

THE BEHAVIOR OF SPATIALLY HOMOGENEOUS AND INHOMOGENEOUS AUTOCATALYSIS SYSTEM AT INFINITY

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ABSTRACT. This paper is devoted to the study of a two dimensional spatially homogeneous and inhomogeneous autocatalysis dynamical system. Using the infinite singularity theory and projecting on Poincare hemisphere, spatially homogeneous and inhomogeneous autocatalysis system behavior at infinity explains.

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1. INTRODUCTION

The autocatalysis models have received much attention in the studies of morphogenesis, population dynamics and autocatalysis oxidation reactions [1, 3]. If the reaction rates are the same and the initial concentration of reactant is fixed, then we can obtain an arbitrary order autocatalysis model

$$(1.1) \quad \begin{cases} \frac{du(t)}{dt} = a - u(t)v^p(t), & t > 0, \\ \frac{dv(t)}{dt} = u(t)v^p(t) - v(t), & t > 0, \\ u(0) = u_0 > 0, & v(0) = v_0 > 0, \end{cases}$$

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where the parameter $a > 0$ is taken as the initial concentration of the reaction precursor, the variables u and v represent the densities of the reactant and autocatalyst respectively and are usually considered to be non-negative.

Several researchers have studied Hopf bifurcation for this system. For example see [4, 5, 6, 7]. Some of them choose a as the Hopf bifurcation parameter and some choose p as the Hopf bifurcation parameter, but choosing a is physically more meaningful than using the parameter p (see [4] for the selkov model). We organize this paper in two parts:

One section gives some mathematical concepts which are going to be used in other sections and then we analyze the singularity of infinite Spatially homogeneous and inhomogeneous autocatalysis system.

2. PRELIMINARIES

In this section we shall state some mathematical concepts and basic theorems. To study the behavior of the trajectories of a planar system by large r , the stereographic projection can be used. A better approach to studying the behavior of trajectories "at infinity" is to use the so-called Poincar'e sphere where we project from the center of the unit sphere

$$S^2 = \{(X, Y, Z) \in R^3 \mid X^2 + Y^2 + Z^2 = 1, \}$$

onto the (x, y) -plane tangent to S^2 at either the north or south pole.

Consider a flow defined by a dynamical system on $\mathbb{R}^2$

$$(2.1) \quad \begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y). \end{cases}$$

Where P and Q are polynomial functions of x and y . Let m denotes the maximum degree of the terms in P and Q . This system can be written in the form of a single differential equation

$$(2.2) \quad ZQ * dX - ZP * dY + (YP * -XQ*)dZ = 0,$$

where

$$P^*(X, Y, Z) = Z^m P\left(\frac{X}{Z}, \frac{Y}{Z}\right),$$

and

$$Q^*(X, Y, Z) = Z^m Q\left(\frac{X}{Z}, \frac{Y}{Z}\right),$$

are polynomials in (X, Y, Z) , see [8].

Theorem 2.1. *The equilibrium points at infinity for the m th degree polynomial system 2.1 occur at the points $(X, Y, 0)$ on the circle $X^2 + Y^2 = 1$ of the Poincare sphere and we have*

$$(2.3) \quad XQ_m(X, Y) - YP_m(X, Y) = 0.$$

For the proof see [8].

The behavior of the solution curves defined by 2.2 in a neighborhood of any equilibrium point at infinity can be determined by projecting that neighborhood onto a plane tangent to S^2 at that point. Actually to determine the behavior of the flow in a neighborhood of any equilibrium point on the equator of S^2 , one can project the hemisphere with $X > 0$ onto the plane $X = 1$ and project the hemisphere with $Y > 0$ onto the plane $Y = 1$. This follows because when m is odd, the flow on S^2 defined by 2.2 is topologically equivalent at antipodal points of S^2 and if m is even it is topologically equivalent, with the direction of the flow reversed. We set $X = 1$ and $dX = 0$ to project the flow on S^2 defined by 2.2 onto the plane $X = 1$. Similarly by setting $Y = 1$ and $dY = 0$ in 2.2 we can project the flow defined by 2.2 onto the plane $Y = 1$.

