

ON CURVES OF N_k -SLANT HELIX AND N_k -CONSTANT PRECESSION IN MINKOWSKI 3-SPACE

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ABSTRACT. In this paper, firstly N_k -timelike (spacelike) slant helix in 3-dimensional Minkowski space $\mathbb{E}_1^3$ is given by using an alternative coordinate frame which is obtained with the help of principal direction curves to Frenet frame $\{T, N, B\}$. Secondly, the axis of slant helix according to different conditions of curves is obtained. Lastly, curves of N_k -timelike (spacelike) constant precession which is a subfamily of N_k -slant helix are introduced.

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1. INTRODUCTION

The helix is generally known as a curve in crystal structure of molecules, DNA double, carbon nanotubes, form of plants. In the geometry, the definition of helix is a curve that which tangent makes a constant angle with a fixed straight line called the axis of the general helix. The study of these curves in the Euclidean 3-space based on 1802 with the Lancret [10]. On the other hand, the slant helices are studied by Izumiya and Takeuchi. And if any curve γ is called a slant helix,

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then the principal normal lines of γ make a constant angle with a fixed direction [7]. Moreover, They showed that γ is a slant helix in the Euclidean 3-space if and only if the geodesic curvature of the principal normal of a space curve γ is a constant function.

In there dimensional Euclidean space, using the new orthonormal frame which is obtained under a process with the help of k -principal direction curves of γ , we introduced a spatial curve family called N_k -slant helix which has the property that the principal normal vector of k -principal direction curves of γ makes a constant angle with a fixed direction [1]. Besides, the given generalization family contains the slant helix and C -slant helix which has the property that the second frame vector makes a constant angle with a fixed direction in terms of the frame $\{N, C, W\}$.

In this paper, we research the N_k -slant helix theory for spacelike and timelike spatial curves in Minkowski 3-space. With the help of orthonormal frame of k -timelike (spacelike) principal curve the characterization of main curve is obtained as a N_k -timelike (spacelike) slant helix which is the principal normal vector of k -principal curve make a constant angle with a fixed direction. Also we give the axis of slant helix according to the different situation of the curve and define curve of N_k -timelike (spacelike) constant precession which has the property that the Darboux vector of γ_k makes a constant angle with a fixed direction $\vec{u}$ which is the obtained for all cases and constant speed with Minkowski 3-space. Finally, we gave an effective example according to the theory.

2. PRELIMINARIES

Let $\mathbb{E}_1^3$ be Minkowski 3-space which endowed with the standard flat metric given by

$$(2.1) \quad \langle \vec{u}, \vec{v} \rangle = \vec{u}_1 \vec{v}_1 + \vec{u}_2 \vec{v}_2 - \vec{u}_3 \vec{v}_3$$

where $\vec{u} = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$ and $\vec{v} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ are the usual coordinate system in $\mathbb{E}_1^3$. Due to semi-Riemannian metric there are three different kind of vectors, namely spacelike, timelike and lightlike (null) depending on the properties $\langle \vec{u}, \vec{u} \rangle > 0$, $\langle \vec{u}, \vec{u} \rangle < 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$, respectively, for any vector $\vec{u}$ in $\mathbb{E}_1^3 - \{0\}$. These can be generalized for curves depending on the casual character of their tangent vectors, that is, the curve γ is called a spacelike (resp. timelike and lightlike) if its velocity vector $\gamma'(t)$ is spacelike (resp. timelike and lightlike) for any $t \in I$. The

norm of a vector $\vec{u}$ is given by $\|\vec{u}\| = \sqrt{|\langle \vec{u}, \vec{u} \rangle|}$. Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal, if $\langle \vec{u}, \vec{u} \rangle = 0$, see [5].

Let $\vec{u}$ and $\vec{v}$ be vectors in Minkowski 3-space. The Lorentzian angle between $\vec{u}$ and $\vec{v}$ are defined as follows, see [6]:

i) If $\vec{u}$ and $\vec{v} \in \mathbb{E}_1^3$ are spacelike vectors that span a spacelike plane,

$$(2.2) \quad \langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \|\vec{v}\| \cos \varphi(u, v)$$

ii) If $\vec{u}$ and $\vec{v} \in \mathbb{E}_1^3$ are spacelike vectors that span a timelike plane,

$$(2.3) \quad \langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \|\vec{v}\| \cosh \varphi(u, v)$$

iii) If $\vec{u}$ and $\vec{v} \in \mathbb{E}_1^3$ are timelike vectors in the same time-cone,

$$(2.4) \quad \langle \vec{u}, \vec{v} \rangle = -\|\vec{u}\| \|\vec{v}\| \cosh \varphi(u, v)$$

iv) If $\vec{u}$ is a spacelike vector and $\vec{v}$ is a future-directed timelike vector,

$$(2.5) \quad \langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \|\vec{v}\| \sinh \varphi(u, v)$$

where $\varphi(u, v) \in \mathbb{R}$ is the Lorentzian angle between $\vec{u}$ and $\vec{v}$.

