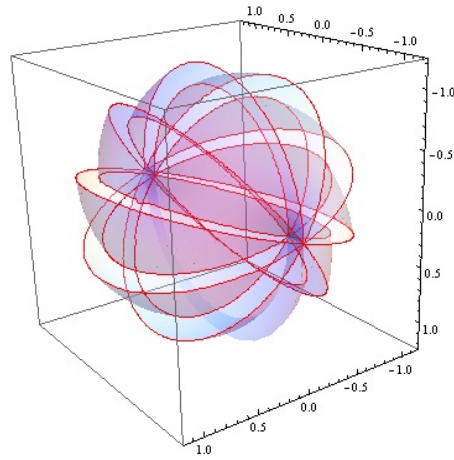


FIGURE 1

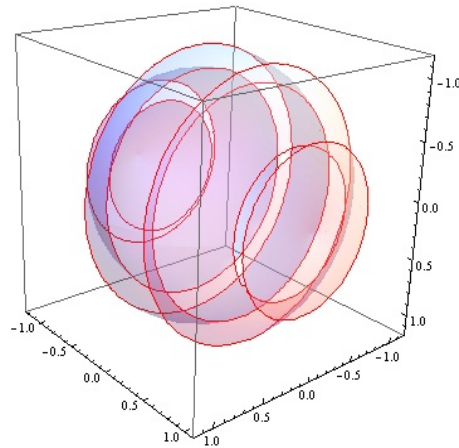


FIGURE 2

equations. Additionally, we obtain necessary and sufficient condition for a binormal spherical curves as surfaces to be a Bonnet surface.

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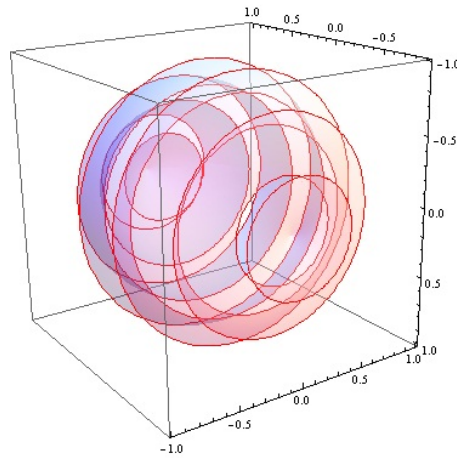


FIGURE 3

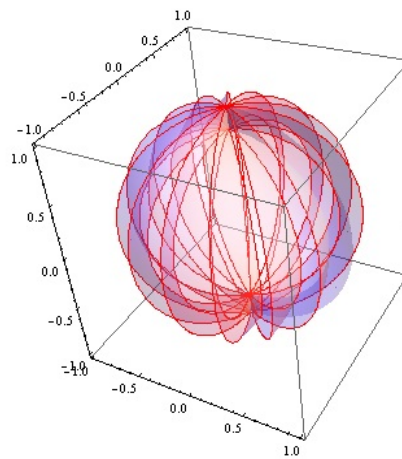


FIGURE 4

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