Theorem 2.2. *The flow defined by 2.2 in a neighborhood of any equilibrium point of 2.2 on the circle $X^2 + Y^2 = 1$ of the Poincare sphere S^2 , except the points $(0, 1, 0)$, is topologically equivalent to the flow of the system*

$$(2.4) \quad \begin{cases} \pm \dot{y} = yz^m P\left(\frac{1}{z}, \frac{y}{z}\right) - z^m Q\left(\frac{1}{z}, \frac{y}{z}\right) \\ \pm \dot{z} = z^{m+1} P\left(\frac{1}{z}, \frac{y}{z}\right). \end{cases}$$

The signs being determined by the flow on the equator of S^2 as determined in Theorem 2.1. Similarly, the flow defined by 2.2 in a neighborhood of any equilibrium point of 2.2 on the equator of S^2 , except the points $(1, 0, 0)$, is topologically equivalent

to the flow of the system

$$(2.5) \quad \begin{cases} \pm \dot{x} = xz^m P(\frac{x}{z}, \frac{1}{z}) - z^m Q(\frac{x}{z}, \frac{1}{z}) \\ \pm \dot{z} = z^{m+1} P(\frac{x}{z}, \frac{1}{z}). \end{cases}$$

The signs being determined by the flow on the equator of S^2 as determined in Theorem 2.1 .

Remark 2.3. Equilibrium point of 2.4 at $(y_0, 0)$ corresponds to a critical point of 2.2 at the point

$$(\frac{1}{\sqrt{1+y_0^2}}, \frac{y_0}{\sqrt{1+y_0^2}}, 0)$$

on S^2 ; and a equilibrium point of 2.5 at $(x_0, 0)$ corresponds to a critical point of 2.2 at the point

$$(\frac{x_0}{\sqrt{1+x_0^2}}, \frac{1}{\sqrt{1+x_0^2}}, 0)$$

on S^2 .

In a 2- dimentions system like 2.1 if the Jacobian matrix of system has only one zero eigenvalue then the system 2.1 can be written in normal form:

$$(2.6) \quad \begin{cases} \dot{x} = p_2(x, y) \\ \dot{y} = y + q_2(x, y) \end{cases}$$

where p_2 and q_2 are analytic near the origin and have expansions that begin with second-degree terms.

Theorem 2.4. Let the origin be an isolated equilibrium point for the system 2.6 . Let $y = \phi(x)$ be such that $y + q_2(x, y) = 0$, in a neighborhood of the origin and let the expansion of the $\psi(x) = p_2(x, \phi(x))$ in a neighborhood of $x = 0$ be in the form $\psi(x) = a_m x^m + \dots$ where $m > 2$ and $a \neq 0$.

- (1) If m be odd and $a_m > 0$, then the origin is an unstable node.
- (2) If m be odd and $a_m < 0$, then the origin is a saddle.
- (3) If m be even, the origin is a saddle-node.

For more details see [8]. In addition for equilibrium points we can use the below theorem: Consider the following system

$$(2.7) \quad \dot{x} = f(x),$$

with $x \in R^2$ and smooth function f .

Theorem 2.5. *Let $\delta = \det J(x_0, y_0)$ and $\tau = \text{trace} J(x_0, y_0)$ and consider the linear system.*

- (1) *If $\delta < 0$ then 2.7 has a saddle at the point (x_0, y_0) .*
- (2) *If $\delta > 0$ and $\tau^2 - 4\delta > 0$ then 2.7 has a node at the point (x_0, y_0) ; it is stable if $\tau < 0$ and unstable if $\tau > 0$.*
- (3) *If $\delta > 0$, $\tau^2 - 4\delta < 0$ then the origin is a focus for 2.7; in addition it is stable if $\tau < 0$ and unstable if $\tau > 0$.*
- (4) *If $\delta > 0$ and $\tau = 0$ then 2.7 has a center at the point (x_0, y_0) .*

3. THE BEHAVIOR OF SPATIALLY HOMOGENEOUS AND INHOMOGENEOUS AUTOCATALYSIS SYSTEM AT INFINITY

In this section the singularities of 1.1 at infinity are studied.