We denote by $\{T(s), N(s), B(s)\}$ the moving positive directed Frenet frame along the unit speed curve γ in Minkowski 3-space. Then the following Frenet formulas can be given, where $T(s) = \gamma'(s)$, $N(s) = \frac{\gamma''(s)}{\|\gamma''(s)\|}$ and $B(s) = T(s) \times N(s)$.

If γ is spatial curve in Minkowski 3-space, then Frenet formulas can be given as follows, see [5]:

$$(2.6) \quad \begin{aligned} T' &= \kappa N, \\ N' &= -\varepsilon_0 \varepsilon_1 \kappa T + \tau B, \\ B' &= -\varepsilon_1 \varepsilon_2 \tau N, \end{aligned}$$

where κ and τ are curvature and torsion of the timelike curve γ , respectively and

$$(2.7) \quad \langle T, T \rangle = \varepsilon_0, \quad \langle N, N \rangle = \varepsilon_1, \quad \langle B, B \rangle = \varepsilon_2, \quad \langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0.$$

All of this study, for the sake of generality we used the future directed timelike vector.

Definition 1. Let $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a unit speed space curve with $\kappa(s) \neq 0$. Then γ is called a slant helix if there exists a non-zero constant vector field $\vec{u}$ in $\mathbb{E}_1^3$ such that the function $\langle N(s), \vec{u} \rangle$ is constant [4]. A curve $\gamma : I \longrightarrow \mathbb{E}_1^3$ with $\kappa(s) \neq 0$, is

called a helix if the tangent lines of γ make a constant angle with a fixed direction ,that is, $\langle T(s), \vec{u} \rangle$ is a constant function [9].

Theorem 1. Let γ be a unit speed timelike curve in $\mathbb{E}_1^3$. Then γ is a slant helix if and only if either one the next two functions

$$(2.8) \quad \sigma(s) = \frac{\kappa^2}{(\tau^2 - \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)' \quad \text{or} \quad \sigma(s) = \frac{\kappa^2}{(\kappa^2 - \tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)'$$

is constant everywhere $\tau^2 - \kappa^2$ does not vanish.

If γ be a unit speed spacelike curve in $\mathbb{E}_1^3$. Then the following two conditions can be given:

(a) If the normal vector of γ is spacelike, then γ is a slant helix if and only if either one the next two functions

$$(2.9) \quad \sigma(s) = \frac{\kappa^2}{(\tau^2 - \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)' \quad \text{or} \quad \sigma(s) = \frac{\kappa^2}{(\kappa^2 - \tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)'$$

is constant everywhere $\tau^2 - \kappa^2$ does not vanish.

(b) If the normal vector of γ is timelike, then γ is a slant helix if and only if the function

$$(2.10) \quad \sigma(s) = \frac{\kappa^2}{(\tau^2 + \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)'$$

is constant [4].

The Darboux vector W is the areal velocity vector of the Frenet frame of a space curve and it can be expressed as in terms of the Frenet apparatus as follows:

If spatial curve γ is timelike ($\varepsilon_0 = -1$), then

$$(2.11) \quad W = -(\tau T + \kappa B)$$

If spatial curve γ is spacelike ($\varepsilon_0 = 1$) and its principal normal vector spacelike or timelike, then

$$(2.12) \quad W = -\tau T + \kappa B.$$

3. k -PRINCIPAL DIRECTION CURVE AND N_k -TIMELIKE(SPACELIKE) SLANT HELIX IN MINKOWSKI 3-SPACE

In this section, it will be given the generalization of slant helix called N_k -timelike (spacelike) slant helix in Minkowski 3-space with the help of principal direction

curves. Moreover substantial relation will be obtained between characterization of curve and alternative coordinate frame $\{T_k, N_k, B_k\}$.

Definition 2. Let $\gamma(s)$ be a regular unit-speed curve in terms of $\{T, N, B\}$, then the integral curves of $T(s), N(s)$ and $B(s)$ are called the tangent direction curve, principal direction curve and binormal direction curve of γ , respectively [3].

Actually, in [1], it is showed that any $(n+2)$ -differentiable curve γ has k -principal direction curve for $1 \leq k \leq n$ in Euclidean space. On the other hand in Minkowski space, the following definitions and theorems can be obtained.

Definition 3. Let $\gamma(s)$ be a unit-parametrized and $(n+2)$ -differentiable curve in terms of $\{T, N, B\}$ in Minkowski 3-space and γ_0 be the tangent direction curve of γ . Then the k -principal direction curve of γ is defined as

$$(3.1) \quad \gamma_k(s) = \int N_{k-1} ds, \quad 1 \leq k \leq n,$$

where N_{k-1} is the principal normal vector of γ_{k-1} . Also γ is called the main curve of γ_k .

The Frenet frame and curvatures of γ_k can be given as

$$(3.2) \quad T_k = N_{k-1}, \quad N_k = \frac{N'_{k-1}}{\|N'_{k-1}\|}, \quad B_k = T_k \times N_k$$

$$(3.3) \quad \kappa_k = \sqrt{|\kappa_{k-1}^2 \pm \tau_{k-1}^2|}, \quad \tau_k = \sigma_{k-1} \kappa_k,$$

where $N_{k-1}, \kappa_{k-1}, \tau_{k-1}, \sigma_{k-1}$ are the principal normal vector, curvature, torsion and geodesic curvature of N_{k-1} , respectively, and B_k is the unit Darboux vector of γ_{k-1} . Also, the norm of the Darboux vector W_k can be given as $\sqrt{|\kappa_{k-1}^2 - \tau_{k-1}^2|}$ or $\sqrt{\kappa_{k-1}^2 + \tau_{k-1}^2}$ for timelike γ_{k-1} or spacelike γ_{k-1} , respectively.

Specially, for $k = 1$, the first principal direction curve γ_1 is

$$(3.4) \quad \gamma_1 = \int N(s) ds \text{ with the frame } \left\{ T_1 = N, N_1 = \frac{N'}{\|N'\|}, B_1 = T \times N \right\}$$

where the frame $\{T_1, N_1, B_1\}$ is equal to $\{N, C, W\}$ in terms of Minkowski metric.

Also, for $k = 2$, the second principal direction curve γ_1 is

$$(3.5) \quad \gamma_2 = \int N_1(s) ds \text{ with the frame } \left\{ T_2 = N_1, N_2 = \frac{N'_1}{\|N'_1\|}, B_2 = T_1 \times N_1 \right\}.$$

This generalization to be continued for $2 < k \leq n$. Unfortunately, the characterization of principal direction curves in terms of being timelike or spacelike is difficult to classify since the ramification of normal vector field of γ_k .

We denote by $\{T_k(s), N_k(s), B_k(s)\}$ the moving Frenet frame along the unit speed curve γ_k in the Minkowski space $\mathbb{E}_1^3$, then the following formulas can be given:

$$(3.6) \quad \begin{pmatrix} T'_k \\ N'_k \\ B'_k \end{pmatrix} = \begin{pmatrix} 0 & \kappa_k & 0 \\ -\varepsilon_0 \varepsilon_1 \kappa_k & 0 & \tau_k \\ 0 & -\varepsilon_1 \varepsilon_2 \tau_k & 0 \end{pmatrix} \begin{pmatrix} T_k \\ N_k \\ B_k \end{pmatrix}$$

where κ_k, τ_k are curvature, torsion of the curve γ_k , respectively, and

$$\langle T_k, T_k \rangle = \varepsilon_0, \quad \langle N_k, N_k \rangle = \varepsilon_1, \quad \langle B_k, B_k \rangle = \varepsilon_2, \quad \langle T_k, N_k \rangle = \langle T_k, B_k \rangle = \langle N_k, B_k \rangle = 0.$$

The Darboux vector W_k provides the following equations

$$(3.7) \quad \begin{aligned} W_k \times T_k &= T'_k \\ W_k \times N_k &= N'_k \\ W_k \times B_k &= B'_k. \end{aligned}$$

Also, the unit Darboux vector of γ_k is denoted as $\overline{W}_k$, $1 \leq k \leq n$. Now, we talk about relation between k -principal direction curve and its main curve.