3.1. Lemma *Equilibrium points of system 1.1 at infinity are: $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}, 0)$.*

Proof: According to Theorem 2.1 critical points of system 1.1 at infinite are satisfied to algebraic equation as follows:

$$XQ_{p+1}(X, Y) - YP_{p+1}(X, Y) = X^2Y^p + XY^{p+1} = 0$$

i.e., equilibrium points are on the equator of S^2 at $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}, 0)$.

According to Theorem 2.2, the behavior in a neighborhood of the critical point $(0, 1, 0)$ is determined by the behavior of the system

$$\begin{cases} -\dot{y} = yaz^{p+1} - (yz)^{p+1} - y^p + z^py \\ -\dot{z} = az^{p+2} - zy^p, \end{cases}$$

or equivalently

$$(3.1) \quad \begin{cases} \dot{y} = -yaz^{p+1} + (yz)^{p+1} + y^p - z^py \\ \dot{z} = -az^{p+2} + zy^p. \end{cases}$$

3.2. Lemma For system 1.1 ,

(i) If $p < 0$ then the critical point $(0, 1, 0)$, is a saddle point.

(ii) If $p > 0$ then the critical point $(0, 1, 0)$ always has an unstable node.

Proof: Consider the system 3.1 with the following Jacobian matrix

$$(3.2) \quad J = \begin{bmatrix} a_{11} & a_{12} \\ pzy^{p-1} & -a(p+2)z^{p+1} + y^p \end{bmatrix}$$

where

$$a_{11} = -az^{p+1} + (p+1)p^pz^{p+1} + py^{p-1} - z^p$$

and

$$a_{12} = -(p+1)ayz^p + (p+1)z^py^{p+1} - pyz^{p-1}.$$

(i) According to Theorem 2.2 , we have

$$(3.3) \quad J(1, 0) = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}.$$

so $\tau = \text{trace}J(1, 0) = p + 1$ and $\delta = \det J(1, 0) = p$. Consequently, according to Theorem 2.5 result is obtained.

(ii) Since $\tau^2 - 4\delta = p^2 - 2p + 1$ is always positive, Thus, according to Theorem 2.5 critical point $(0, 1, 0)$ for every $p > 0$ is always an unstable node for the system 1.1 will be.

3.3. Lemma For system 1.1 ,

(i) If $p > 0$ then the critical point $(0, -1, 0)$, is a saddle point.

(ii) If $p < 0$ and p odd then the critical point $(0, -1, 0)$ always has a stable node. Also if $p < 0$ and p even then the critical point $(0, -1, 0)$ always has an unstable node.

Proof: Similar to 3.1 by using theorem 2.2 and 2.5 the results will be achieved. Also according to theorem 2.2, the flow in some neighborhoods of the equilibrium points $(\pm 1, 0, 0)$ are similar to the flow of system

$$\begin{cases} -\dot{x} = (a-1)x - x^2 + z^p, \\ -\dot{z} = a - x. \end{cases}$$

or equivalently

$$(3.4) \quad \begin{cases} \dot{y} = x^2 + (1-a)x - z^p \\ \dot{z} = x - a. \end{cases}$$

Consider the system 3.4 with the following Jacobian matrix

$$(3.5) \quad J = \begin{bmatrix} 2x - a + 1 & -pz^{p-1} \\ 1 & 0 \end{bmatrix},$$

similar to above discussion we can find that the critical points $(1, 0, 0)$ and $(-1, 0, 0)$ are saddle node points.

3.4. Remark: According to remark 2.3 critical points $(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}, 0)$ are node or saddle-node.

3.5. Theorem Spatially homogeneous and inhomogeneous autocatalysis dynamical system does not have any infinite periodic behavior.

Proof: According to Lemma 3.2 and 3.3 because infinite critical points are node or saddle or saddle-node, so they can not have periodic behaviors.

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