Definition 4. Let γ be a $(n+2)$ -differentiable timelike(spacelike) spatial curve, γ_0 and γ_k be the tangent direction curve and k -principal direction curves of γ , $1 \leq k \leq n$, respectively. The curve γ is called N_k -timelike(spacelike) slant helix which has the property that the principal normal vector of γ_k makes a constant angle with a fixed line called the axis.

Remark 1. As it is known if the normal vector of any spatial curve makes a constant angle with a fixed direction, then the curve is called a slant helix. The condition of a curve, in terms of the alternative frame $\{T_k, N_k, B_k\}$, to be a slant helix is $k = 0$.

After giving the necessary definitions, we will give the following theories and results according to the types of curves in Minkowski 3-space, and complete the theory.

Theorem 2. *Let γ be a $(n+2)$ -differentiable timelike(spacelike) curve in Minkowski 3-space, and γ_k be the k -principal direction curves of γ , $1 \leq k \leq n$. The curve γ is a N_k -timelike(spacelike) slant helix if and only if the k -principal direction curve γ_k is a slant helix.*

Proof. Let γ_k be a unit speed curve in Minkowski 3-space. From definition 4., if γ is a N_k -timelike(spacelike) slant helix, then the principal normal vector of γ_k makes a constant angle with a fixed direction $\vec{u}$

$$\langle N_k, \vec{u} \rangle = \text{constant}$$

So, this definition coincides with being the slant helix of γ_k .

This completes the proof. $\square$

Corollary 1. *The main curve γ is a N_k -timelike slant helix if and only if $\sigma_k = \frac{\kappa_k^2 \left(\frac{\tau_k}{\kappa_k} \right)'}{(\tau_k^2 - \kappa_k^2)^{3/2}}$ or $\sigma_k = \frac{\kappa_k^2 \left(\frac{\tau_k}{\kappa_k} \right)'}{(\kappa_k^2 - \tau_k^2)^{3/2}}$ which is the geodesic curvature of N_k are a constant function everywhere $\tau_k^2 - \kappa_k^2$ does not vanish.*

Corollary 2. *The main curve γ is a N_k -spacelike slant helix with the spacelike normal vector if and only if $\sigma_k = \frac{\kappa_k^2 \left(\frac{\tau_k}{\kappa_k} \right)'}{(\tau_k^2 - \kappa_k^2)^{3/2}}$ or $\sigma_k = \frac{\kappa_k^2 \left(\frac{\tau_k}{\kappa_k} \right)'}{(\kappa_k^2 - \tau_k^2)^{3/2}}$ which is the geodesic curvature of N_k are a constant function everywhere $\tau_k^2 - \kappa_k^2$ does not vanish.*

Corollary 3. *The main curve γ is a N_k -spacelike slant helix with the timelike normal vector if and only if $\sigma_k = \frac{\kappa_k^2 \left(\frac{\tau_k}{\kappa_k} \right)'}{(\tau_k^2 + \kappa_k^2)^{3/2}}$ which is the geodesic curvature of N_k is a constant function.*

Theorem 3. *Let γ be a $(n+2)$ -differentiable spatial curve, γ_k be the k -principal direction curve of γ , $1 \leq k \leq n$. The curve γ is a N_k -timelike(spacelike) slant helix if and only if the tangent indicatrix of γ_k is a helix.*

Proof. Let γ be a N_k -timelike(spacelike) slant helix, then from the theorem 2 the k -principal direction curve γ_k is a slant helix. So, it is clear that tangent indicatrix T_k is a helix.

Conversely, if T_k is a timelike(spacelike) helix so the rate its curvature and torsion is constant function. This means that,

$$(3.8) \quad \frac{\tau_T}{\kappa_T} = \frac{\kappa_k^2 \left(\frac{\tau_k}{\kappa_k} \right)'}{(\tau_k^2 - \kappa_k^2)^{3/2}}, \quad \frac{\tau_T}{\kappa_T} = \frac{\kappa_k^2 \left(\frac{\tau_k}{\kappa_k} \right)'}{(\kappa_k^2 - \tau_k^2)^{3/2}} \text{ or } \frac{\tau_T}{\kappa_T} = \frac{\kappa_k^2 \left(\frac{\tau_k}{\kappa_k} \right)'}{(\tau_k^2 + \kappa_k^2)^{3/2}}$$

where the ratios are defined according to type of the k -principal direction curve γ_k . Thus from the Eq.(3.8) and theorem 2, it is obvious that γ is a N_k -slant helix.

This completes the proof. $\square$

4. THE AXIS OF N_k -TIMELIKE(SPACELIKE) SLANT HELIX

The alternative frame $\{T_k, N_k, B_k\}$ which is parametrized with s the parameter of the main curve γ is an orthonormal basis for Minkowski 3-space. In this section, with the aid of this frame, we arrange the axis of the N_k -timelike(spacelike) slant helix according to the different conditions of γ and types of axis of the principal direction curve γ_k .

Case 1. Let the main curve γ be a N_k -timelike(spacelike) slant helix with timelike k -principal direction curve γ_k in Minkowski 3-space. In this condition the axis of γ is either timelike or spacelike with different constant functions $\langle N_k, \vec{u} \rangle = \cos \theta$, $\langle N_k, \vec{u} \rangle = \cosh \theta$ or $\langle N_k, \vec{u} \rangle = \sinh \theta$.

Now, we give the following theorems to show that difference between the equations of the axis of γ in terms of the Darboux vector and the spacelike principal normal vector N_k .

Theorem 4. *Let the main curve γ be a N_k -timelike(spacelike) slant helix with the spacelike axis $\vec{u}$ in Minkowski 3-space. If N_k and the axis $\vec{u}$ span a spacelike plane or a timelike plane, then the axis of γ is defined as*

$$(4.1) \quad \vec{u} = -\sin \theta \overline{W}_k + \cos \theta N_k \text{ or } \vec{u} = -\sinh \theta \overline{W}_k + \cosh \theta N_k$$

respectively, where $\overline{W}_k$ is the unit Darboux vector of γ_k .

Proof. Let the main curve γ be a N_k -timelike(spacelike) slant helix with the spacelike axis $\vec{u}$, then

$$(4.2) \quad \langle N_k, \vec{u} \rangle = \cos \theta$$

where $Sp \{N_k, \vec{u}\}$ is a spacelike plane.

After the twice derivatives of Eq. (4.2),

$$(4.3) \quad \kappa_k \langle T_k, \vec{u} \rangle + \tau_k \langle B_k, \vec{u} \rangle = 0,$$

$$(4.4) \quad \kappa'_k \langle T_k, \vec{u} \rangle + \tau'_k \langle B_k, \vec{u} \rangle = (\tau_k^2 - \kappa_k^2) \cos \theta,$$

then from the Eq. (4.3) and (4.4) it is obtain that,

$$\langle T_k, \vec{u} \rangle = -\frac{\tau_k (\tau_k^2 - \kappa_k^2) \cos \theta}{\kappa_k^2 \left(\frac{\tau_k}{\kappa_k} \right)'} \text{ and } \langle B_k, \vec{u} \rangle = \frac{(\tau_k^2 - \kappa_k^2) \cos \theta}{\kappa_k \left(\frac{\tau_k}{\kappa_k} \right)'}$$

Hence,

$$\begin{aligned} \vec{u} &= (\lambda_1 T_k + \lambda_2 N_k + \lambda_3 B_k), \\ &= \left(\frac{\tau_k}{\sigma_k \sqrt{|\kappa_k^2 - \tau_k^2|}} T_k + N_k + \frac{\kappa_k}{\sigma_k \sqrt{|\kappa_k^2 - \tau_k^2|}} B_k \right) \cos \theta, \\ &= \left(\frac{1}{\sigma_k} \left(\frac{\tau_k T_k + \kappa_k B_k}{\sqrt{|\kappa_k^2 - \tau_k^2|}} \right) + N_k \right) \cos \theta, \end{aligned}$$

Since $\vec{u}$ is spacelike, $\langle \vec{u}, \vec{u} \rangle = -\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$ holds. If necessary operations are done, $\frac{1}{\sigma_k} = \tan \theta$ is provided. So the axis of γ is obtained as

$$\vec{u} = -\sin \theta \overline{W}_k + \cos \theta N_k.$$

On the other hand, while $Sp \{N_k, \vec{u}\}$ is a timelike plane, then $\langle N_k, \vec{u} \rangle = \cosh \theta$ and $\frac{1}{\sigma_k} = \tanh \theta$. So, the proof is similarly done and the axis is obviously obtained that

$$\vec{u} = -\sinh \theta \overline{W}_k + \cosh \theta N_k.$$

This proof is completed. $\square$

Theorem 5. Let γ the main curve of γ_k be a N_k -timelike(spacelike) slant helix with the timelike axis $\vec{u}$ in Minkowski 3-space. The axis of γ is

$$(4.5) \quad \vec{u} = -\cosh \theta \overline{W}_k + \sinh \theta N_k$$

where $N_k, \overline{W}_k$ are the principal normal vector, the unit Darboux vector of γ_k .

Proof. Let $\vec{u}$ be timelike axis, $\langle \vec{u}, \vec{u} \rangle = -\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = -1$ for the main curve γ . Also γ is a N_k -timelike(spacelike) slant helix with $\frac{1}{\sigma_k} = \coth \theta$. After similar operations such as theorem 4, the axis is obtained in form

$$\vec{u} = -\cosh \theta \overline{W}_k + \sinh \theta N_k.$$

Case 2. Let the main curve γ be a N_k –timelike(spacelike) slant helix and k –principal direction curve γ_k be spacelike with N_k the future directed timelike normal vector field. In this instance the axis of γ is specified with the constant functions $\langle N_k, \vec{u} \rangle = \sinh \theta$ or $\langle N_k, \vec{u} \rangle = -\cosh \theta$. $\square$

Theorem 6. Let γ the main curve of γ_k be a N_k –timelike(spacelike) slant helix in Minkowski 3–space. The axis of γ is

$$(4.6) \quad \vec{u} = -(\cosh \theta \bar{W}_k + \sinh \theta N_k) \quad \text{or} \quad \vec{u} = \sinh \theta \bar{W}_k + \cosh \theta N_k$$

when $\vec{u}$ is spacelike or timelike, respectively and $\bar{W}_k$ is the unit Darboux vector of γ_k .

Proof. Let the main curve γ be a N_k –spacelike slant helix with the axis $\vec{u}$, then

$$(4.7) \quad \langle N_k, \vec{u} \rangle = \sinh \theta$$

After the twice derivatives of Eq. (4.7),

$$(4.8) \quad \kappa_k \langle T_k, \vec{u} \rangle + \tau_k \langle B_k, \vec{u} \rangle = 0,$$

$$(4.9) \quad \kappa'_k \langle T_k, \vec{u} \rangle + \tau'_k \langle B_k, \vec{u} \rangle = -(\kappa_k^2 + \tau_k^2) \sinh \theta,$$

then from the Eq. (4.8) and (4.9) it is obtain that

$$\langle T_k, \vec{u} \rangle = \frac{\tau_k (\kappa_k^2 + \tau_k^2) \sinh \theta}{\kappa_k^2 \left(\frac{\tau_k}{\kappa_k} \right)'} \quad \text{and} \quad \langle B_k, \vec{u} \rangle = -\frac{(\kappa_k^2 + \tau_k^2) \sinh \theta}{\kappa_k \left(\frac{\tau_k}{\kappa_k} \right)'}$$

Hence,

$$\begin{aligned} \vec{u} &= (\lambda_1 T_k + \lambda_2 N_k + \lambda_3 B_k), \\ &= \left(\frac{\tau_k}{\sigma_k \sqrt{\kappa_k^2 + \tau_k^2}} T_k - N_k - \frac{\kappa_k}{\sigma_k \sqrt{\kappa_k^2 + \tau_k^2}} B_k \right) \sinh \theta, \\ &= \left(\frac{1}{\sigma_k} \left(\frac{\tau_k T_k - \kappa_k B_k}{\sqrt{\kappa_k^2 + \tau_k^2}} \right) - N_k \right) \sinh \theta, \end{aligned}$$

Since $\vec{u}$ is spacelike, $\langle \vec{u}, \vec{u} \rangle = -\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$ holds. If necessary arrangements are made, $\frac{1}{\sigma_k} = \tan \theta$ is provided. So the axis of γ is obtained as

$$\vec{u} = -(\cosh \theta \bar{W}_k + \sinh \theta N_k).$$

On the other hand, if the axis $\vec{u}$ is timelike and the normal indicatrix N_k is in the same timelike cone with $\vec{u}$, then $\langle N_k, \vec{u} \rangle = -\cosh \theta$ and $\frac{1}{\sigma_k} = \tanh \theta$. After similar calculations, the axis is

$$\vec{u} = \sinh \theta \overline{W}_k + \cosh \theta N_k.$$

This completes the proof. $\square$

Case 3. Let the main curve γ be a N_k -timelike(spacelike) slant helix and k -principal direction curve γ_k be spacelike with N_k the spacelike normal vector field. If the axis of γ is spacelike or timelike, then the constant function is $\langle N_k, \vec{u} \rangle = \cos \theta$ $\langle N_k, \vec{u} \rangle = \cosh \theta$ or $\langle N_k, \vec{u} \rangle = \sinh \theta$, respectively.

Theorem 7. *Let γ the main curve of γ_k be a N_k -timelike(spacelike) slant helix in Minkowski 3-space. If N_k which is the unit spacelike normal vector of γ_k and the spacelike axis $\vec{u}$ span a spacelike plane or a timelike plane, then the axis of γ is defined as*

$$(4.10) \quad \vec{u} = \sin \theta \overline{W}_k + \cos \theta N_k \text{ or } \vec{u} = -\sinh \theta \overline{W}_k + \cosh \theta N_k$$

respectively, where $\overline{W}_k$ is the unit Darboux vector of γ_k .

Proof. From the other proofs, the proof is clear. $\square$

Theorem 8. *Let γ the main curve of γ_k be a N_k -timelike(spacelike) slant helix in Minkowski 3-space. The axis of γ is*

$$(4.11) \quad \vec{u} = -\cosh \theta \overline{W}_k + \sinh \theta N_k$$

where $\vec{u}$ is the future directed timelike vector.

Proof. The proof is clear. $\square$

Corollary 4. *If any spatial curve γ is a N_k -timelike(spacelike) slant helix in Minkowski 3-space with the axis $\vec{u}$ which is for the condition described above, then it is obviously seen that $\langle \overline{W}_k, \vec{u} \rangle$ is a constant function where $\overline{W}_k$ is the Darboux vector of the k -principal direction curve of γ . Thus, the curve γ is also called $\overline{W}_k$ -timelike(spacelike) Darboux helix in Minkowski 3-space.*

5. CURVES OF N_k —TIMELIKE(SPACELIKE) CONSTANT PRECESSION

Any curve of constant precession is defined as the Darboux vector of any spatial curve revolves about a fixed direction $\vec{u}$ called the axis with constant angle and constant speed in Euclidean space [8]. In other words, curve of constant precession is a special slant helix with the constant-length Darboux vector. We define curve of N_k —timelike(spacelike) constant precession according to the different conditions of curves with Minkowski 3—space in this section.

Definition 5. Let γ be a $(n+2)$ —differentiable any spatial curve timelike(spacelike) in Minkowski 3—space, γ_k be k —principal direction curves of γ , $1 \leq k \leq n$. The curve γ is called curve of N_k —timelike(spacelike) constant precession which has the property that $\langle W_k, \vec{u} \rangle$ is constant function where W_k and $\vec{u}$ are the Darboux vector and fixed direction of γ_k , respectively in terms of $\{T_k, N_k, B_k\}$.

Corollary 5. If γ is a N_k —timelike(spacelike) slant helix with constant speed the Darboux vector in terms of $\{T_k, N_k, B_k\}$ in Minkowski 3—space, then γ is a curve of N_k —timelike(spacelike) constant precession.

Now, lets take an example on the curve of N_k —timelike(spacelike) constant precession.

Example 1. Let $\gamma(s)$ be a any unit speed spatial curve in Minkowski 3—space with curvatures $\kappa(s) = 4 \sin 5s \cosh(\frac{4}{5} \sin 5s)$, $\tau(s) = 4 \sin 5s \sinh(\frac{4}{5} \sin 5s)$ and the curve γ specifies neither helix nor slant helix in terms of the Frenet frame $\{T(s), N(s), B(s)\}$. Now lets obtain principal direction curves of γ and characterize itself. And with the solution of the differential systems (3.3) the curvatures of γ_1 are obtained as

$$\kappa_1 = 4 \sin 5s \text{ and } \tau_1 = 4 \cos 5s$$

So, the spacelike principal direction curve γ_1 which is specified uniquely by κ_1, τ_1 is obtained with the help of Taylor expansion as follows

$$\gamma_1(s) = \left(\frac{1}{24} \sin 8s + \frac{2}{3} \sin 2s, -\frac{1}{24} \cos 8s + \frac{2}{3} \cos 2s, \frac{4}{15} \sin 5s \right).$$

The tangent, principal, binormal vector field of the curve γ_1 are

$$\begin{aligned} T_1(s) &= \left(\frac{1}{3} \cos 8s + \frac{4}{3} \cos 2s, \frac{1}{3} \sin 8s - \frac{4}{3} \sin 2s, \frac{4}{3} \cos 5s \right), \\ N_1(s) &= \left(-\frac{4}{3} \cos 3s, -\frac{4}{3} \sin 3s, -\frac{5}{3} \right), \\ B_1(s) &= \left(\frac{1}{3} \sin 8s + \frac{4}{3} \sin 2s, -\frac{1}{3} \cos 8s + \frac{4}{3} \cos 2s, \frac{4}{3} \sin 5s \right), \end{aligned}$$

respectively.

From Theorem 1, γ_1 which is the first principal direction curve of γ is a slant helix so γ is a N_1 -timelike(spacelike) slant helix. Moreover the principal direction curve γ_1 has constant speed Darboux vector so γ is called curve of N_1 -constant precession with the axis

$$\begin{aligned} \vec{u} &= \sinh \phi \overline{W}_1 + \cosh \phi N_1 \\ &= (0, 0, 1). \end{aligned}$$

Now let's obtain the second principal direction curves of γ and characterize itself. Using the equation (3.1), the second principal direction curve of γ is $\gamma_2 = \int N_1(s) ds$. So, γ_2 is a timelike curve with a constant precession

$$\gamma_2(s) = \left(-\frac{4}{9} \sin 3s, \frac{4}{9} \cos 3s, -\frac{5}{3}s \right).$$

with curvatures $\kappa_2 = 4$, $\tau_2 = 5$. Hence, the tangent, principal, binormal vector field of the curve γ_2 are

$$\begin{aligned} T_2(s) &= \left(-\frac{4}{3} \cos 3s, -\frac{4}{3} \sin 3s, -\frac{5}{3} \right), \\ N_2(s) &= (\sin 3s, -\cos 3s, 0), \\ B_2(s) &= \left(-\frac{5}{3} \cos 3s, -\frac{5}{3} \sin 3s, -\frac{4}{3} \right) \end{aligned}$$

where $T_2 = N_1(s)$, B_2 is unit Darboux vector field of γ_2 .

From the theorem 2, γ_2 is a special (timelike/spacelike) slant helix with the director vector $\vec{u} = (0, 0, 1)$ and the fixed angle $\theta = 0$, that is, $\langle N_2, \vec{u} \rangle = \sinh \theta = 0$. So, γ is a N_2 -(timelike/spacelike) slant helix with the axis

$$\begin{aligned} \vec{u} &= \cosh \theta \overline{W}_2 + \sinh \theta N_2 \\ &= (0, 0, 1) \end{aligned}$$

where $\overline{W}_2 = \frac{\tau_2 T_2 + \kappa_2 B_2}{\sqrt{\tau_2^2 - \kappa_2^2}} = (0, 0, 1)$ (From theorem 5). Moreover the second principal curve γ_2 is a curve of constant precession with constant speed Darboux, then γ is called curve of N_2 –(timelike/spacelike) constant procession. Because, $T_2(s)$ tangent indicatrix of γ_2 is a helix and there is the fixed angle between tangent and the fixed direction.

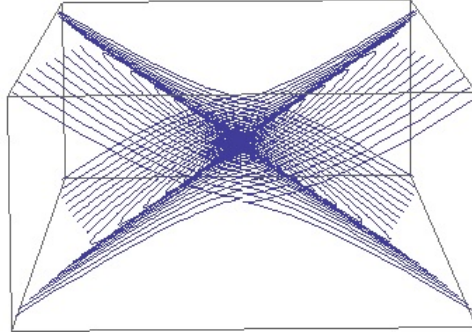


Figure 1. The main curve $\gamma(s)$

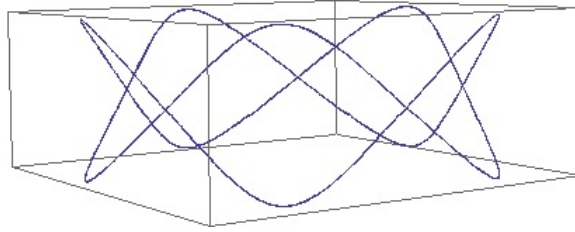


Figure 2. The first principal curve $\gamma_1(s)$

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